delta-k

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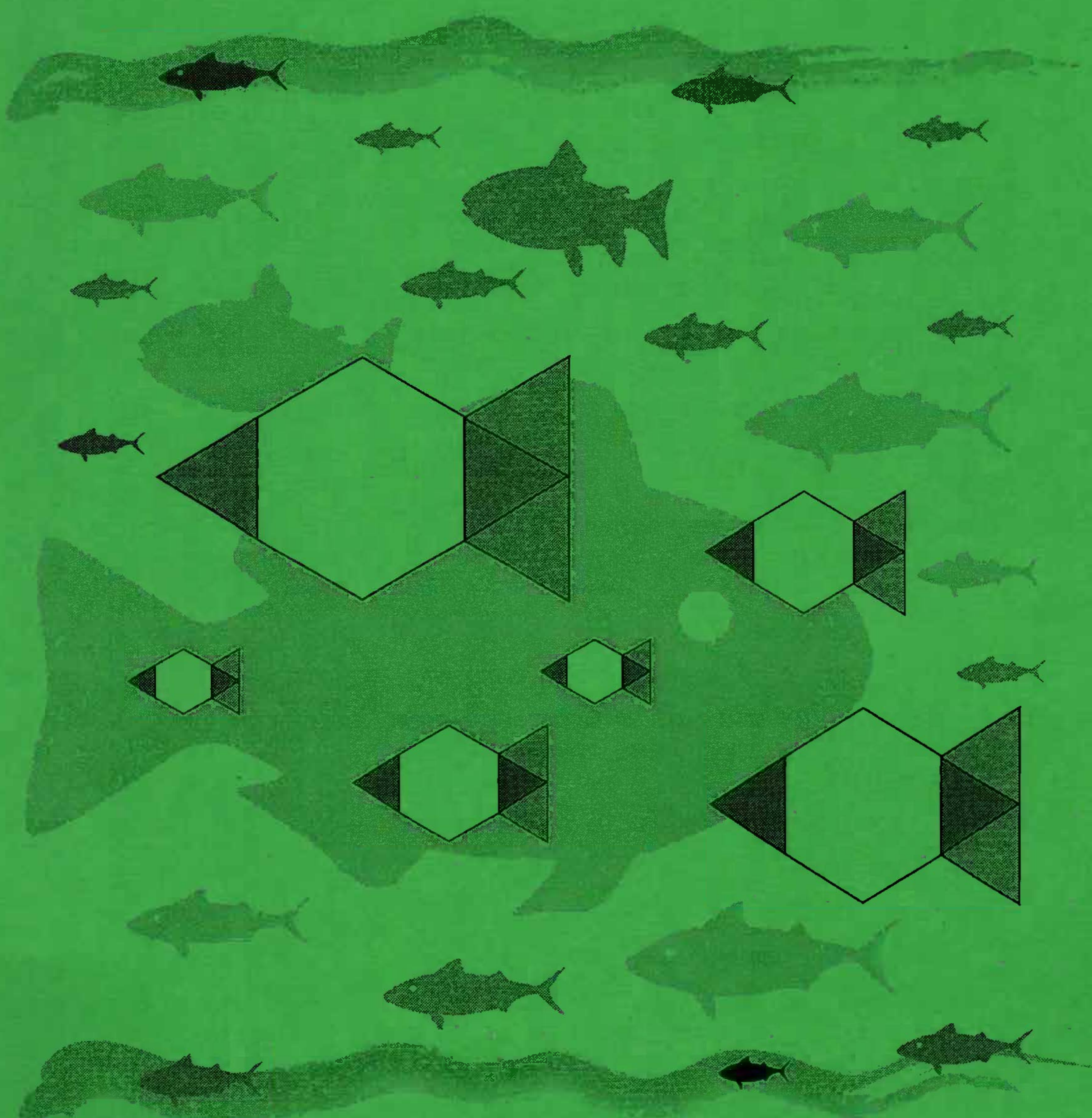

GUIDELINES FOR MANUSCRIPTS

delta-K is a professional journal for mathematics teachers in Alberta. It is published to

- promote the professional development of mathematics educators, and
- stimulate thinking, explore new ideas and offer various viewpoints.

Submissions are requested that have a classroom as well as a scholarly focus. They may include

- personal explorations of significant classroom experiences;
- descriptions of innovative classroom and school practices;
- reviews or evaluations of instructional and curricular methods, programs or materials;
- discussions of trends, issues or policies;
- a specific focus on technology in the classroom; and
- a focus on the curriculum, professional and assessment standards of the NCTM.

