Using Telescoping Terms to Derive Formulae for Sums of Powers of the First *n* Natural Numbers

Darryl Smith

The excellent article by A. Craig Loewen titled "Sums of Arithmetic Sequences: Several Problems and a Manipulative" in the June 2004 issue (Volume 41, Number 2) of *delta-K* reminded me of Riemann sums and the formulae that are so important in those problems. For example, to calculate the value of the integral

$$\int_1^2 x^2 dx = \lim_{n\to\infty}\sum_{i=1}^{\infty} f(x_i)\Delta x_i \ ,$$

the interval must be partitioned from $x = 1$ to $x = 2$ into n subintervals, where the width of each subinterval, Δx_i, is given by the expression $\frac{2-1}{n}$ or $\frac{1}{n}$. Then,

$$\int_1^2 x^2 dx$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}(f(x_i)\Delta x_i)$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(f(1+\frac{i}{n})\frac{1}{n}\right)$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left((1+\frac{i}{n})^2\frac{1}{n}\right)$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left((1+\frac{2i}{n}+\frac{i^2}{n^2})\frac{1}{n}\right)$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{1}{n}\times 1+\frac{2}{n^2}\times i+\frac{1}{n^3}\times i^2\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n}1+\frac{2}{n^2}\sum_{i=1}^{n}i+\frac{1}{n^3}\sum_{i=1}^{n}i^2\right) .$$

At this point, some of the formulae for sums of powers of the first n natural numbers are required, such as

$$\sum_{i=1}^{n}1 = n \ ,$$

$$\sum_{i=1}^{n}i = \frac{n(n+1)}{2} \quad \text{and}$$

$$\sum_{i=1}^{n}i^2 = \frac{n(n+1)(2n+1)}{6} .$$

Other such formulae will appear later in this article. At first, I would introduce the required formulas, verify them and prove them by mathematical induction. However, it always concerned me that I did not have an algebraic method of determining these formulae in my bag of tricks.

To complete the above integral, substitutions of the required formulae are made into the last statement to obtain

$$= \lim_{n\to\infty}\left(\frac{1}{n}\times n+\frac{2}{n^2}\times\frac{n(n+1)}{2}+\frac{1}{n^3}\times\frac{n(n+1)(2n+1)}{6}\right)$$

$$= \lim_{n\to\infty}\left(1+\frac{n+1}{n}+\frac{1}{6}\times\frac{n+1}{n}\times\frac{2n+1}{n}\right)$$

$$= 1+1+\frac{2}{6}$$

$$= \frac{7}{3} .$$

That is, the value of the definite integral $\int_1^2 x^2 dx$ is exactly $\frac{7}{3}$. Of course, since $x^2 > 0$ for all values $x \in [1,2]$, the value of the integral is also the area enclosed by the function $y = x^2$ and the x-axis between $x = 1$ and $x = 2$.

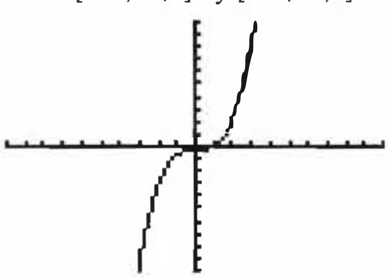
The result can be verified on the home screen of the TI-83 calculator using MATH9, which pastes in the numerical integration function fnInt. The format of the argument for this function is fnInt (function, independent variable, lower limit, upper limit).

```
fnInt(X²,X,1,2)
        2.333333333
█
```

