A New Approach to Partial Fractions

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Algebra courses give instruction for adding or subtracting two rational functions, or ratios of polynomials. One way to do this is to find a common denominator, rewrite each rational function using this common denominator, add or subtract the numerators and simplify the result. In integral calculus, it is often necessary to reverse this process; that is, to rewrite a given rational function as a sum of several simpler rational functions, or partial fractions (see, for instance, Stewart 2001, 405–7). It is possible to describe and justify a method for carrying out this rewriting at the level of a typical precalculus course (see, for instance, Dobbs and Peterson 1993, 197–204). Because this method is designed to be generally applicable, it tends to be rather time-consuming. Moreover, this method consists of several steps, requires the user to introduce a considerable amount of notation (variables that must be solved for in systems of equations) and often taxes students' memories and patience. Although some shortcuts are known, they can fail to lead to a complete solution, such as when the denominator of the given rational function in lowest terms has a multiple root. One currently fashionable way to address this situation is to use computer algebra systems (CAS) to find partial-fraction decompositions. This article will introduce a new paper-and-pencil algorithm for obtaining partial-fraction decompositions. The next two sections will introduce this new approach as an iterative method based on a few easy-to-remember strategies.

Like other paper-and-pencil algorithms for obtaining partial-fraction decompositions, this method is quickest when applied to decomposing rational functions whose denominators are expressed as products of linear factors. This new method has the following advantages. When used on complex calculus problems and differential equations, technology is not required to combat impatience. The new method does not require the user to introduce any notation. Data whose denominators have multiple roots do not pose any additional difficulty. Like traditional approaches to partial-fraction decomposition, the new method can reinforce several topics in the typical precalculus course, including arithmetic operations on rational functions and factorization of quadratic polynomials. In addition, the new method can reinforce the binomial theorem and, if covered, mathematical induction. Moreover, its treatment of irreducible quadratic factors can reinforce arithmetic with complex numbers.

The New Method When Denominators Have Only Linear Factors

The task is to decompose a rational function that is in lowest terms. In this section, we consider the case where the denominator of the given rational function has only linear factors. Notice that a linear factor $ax+b$ can be rewritten as $a(x-c)$, where $c=-\frac{b}{a}$. Thus, there is no harm in assuming that the linear factors of the denominator are of the form $x-d$ for various constants d.

One paradigm will be enough to point the way. Our experience in algebra courses with the addition and subtraction of rational functions tells us that

$$\frac{A}{x-a}+\frac{B}{x-b}=\frac{(A+B)x-(Ab+Ba)}{(x-a)(x-b)}.$$

Therefore, to rewrite

$$\frac{1}{(x-a)(x-b)} \text{ as}$$

$$\frac{A}{x-a}+\frac{B}{x-b},$$

$(A+B)x-(Ab+Ba)$ must equal 1. In particular, the coefficient of x must be 1; that is, $A+B=0$ or, equivalently, $B=-A$. Although this analysis has the flavour of the traditional method for partial-fraction decomposition, all that needs to be remembered from the analysis thus far is that, if $a \neq b$, then

$$\frac{1}{(x-a)(x-b)}$$

can be rewritten as

$$\frac{A}{x-a}-\frac{A}{x-b}$$

for the constant A, which can be determined using the rules for the addition and subtraction of rational functions. For instance, to decompose

$$\frac{1}{(x-2)(x+3)}\text{, use}$$

$$\frac{A}{x-2}-\frac{A}{x+3},$$

where $A(x+3)-A(x-2)=1$; that is, $5A=1$, or $A=\frac{1}{5}$. The upshot is the partial-fraction decomposition

$$\frac{1}{(x-2)(x+3)}=\frac{1/5}{x-2}-\frac{1/5}{x+3}.$$

The above method is memorable, and can be summarized with the following general formula: If $a \neq b$, then