Manuscript Guidelines

1. All manuscripts should be typewritten, double-spaced and properly referenced.
2. Submit work electronically, preferably in Microsoft Word format.
3. Pictures or illustrations should be clearly labelled and placed where you want them to appear in the article. A caption and photo credit should accompany each photograph.
4. If any student sample work is included, please provide a release letter from the student's parent allowing publication in the journal.
5. Limit your manuscripts to no more than eight pages double-spaced.
6. A 250–350-word abstract should accompany your manuscript for inclusion on the Mathematics Council's website.
7. Letters to the editor or reviews of curriculum materials are welcome.
8. *delta-K* is not refereed. Contributions are reviewed by the editor(s), who reserve the right to edit for clarity and space. **The editor shall have the final decision to publish any article.** Send manuscripts to A. Craig Loewen, Editor, 414 25 Street S, Lethbridge, AB T1J 3P3; fax (403) 329-2412, e-mail loewen@uleth.ca.

Submission Deadlines

delta-K is published twice a year. Submissions must be received by August 31 for the fall issue and December 15 for the spring issue.

MCATA Mission Statement

Providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics.

delta-k

Volume 42, Number 1 December 2004

CONTENTS

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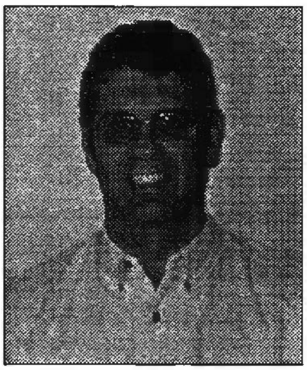
From the President's Pen



The first thing I need to inform you of is that we are making *delta-K* a refereed journal. *delta-K* has been, for many years, the academic publication of MCATA. It has maintained a high quality in both the articles published and its presentation and has garnered respect and praise from all corners. But there is always room to improve. With direction from editor Craig Loewen and the MCATA executive, we plan to take *delta-K* to a new level by incorporating a refereeing process for submitted articles. The details of this process will be worked out at our regular meetings throughout the year.

Also on the topic of change, I notice that the Western and Northern Canadian Protocol (WNCP) is beginning preliminary work on revisiting mathematics curriculum for Grades K–12. This initiative will assess whether major revisions are needed at each level. At the secondary level, Grade 7 is due for implementation in September 2006, moving up one grade every year until Grade 12 implementation in September 2011.

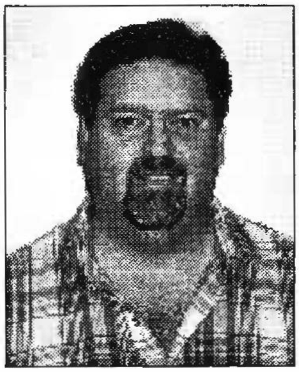
Mathematics education is certainly not a static topic. The applications change, the technology used to assist understanding changes, connections to other curricular areas improve and so on. In fact, over the years, mathematics education has seemed to be an evolving, adapting entity, almost a living and breathing organism. We as educators are a major force in this process. Tremendous work was done by teachers in the last curriculum revision to make mathematics more relevant and meaningful to Alberta students. We must be prepared to do so again over the next several years if we are to continue our commitment to provide the highest quality mathematics education in our schools.

Finally, if you are looking for some interesting reading on mathematics curriculum, try the following:

- *What the Numbers Say: A Field Guide to Mastering Our Numerical World* (Broadway Books, 2003), a numeracy argument by Derrick Niederman and David Boyum
- *To Infinity and Beyond* (Princeton University Press, 1987), a study of the concept of infinity and undefined values, by Eli Maor

Len Bonifacio

EDITORIAL



This is my second issue as the new editor of *delta-K*, and I wanted to start by thanking several people who have provided such great assistance during the past year. In particular, I want to thank the Mathematics Council executive, who have shown continual interest and are always ready to provide support when needed. Also, thanks must be given to Karen Virag and her excellent staff for all they do to present the journal in such a professional format.

Beginning with the next issue of *delta-K*, Gladys Sterenberg will be joining me as coeditor. Gladys has been a teacher in the Lethbridge area for a number of years and has served as an instructor and supervisor in the Faculty of Education at the University of Lethbridge, specializing in mathematics. Gladys completed her bachelor's and master's degrees at the University of Lethbridge and is currently engaged in her doctoral studies at the University of Alberta. I am delighted to have Gladys as part of this team and I look forward to her many contributions as both author and editor in the issues to come.