When faced with evaluating an integral-defined function, such as $f(x)=\int_1^x t^2dt$, simply use the format described above to define the function in a convenient location, such as y_1, as shown below. In function mode, the only independent variable recognized by the TI-83 is x. That is, the t used in the defined function $f(x)=\int_1^x t^2dt$ will be replaced by x.

```
Plot1 Plot2 Plot3
\Y1■fnInt(X²,X,1
,X)█
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```

The graph of the function $f(x)=\int_1^x t^2dt$ is graphed below using Zoom-6, which defines the viewing window [-10,10,1] by [-10,10,1].



Notice that the graph of $f(x)=\int_1^x t^2dt$ appears to have a zero at $x=1$, which is consistent with the value of the definite integral $f(1)=\int_1^1 t^2dt=0$. The values of these definite integrals are easily obtained using the table feature of the TI-83, where the numbers in the column labelled y_1 are the values of definite integrals that are members of the range of the function $f(x)=\int_1^x t^2dt$. The domain for this function is $x\in R$. It is left to the reader to verify that, if $x<1$, then $f(x)=\int_1^x t^2dt<0$.

X	Y1
1	0
2	2.3333
3	8.6667
4	21
5	41.333
6	71.667
7	114

X=1

The focus of this article is not Riemann sums and integrals but, rather, an algebraic technique using the properties of summation and telescoping terms to directly derive formulae for the sums of powers of the first n natural numbers involved in Riemann sums.

Consider the series consisting of n terms, each of which is 1, so that we have 1 + 1 + 1 + . . . + 1. Because there are n-identical 1s, the sum is obviously n, so we can write $\sum_{i=1}^{n}1=n$. The same result could be obtained by treating the above series as arithmetic with a common difference of $d=0$.

To derive a formula for the sum of the first power of the first n natural numbers, we have $1+2+3+\ldots+n$. This series is arithmetic with $d=1$, and applying the formula

$S_n=\frac{n}{2}[2a+(n-1)d]$ gives the result

$$\sum_{i=1}^{n}i=\frac{n(n+1)}{2}.$$

For an alternative approach, consider the expression $(i+1)^2-i^2=2i+1$. We can use the properties of summation to obtain

$$\sum_{i=1}^{n}\left((i+1)^2-i^2\right)=\sum_{i=1}^{n}(2i+1)$$

$$\Leftrightarrow (1+1)^2-1^2+(2+1)^2-2^2+$$
$$(3+1)^2-3^2+\ldots+(n+1)^2-n^2$$
$$=2\sum_{i=1}^{n}i+\sum_{i=1}^{n}1$$

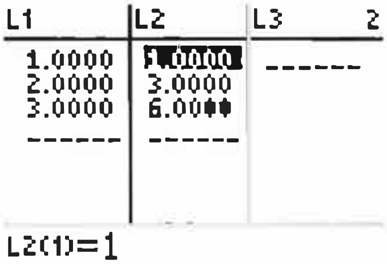
Notice that, in the expansion of the left-hand side, all the terms cancel except $(n+1)^2$ and $-(1)^2$; that is, the terms telescope, leaving just two terms. Because we have previously determined that $\sum_{i=1}^{n}1=n$, we obtain $(n+1)^2-1^2=2\sum_{i=1}^{n}i+n$. Solve for $\sum_{i=1}^{n}i$ to obtain

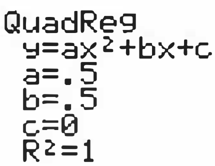
$$\sum_{i=1}^{n}i=\frac{(n+1)^2-1^2-n}{2}=\frac{n^2+n}{2}$$

$$\sum_{i=1}^{n}i=\frac{n(n+1)}{2}$$

as before.

A calculator approach can also be taken. Using STAT mode on the TI-83, simply enter at least the first three terms of the natural number sequence 1, 2, 3, . . . in List1, and a matching number of terms for the sequence of partial sums 1, 3, 6, . . . in List2, as shown. Because the quadratic regression involves the parameters a, b and c, at least three data points are required.





Performing a quadratic regression gives

$$y = 0.5x^2 + 0.5x + 0 = \frac{x(x+1)}{2}$$