$$\frac{1}{(x-a)(x-b)}=\frac{1/(-b+a)}{x-a}\frac{1/(-b+a)}{x-b}. \qquad (1.1)$$

Let us next consider the case that the given rational function has a constant numerator and denominator with more than two linear factors. If the denominator does not have a multiple root (that is, if the denominator does not have a repeated linear factor), the above reasoning only has to be repeated several times. More precisely, to decompose a rational function of the form

$$\frac{1}{(x-a)g(x)},$$

decompose the simpler rational function $\frac{1}{g(x)}$, multiply the resulting sum of partial fractions by $\frac{1}{x-a}$, expand by using the distributive property and then decompose each of the resulting simpler rational functions. The success of the method depends on reducing the problem to a set of simpler rational functions at each step. The rigorous mathematical explanation is that the method uses mathematical induction on the number of linear factors in the denominator. However, most students are fully convinced (and should be) by the following kind of example.

To obtain the partial-fraction decomposition of

$$\frac{1}{(x-2)(x+1)(x-4)},$$

reason as follows. First, as in the general formula from the first decomposition problem, we have

$$\frac{1}{(x+1)(x-4)}=\frac{-1/5}{x+1}+\frac{1/5}{x-4}.$$

Next, multiplying through by $\frac{1}{x-2}$, we find that

$$\frac{1}{(x-2)(x+1)(x-4)}=\frac{1}{x-2}\left(\frac{-1/5}{x+1}+\frac{1/5}{x-4}\right)$$

$$=-\frac{1}{5}\left(\frac{1}{(x-2)(x+1)}\right)+\frac{1}{5}\left(\frac{1}{(x-2)(x-4)}\right).$$

Finally, by again using the earlier general formula and combining like terms, we obtain

$$-\frac{1}{5}\left(\frac{1/3}{x-2}-\frac{1/3}{x+1}\right)+\frac{1}{5}\left(\frac{-1/2}{x-2}+\frac{1/2}{x-4}\right)$$

$$=\frac{-1/6}{x-2}+\frac{1/15}{x+1}+\frac{1/10}{x-4}.$$

To handle nonconstant numerators, we only need to rewrite each occurrence of x in the numerator as $(x-a)+a$, where $x-a$ is a factor of the denominator, and then to rewrite each occurrence of x^n in the numerator by using the binomial theorem to expand $((x-a)+a)^n$. In effect, we are rewriting the numerator, which is a polynomial in the variable x, as a polynomial in the variable $x-a$. (This part of the method is an excellent preparation activity for calculus, in which students could learn, more generally, to approximate an n–times differentiable function by its nth Taylor polynomial centred at a.) Once again, the following sufficiently complicated example can be given to convince a class that the method works in general.

Let's use the above strategy to find the partial-fraction decomposition of

$$\frac{3x^2-7x+5}{(x-2)(x+1)(x-4)}.$$

The numerator is rewritten as

$$3((x-2)+2)^2-7((x-2)+2)+5=$$

$$3((x-2)^2+4(x-2)+4)-7(x-2)-9=$$

$$3(x-2)^2+5(x-2)+3 \quad .$$

Thus, by again using what is known about addition and subtraction of rational functions and then cancelling common factors of corresponding numerators and denominators, we find that

$$\frac{3x^2-7x+5}{(x-2)(x+1)(x-4)}=\frac{3(x-2)^2+5(x-2)+3}{(x-2)(x+1)(x-4)}=$$

$$\frac{3(x-2)}{(x+1)(x-4)}+\frac{5}{(x+1)(x-4)}+\frac{3}{(x-2)(x+1)(x-4)}.$$

We have already seen how to decompose each of the last two terms on the right-hand side as sums of partial fractions. The remaining term,

$$\frac{3(x-2)}{(x+1)(x-4)},$$

is simpler than the original problem and can now be written as a sum of partial fractions by another application of the above method, as follows:

$$\frac{3(x-2)}{(x+1)(x-4)}=\frac{3x-6}{(x+1)(x-4)}=\frac{3((x+1)-1)-6}{(x+1)(x-4)}$$