It is with regret that I inform you of the passing of John Percevault, a former editor of *delta-K* and someone familiar to many of our members. John served in the Faculty of Education at the University of Lethbridge for several years as the mathematics specialist and as a faculty administrator. John organized and led several activities as a member of the Mathematics Council's South Regional and presented regularly at MCATA and NCTM conferences. He was highly committed to the reconceptualization of mathematics teaching and learning, and he showed a special interest in the improvement of problem-solving instruction. He was awarded the distinction of Mathematics Teacher of the Year in 1986. John Percevault will be remembered for his collaborative nature, his years of volunteer service with and leadership in the Mathematics Council, as well as his many contributions within the broader mathematics and educational communities in Alberta.

This issue includes a wonderful range of articles with topics addressing instruction from the earliest grades through to university calculus. Inside, you will find many problems and teaching ideas and lots to keep your mathematical skills honed. Enjoy!

A. Craig Loewen

Using Telescoping Terms to Derive Formulae for Sums of Powers of the First *n* Natural Numbers

Darryl Smith

The excellent article by A. Craig Loewen titled "Sums of Arithmetic Sequences: Several Problems and a Manipulative" in the June 2004 issue (Volume 41, Number 2) of *delta-K* reminded me of Riemann sums and the formulae that are so important in those problems. For example, to calculate the value of the integral

$$\int_1^2 x^2 dx = \lim_{n\to\infty}\sum_{i=1}^{\infty} f(x_i)\Delta x_i ,$$

the interval must be partitioned from $x = 1$ to $x = 2$ into n subintervals, where the width of each subinterval, Δx_i, is given by the expression $\frac{2-1}{n}$ or $\frac{1}{n}$. Then,

$$\int_1^2 x^2 dx$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}(f(x_i)\Delta x_i)$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(f(1+\frac{i}{n})\frac{1}{n}\right)$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left((1+\frac{i}{n})^2\frac{1}{n}\right)$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left((1+\frac{2i}{n}+\frac{i^2}{n^2})\frac{1}{n}\right)$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{1}{n}\times 1+\frac{2}{n^2}\times i+\frac{1}{n^3}\times i^2\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n}1+\frac{2}{n^2}\sum_{i=1}^{n}i+\frac{1}{n^3}\sum_{i=1}^{n}i^2\right).$$

At this point, some of the formulae for sums of powers of the first n natural numbers are required, such as

$$\sum_{i=1}^{n}1 = n ,$$

$$\sum_{i=1}^{n}i = \frac{n(n+1)}{2} \text{ and}$$

$$\sum_{i=1}^{n}i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Other such formulae will appear later in this article. At first, I would introduce the required formulas, verify them and prove them by mathematical induction. However, it always concerned me that I did not have an algebraic method of determining these formulae in my bag of tricks.

To complete the above integral, substitutions of the required formulae are made into the last statement to obtain

$$= \lim_{n\to\infty}\left(\frac{1}{n}\times n+\frac{2}{n^2}\times\frac{n(n+1)}{2}+\frac{1}{n^3}\times\frac{n(n+1)(2n+1)}{6}\right)$$

$$= \lim_{n\to\infty}\left(1+\frac{n+1}{n}+\frac{1}{6}\times\frac{n+1}{n}\times\frac{2n+1}{n}\right)$$

$$= 1+1+\frac{2}{6}$$

$$= \frac{7}{3}.$$

That is, the value of the definite integral $\int_1^2 x^2 dx$ is exactly $\frac{7}{3}$. Of course, since $x^2 > 0$ for all values $x \in [1,2]$, the value of the integral is also the area enclosed by the function $y = x^2$ and the x-axis between $x = 1$ and $x = 2$.