with $R^2 = 1$. Because it is a sequence, observe the condition that $x \in N$.

Generally, to determine an expression for the sum of the first n terms of the kth power of the natural numbers, the following expression can be used:

$$(i+1)^{k+1} - i^{k+1} = i^{k+1} + (k+1)i^k + \frac{(k+1)k}{2!}i^{k-1} + \ldots$$
$$+1 - i^{k+1}.$$

The first and last terms on the right will cancel, so we have

$$\sum_{i=1}^{n}\left((i+1)^{k+1} - i^{k+1}\right) = \sum_{i=1}^{n}\left((k+1)i^k + \frac{(k+1)k}{2!}i^{k-1} + \ldots + 1\right).$$

In expanding the left-hand side of this expression, the terms will always telescope. Simply substitute previously determined expressions into the right-hand side and solve for the expression $\sum_{i=1}^{n} i^k$.

To again illustrate the technique, we determine a closed form for $\sum_{i=1}^{n} i^2$. Beginning with the expression $(i+1)^3 - i^3 = 3i^2 + 3i + 1$, we obtain

$$\sum_{i=1}^{n}\left((i+1)^3 - i^3\right) = \sum_{i=1}^{n}\left(3i^2 + 3i + 1\right)$$

$$\Leftrightarrow (1+1)^3 - 1^3 + (2+1)^3 - 2^3 + (3+1)^3 - 3^3 + \ldots$$

$$+(n+1)^3 - n^3 = 3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1$$

$$\Leftrightarrow (n+1)^3 - 1^3 = 3\sum_{i=1}^{n} i^2 + 3\frac{n(n+1)}{2} + n$$

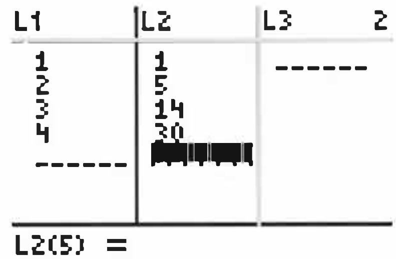
$$\Leftrightarrow n^3 + 3n^2 + 3n = 3\sum_{i=1}^{n} i^2 + 3\frac{n(n+1)}{2} + n$$

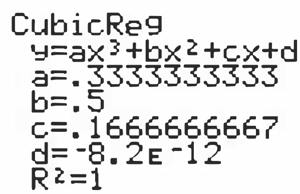
$$\Leftrightarrow 2n^3 + 6n^2 + 6n = 6\sum_{i=1}^{n} i^2 + 3n^2 + 3n + 2n$$

$$\Leftrightarrow 6\sum_{i=1}^{n} i^2 = 2n^3 + 3n^2 + n = n(2n^2 + 3n + 1)$$

$$\Leftrightarrow \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

To obtain the same result on the TI-83, a cubic regression using at least four data points is needed, where $L2_1 = 1^2$, $L2_2 = 1^2 + 2^2$ and so on.





The calculator gives the result $y = 0.333333\ldots x^3 + 0.5x^2 + 0.1666666\ldots x$, with $R^2 = 1$. Since the coefficients are $\frac{1}{3}$, $\frac{1}{2}$ and $\frac{1}{6}$ respectively, we have

$$y = \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{6}x,$$

which can be expressed in the more familiar and convenient form

$$y = \frac{x(x+1)(2x+1)}{6}, x \in N.$$

Consider $(i+1)^4 - i^4 = 4i^3 + 6i^2 + 4i$. Using the properties of summation, we obtain

$$\sum_{i=1}^{n}\left((i+1)^4 - i^4\right) = \sum_{i=1}^{n}\left(4i^3 + 6i^2 + 4i + 1\right)$$

$$\Leftrightarrow (1+1)^4 - 1^4 + (2+1)^4 - 2^4 + (3+1)^4 - 3^4 + \ldots$$

$$+(n+1)^4 - n^4 = 4\sum_{i=1}^{n} i^3 + 6\sum_{i=1}^{n} i^2 + 4\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1$$

$$\Leftrightarrow (n+1)^4 - 1^4 = 4\sum_{i=1}^{n} i^3 + 6\frac{n(n+1)(2n+1)}{6} +$$

$$\Leftrightarrow n^4 + 4n^3 + 6n^2 + 6n = 4\sum_{i=1}^{n} i^3 + 2n^3 + 3n^2 +$$

$$\Leftrightarrow n^4 + 2n^3 + n^2 = 4\sum_{i=1}^{n} i^3$$

$$\Leftrightarrow 4\sum_{i=1}^{n} i^3 = n^2(n^2 + 2n + 1)$$

$$\Leftrightarrow \sum_{i=1}^{n} i^3 = \frac{n^2(n^2 + 2n + 1)}{4} = \frac{n^2(n+1)^2}{4}$$

$$\Leftrightarrow \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2 .$$

Expressing the result in this form makes it easy to remember because it is simply the square of the result for the sum of the first n natural numbers.

Regression can be used as before, where $L2_1 = 1^3$, $L2_2 = 1^3 + 2^3$ and so on.

L1	L2	L3 2
1	1	------
2	9	
3	36	
4	100	
5	225	

L2(6) =