$$=\frac{3(x+1)-9}{(x+1)(x-4)}=\frac{3}{x-4}-\frac{9}{(x+1)(x-4)}.$$

How should we proceed if the denominator of the given rational function has a multiple root; that is, if it has a repeated linear factor? It is enough to indicate how to handle rational functions of the form

$$\frac{1}{(x-a)^n g(x)},$$

where $x-a$ is not a factor of $g(x)$ (that is, where a is not a root of $g(x)$), because nonconstant numerators can be handled as above using the "$x=(x-a)+a$" trick. To do so, factor $x-a$ out of the denominator so that the problem is presented as

$$\frac{1}{x-a}\left(\frac{1}{(x-a)^{n-1}g(x)}\right).$$

Notice that the second factor,

$$\frac{1}{(x-a)^{n-1}g(x)},$$

represents a simpler problem (inasmuch as its denominator has a degree that is less than the degree of the original denominator). By mathematical induction on this degree (in practice, by repeatedly factoring linear factors out of the denominator), we can reduce the rational function to a sum of simpler rational functions that are, ultimately, already in partial-fraction form or amenable to being rewritten with the help of the general formula from earlier. The following example can be given to illustrate the method.

Let's indicate how to use the above strategy to produce the partial-fraction decomposition of

$$\frac{1}{(x-2)^2(x+4)^3(x-1)}.$$

Note that the given denominator has degree 6. Factor one of the linear factors out of the denominator, such as $x-2$, so that the problem becomes rewriting

$$\frac{1}{x-2}\left(\frac{1}{g(x)}\right),$$

where $g(x)=(x-2)(x+4)^3(x+1)$. Lurking within $\frac{1}{g(x)}$ is sure to be a simpler problem, which is either already in partial-fraction form or amenable to being treated by the general formula from earlier. By the general formula,

$$\frac{1}{(x-2)(x+1)}=\frac{1/3}{x-2}-\frac{1/3}{x+1}.$$

Therefore, in multiplying by $\frac{1}{(x+4)^3}$, we have

$$\frac{1}{g(x)}=\frac{1/3}{(x-2)(x+4)^3}-\frac{1/3}{(x+1)(x+4)^3},$$

and so the given rational function has been rewritten as

$$\frac{1}{(x-2)g(x)}=\frac{1/3}{(x-2)^2(x+4)^3}-\frac{1/3}{(x+1)(x-2)(x+4)^3}.$$

Notice that both of the terms on the right-hand side have degree 5, and are thus simpler than the original data. By iterating the process once more, we come to a sum of terms, each having denominators with degree 4 at most. Further iterations lead to the desired partial-fraction decomposition. For reasons of space, the details of these calculations will be left to the reader.

When Denominators Have Irreducible Quadratic Factors

In this section, we will look at a new way of finding the partial-fraction decomposition of a rational function in lowest terms whose denominator has at least one irreducible quadratic factor. We begin with a notational simplification, just like at the beginning of the preceding section. Any quadratic polynomial ax^2+bx+c can be rewritten as $a(x^2+dx+e)$, where $d=\frac{b}{a}$ and $e=\frac{c}{a}$. Thus, we can assume that the irreducible quadratic factors of the denominator are of the form x^2+bx+c for various constants b and c.

It would appear at first that, in order to continue using the method introduced in the preceding section, more than one formula analogous to the general formula may need to be developed. This approach is rather similar to the classical method of introducing several variables and solving systems of linear equations. For this reason, this section's problem is best handled by reducing it to that of the preceding section. This can be done, provided that you are prepared to use complex numbers as coefficients of various relevant polynomials. If $g(x)=x^2+bx+c$ is an irreducible factor of the denominator, recall that $g(x)$ can be factored as $(x-r_1)(x-r_2)$, where r_1 and r_2 are the roots of $g(x)$. (If necessary, r_1 and r_2 can be found by using the quadratic formula.) Thus, if the denominator is given as having been factored over the real numbers (as is usual in the classical method for partial-fraction decomposition), we can write that denominator as a product of linear factors, possibly with complex number coefficients. We can then proceed to find a partial-fraction decomposition exactly as in the preceding section.