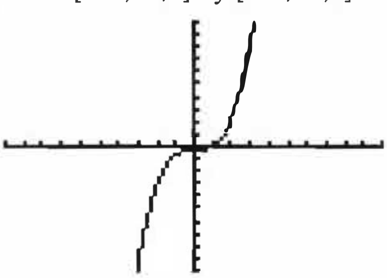
The result can be verified on the home screen of the TI-83 calculator using MATH9, which pastes in the numerical integration function fnInt. The format of the argument for this function is fnInt (function, independent variable, lower limit, upper limit).

```
fnInt(X²,X,1,2)
       2.333333333
■
```

When faced with evaluating an integral-defined function, such as $f(x)=\int_1^x t^2dt$, simply use the format described above to define the function in a convenient location, such as y_1, as shown below. In function mode, the only independent variable recognized by the TI-83 is x. That is, the t used in the defined function $f(x)=\int_1^x t^2dt$ will be replaced by x.

```
Plot1 Plot2 Plot3
\Y1■fnInt(X²,X,1
,X)■
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```

The graph of the function $f(x)=\int_1^x t^2dt$ is graphed below using Zoom-6, which defines the viewing window [-10,10,1] by [-10,10,1].



Notice that the graph of $f(x)=\int_1^x t^2dt$ appears to have a zero at $x = 1$, which is consistent with the value of the definite integral $f(1)=\int_1^1 t^2dt=0$. The values of these definite integrals are easily obtained using the table feature of the TI-83, where the numbers in the column labelled y_1 are the values of definite integrals that are members of the range of the function $f(x)=\int_1^x t^2dt$. The domain for this function is $x\in R$. It is left to the reader to verify that, if $x<1$, then $f(x)=\int_1^x t^2dt<0$.

X	Y1
1	0
2	2.3333
3	8.6667
4	21
5	41.333
6	71.667
7	114

X=1

The focus of this article is not Riemann sums and integrals but, rather, an algebraic technique using the properties of summation and telescoping terms to directly derive formulae for the sums of powers of the first n natural numbers involved in Riemann sums.

Consider the series consisting of n terms, each of which is 1, so that we have 1 + 1 + 1 + . . . + 1. Because there are n-identical 1s, the sum is obviously n, so we can write $\sum_{i=1}^{n}1=n$. The same result could be obtained by treating the above series as arithmetic with a common difference of $d = 0$.

To derive a formula for the sum of the first power of the first n natural numbers, we have 1 + 2 + 3 + . . . + n. This series is arithmetic with $d = 1$, and applying the formula

$S_n=\frac{n}{2}[2a+(n-1)d]$ gives the result

$$\sum_{i=1}^{n}i=\frac{n(n+1)}{2}.$$

For an alternative approach, consider the expression $(i + 1)^2 - i^2 = 2i + 1$. We can use the properties of summation to obtain

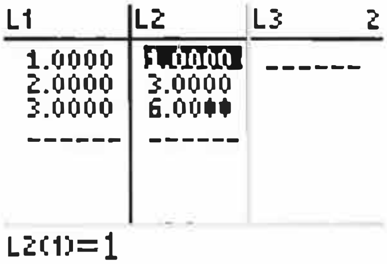
$$\sum_{i=1}^{n}\left((i+1)^2-i^2\right)=\sum_{i=1}^{n}(2i+1)$$
$$\Leftrightarrow (1+1)^2-1^2+(2+1)^2-2^2+$$
$$(3+1)^2-3^2+...+(n+1)^2-n^2$$
$$=2\sum_{i=1}^{n}i+\sum_{i=1}^{n}1$$

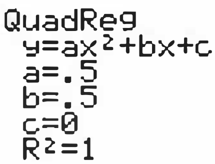
Notice that, in the expansion of the left-hand side, all the terms cancel except $(n + 1)^2$ and $-(1)^2$; that is, the terms telescope, leaving just two terms. Because we have previously determined that $\sum_{i=1}^{n}1=n$, we obtain $(n+1)^2-1^2=2\sum_{i=1}^{n}i+n$. Solve for $\sum_{i=1}^{n}i$ to obtain