```
QuarticReg
 y=ax⁴+bx³+...+e
 a=.25
 b=.5
 c=.2500000002
 d=-3.42E-10
 e=1.81E-10
```

The R^2 value is again 1, and the value of the coefficients d and e in $y = ax^4 + bx^3 + cx^2 + dx + e$ is 0. Therefore, we have $y = 0.25x^4 + 0.5x^3 + 0.25x^2$ or

$$y = \frac{1}{4}x^4 + \frac{1}{2}x^3 + 0.25x^2$$

$$= \frac{x^4 + 2x^3 + x^2}{4}$$

$$= \left[\frac{x(x+1)}{2}\right]^2, x \in N .$$

The expression for $\sum_{i=1}^{n} i^4$ is a bit tedious, but it is achieved using the same technique. We begin with

$$\sum_{i=1}^{n}\left((i+1)^5 - i^5\right) = \sum_{i=1}^{n}\left(5i^4 + 10i^3 + 10i^2 + 5i + 1\right)$$

$$\Leftrightarrow (n+1)^5 - 1^5 = 5\sum_{i=1}^{n} i^4 + 10\sum_{i=1}^{n} i^3 + 10\sum_{i=1}^{n} i^2 + 5\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 .$$

As before, we substitute previously derived expressions into the right-hand side and simplify the left-hand side to obtain

$$5n^4 + 10n^3 + 10n^2 + 5n + 1$$

$$= 5\sum_{i=1}^{n} i^4 + 10\frac{n^2(n+1)^2}{4} + 10\frac{n(n+1)(2n+1)}{6} +$$

$$5\frac{n(n+1)}{2} + n .$$

Multiply through by 6 to clear fractions and isolate the term containing $\sum_{i=1}^{n} i^4$ to obtain

$$30\sum_{i=1}^{n} i^4 = 6n^5 + 15n^4 + 10n^3 - n$$

$$= n(6n^4 + 15n^3 + 10n^2 - 1) .$$

The right-hand side can be factored and then divided through by 30 for the final result of

$$\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2 + 3n - 1)}{30} .$$

Unfortunately, a quintic polynomial regression is beyond the capabilities of the calculator.

Conclusion

The properties of summation and telescoping terms have been used to derive the following:

$$\sum_{i=1}^{n} 1 = n$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

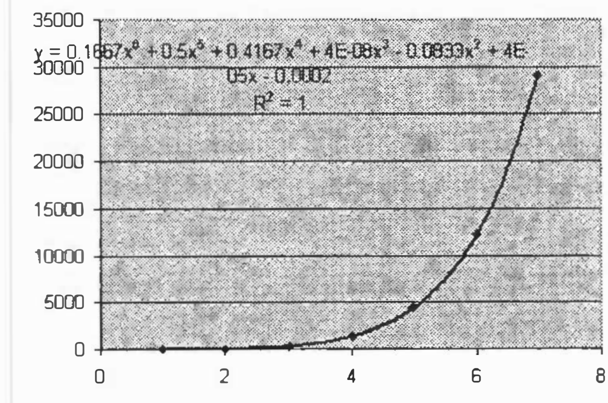
$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

$$\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2 + 3n - 1)}{30} .$$

The next time you teach Riemann sums in your calculus class and it comes time to derive formulae for sums of powers of natural numbers, I encourage you to consider the direct approach using telescoping terms. The technique is rich in algebraic opportunity, such as expanding powers of binomials and exploring the properties of sigma notation and limits. When I carefully and thoroughly worked through the first derivations with the class, the students were capable of doing the last ones by themselves, provided that I gave them a hint as to the required form. More importantly, students always seem impressed by their ability to use what turns out to be rather straightforward algebraic tools to determine some rather impressive identities. Also, a word from the voice of experience: even though the overhead projector was my favourite mode of presentation in class, I always did this lesson on a 20-foot whiteboard. It made it easier to follow the derivations, look back and record the list of formulae as we went. The calculator regressions are interesting and are best done concurrently with the algebraic derivation, but they are insufficient by themselves. Try obtaining these regressions using Excel, a program capable of doing up to degree 6 polynomial regressions. The result given by Excel for $\sum_{i=1}^{n} i^5$ is shown below. In spite of the given value $R^2 = 1$, the result holds only to approximately the fourth term when compared to exact values determined from

$$\sum_{i=1}^{n} i^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}.$$

n	n^5	partial sum	using derivation	regression eqn
1	1	1	1	0.99994004
2	32	33	33	33.00268032
3	243	276	276	276.0272211
4	1024	1300	1300	1300.145563
5	3125	4425	4425	4425.542505
6	7776	12201	12201	12202.59965
7	16807	29008	29008	29012.00339



Darryl Smith is in his third year of retirement after 34 years with the Edmonton Catholic School District, 30 of which were spent at Austin O'Brien High School. His one regret is that technology use did not arrive in the mathematics classroom until the last third of his career. During the past two years, he has had the privilege of working with many excellent teachers from the Edmonton Catholic School District in workshop settings and relishes these opportunities to implement calculator technology into mathematics education.