The price to be paid in using this method is that you must be willing to do a considerable amount of arithmetic with complex numbers. Moreover, it is easy to lose heart during the process as to whether the final result will simplify and have only real-number coefficients. Be assured that this will always work out, in view of the general theorem guaranteeing partial-fraction decompositions over any field (see, for instance, Dobbs and Hanks 1992, 31–33). While keeping track of the various factors that arise, remember that the nonreal complex roots of any real polynomial $g(x)$ arise as pairs of complex numbers that have the same multiplicity as roots of $g(x)$ (see, for instance, Dobbs and Hanks 1992, 54).

Be warned that this method of handling irreducible quadratic factors can be time-consuming. However, as promised, it requires only the method of the preceding section coupled with the above formula for factoring quadratic polynomials and the ability to perform the arithmetic operations on complex numbers. We close with an example that illustrates the new method outlined in this section.

Use the above method to find the partial-fraction decomposition of

$$\frac{x^2-7x-9}{(x^2+4)(x-3)}.$$

Using the "$x=(x-a)+a$" trick, we can rewrite the numerator as

$$((x-3)+3)^2-7((x-3)+3)-9=(x-3)^2-(x-3)-21.$$

Therefore, by distributivity and cancelling,

$$\frac{x^2-7x-9}{(x^2+4)(x-3)}.$$

The first and second terms on the right-hand side are already in partial-fraction form. So, all we have left to do is find the partial-fraction decomposition of

$$\frac{1}{(x^2+4)(x-3)}.$$

Because the roots of x^2+4 are $\pm 2i$, the denominator can be factored as $(x+2i)(x-2i)(x-3)$. Next, by using the method of the preceding section and arithmetic with complex numbers, we find

$$\frac{1}{(x^2+4)(x-3)} = \frac{1}{(x+2i)(x-2i)(x-3)} =$$

$$\frac{1}{x-3}\left(\frac{1}{(x+2i)(x-2i)}\right) = \frac{1}{x-3}\left(\frac{i/4}{x+2i}-\frac{i/4}{x-2i}\right) =$$

$$\left(\frac{i}{4}\right)\left(\frac{1}{(x-3)(x+2i)}-\frac{1}{(x-3)(x-2i)}\right) =$$

$$\left(\frac{i}{4}\right)\left[\left(\frac{1}{2i+3}\right)\left(\frac{1}{x-3}-\frac{1}{x+2i}\right)\right]-\left(\frac{i}{4}\right)\left[\left(\frac{1}{3-2i}\right)\left(\frac{1}{x-3}-\frac{1}{x-2i}\right)\right] =$$

$$\left(\frac{i}{4}\right)\left(\frac{1}{2i+3}-\frac{1}{3-2i}\right)\frac{1}{x-3}-\left(\frac{i}{4}\right)\left(\frac{1}{2i+3}\right)\frac{1}{x+2i}+$$

$$\left(\frac{i}{4}\right)\left(\frac{1}{3-2i}\right)\frac{1}{x-2i}=\frac{1/13}{x-3}-\left(\frac{2+3i}{52}\right)\frac{1}{x+2i}+\left(\frac{-2+3i}{52}\right)\frac{1}{x-2i}.$$

Then, by adding the terms that involve nonreal numbers (the same way that any rational functions are added), the final form of the decomposition is

$$\frac{1}{(x^2+4)(x-3)}=\frac{1/13}{x-3}+\frac{-\left(\frac{2+3i}{52}\right)(x-2i)+\left(\frac{-2+3i}{52}\right)(x+2i)}{(x+2i)(x-2i)}=$$

$$\frac{\frac{1}{13}}{x-3}+\frac{-\frac{1}{13}x-\frac{3}{13}}{x^2+4}.$$

References

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