$$\sum_{i=1}^{n}i=\frac{(n+1)^2-1^2-n}{2}=\frac{n^2+n}{2}$$
$$\sum_{i=1}^{n}i=\frac{n(n+1)}{2}$$

as before.

A calculator approach can also be taken. Using STAT mode on the TI-83, simply enter at least the first three terms of the natural number sequence 1, 2, 3, . . . in List1, and a matching number of terms for the sequence of partial sums 1, 3, 6, . . . in List2, as shown. Because the quadratic regression involves the parameters a, b and c, at least three data points are required.





Performing a quadratic regression gives

$$y = 0.5x^2 + 0.5x + 0 = \frac{x(x+1)}{2}$$

with $R^2 = 1$. Because it is a sequence, observe the condition that $x \in N$.

Generally, to determine an expression for the sum of the first n terms of the kth power of the natural numbers, the following expression can be used:

$$(i+1)^{k+1} - i^{k+1} = i^{k+1} + (k+1)i^k + \frac{(k+1)k}{2!}i^{k-1} + \ldots + 1 - i^{k+1}.$$

The first and last terms on the right will cancel, so we have

$$\sum_{i=1}^{n}\left((i+1)^{k+1} - i^{k+1}\right) = \sum_{i=1}^{n}\left((k+1)i^k + \frac{(k+1)k}{2!}i^{k-1} + \ldots + 1\right).$$

In expanding the left-hand side of this expression, the terms will always telescope. Simply substitute previously determined expressions into the right-hand side and solve for the expression $\sum_{i=1}^{n} i^k$.

To again illustrate the technique, we determine a closed form for $\sum_{i=1}^{n} i^2$. Beginning with the expression $(i+1)^3 - i^3 = 3i^2 + 3i + 1$, we obtain

$$\sum_{i=1}^{n}\left((i+1)^3 - i^3\right) = \sum_{i=1}^{n}\left(3i^2 + 3i + 1\right)$$

$$\Leftrightarrow (1+1)^3 - 1^3 + (2+1)^3 - 2^3 + (3+1)^3 - 3^3 + \ldots$$

$$+ (n+1)^3 - n^3 = 3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1$$

$$\Leftrightarrow (n+1)^3 - 1^3 = 3\sum_{i=1}^{n} i^2 + 3\frac{n(n+1)}{2} + n$$

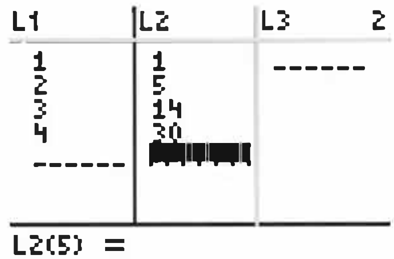
$$\Leftrightarrow n^3 + 3n^2 + 3n = 3\sum_{i=1}^{n} i^2 + 3\frac{n(n+1)}{2} + n$$

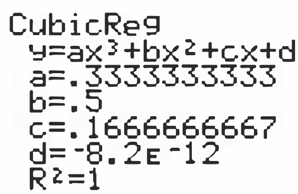
$$\Leftrightarrow 2n^3 + 6n^2 + 6n = 6\sum_{i=1}^{n} i^2 + 3n^2 + 3n + 2n$$

$$\Leftrightarrow 6\sum_{i=1}^{n} i^2 = 2n^3 + 3n^2 + n = n(2n^2 + 3n + 1)$$

$$\Leftrightarrow \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

To obtain the same result on the TI-83, a cubic regression using at least four data points is needed, where $L2_1 = 1^2$, $L2_2 = 1^2 + 2^2$ and so on.





The calculator gives the result $y = 0.333333\ldots x^3 + 0.5x^2 + 0.1666666\ldots x$, with $R^2 = 1$. Since the coefficients are $\frac{1}{3}$, $\frac{1}{2}$ and $\frac{1}{6}$ respectively, we have

$$y = \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{6}x,$$

which can be expressed in the more familiar and convenient form

$$y = \frac{x(x+1)(2x+1)}{6}, x \in N.$$

Consider $(i+1)^4 - i^4 = 4i^3 + 6i^2 + 4i$. Using the properties of summation, we obtain

$$\sum_{i=1}^{n}\left((i+1)^4 - i^4\right) = \sum_{i=1}^{n}\left(4i^3 + 6i^2 + 4i + 1\right)$$

$$\Leftrightarrow (1+1)^4 - 1^4 + (2+1)^4 - 2^4 + (3+1)^4 - 3^4 + \ldots$$

$$+ (n+1)^4 - n^4 = 4\sum_{i=1}^{n} i^3 + 6\sum_{i=1}^{n} i^2 + 4\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1$$

$$\Leftrightarrow (n+1)^4 - 1^4 = 4\sum_{i=1}^{n} i^3 + 6\frac{n(n+1)(2n+1)}{6} +$$

$$\Leftrightarrow n^4 + 4n^3 + 6n^2 + 6n = 4\sum_{i=1}^{n} i^3 + 2n^3 + 3n^2 +$$

$$\Leftrightarrow n^4 + 2n^3 + n^2 = 4\sum_{i=1}^{n} i^3$$

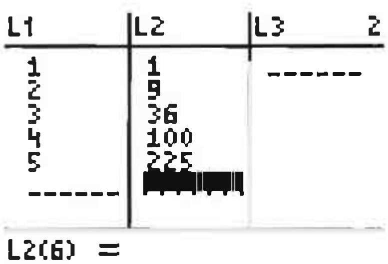
$$\Leftrightarrow 4\sum_{i=1}^{n} i^3 = n^2(n^2 + 2n + 1)$$

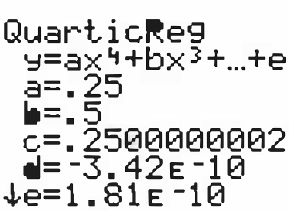
$$\Leftrightarrow \sum_{i=1}^{n} i^3 = \frac{n^2(n^2 + 2n + 1)}{4} = \frac{n^2(n+1)^2}{4}$$

$$\Leftrightarrow \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2 .$$

Expressing the result in this form makes it easy to remember because it is simply the square of the result for the sum of the first n natural numbers.

Regression can be used as before, where $L2_1 = 1^3$, $L2_2 = 1^3 + 2^3$ and so on.





The R^2 value is again 1, and the value of the coefficients d and e in $y = ax^4 + bx^3 + cx^2 + dx + e$ is 0. Therefore, we have $y = 0.25x^4 + 0.5x^3 + 0.25x^2$ or

$$y = \frac{1}{4}x^4 + \frac{1}{2}x^3 + 0.25x^2$$

$$= \frac{x^4 + 2x^3 + x^2}{4}$$

$$= \left[\frac{x(x+1)}{2}\right]^2, x \in N .$$

The expression for $\sum_{i=1}^{n} i^4$ is a bit tedious, but it is achieved using the same technique. We begin with

$$\sum_{i=1}^{n}\left((i+1)^5 - i^5\right) = \sum_{i=1}^{n}\left(5i^4 + 10i^3 + 10i^2 + 5i + 1\right)$$

$$\Leftrightarrow (n+1)^5 - 1^5 = 5\sum_{i=1}^{n} i^4 + 10\sum_{i=1}^{n} i^3 + 10\sum_{i=1}^{n} i^2 + 5\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 .$$

As before, we substitute previously derived expressions into the right-hand side and simplify the left-hand side to obtain

$$5n^4 + 10n^3 + 10n^2 + 5n + 1$$

$$= 5\sum_{i=1}^{n} i^4 + 10\frac{n^2(n+1)^2}{4} + 10\frac{n(n+1)(2n+1)}{6} + 5\frac{n(n+1)}{2} + n .$$

Multiply through by 6 to clear fractions and isolate the term containing $\sum_{i=1}^{n} i^4$ to obtain

$$30\sum_{i=1}^{n} i^4 = 6n^5 + 15n^4 + 10n^3 - n$$

$$= n(6n^4 + 15n^3 + 10n^2 - 1) .$$

The right-hand side can be factored and then divided through by 30 for the final result of

$$\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2 + 3n - 1)}{30} .$$

Unfortunately, a quintic polynomial regression is beyond the capabilities of the calculator.

Conclusion

The properties of summation and telescoping terms have been used to derive the following:

$$\sum_{i=1}^{n} 1 = n$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

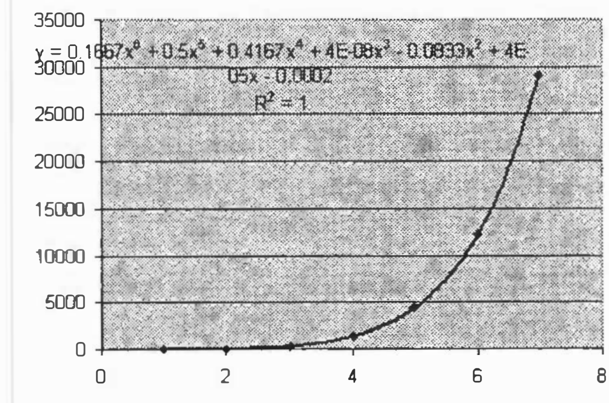
$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

$$\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2 + 3n - 1)}{30} .$$

The next time you teach Riemann sums in your calculus class and it comes time to derive formulae for sums of powers of natural numbers, I encourage you to consider the direct approach using telescoping terms. The technique is rich in algebraic opportunity, such as expanding powers of binomials and exploring the properties of sigma notation and limits. When I carefully and thoroughly worked through the first derivations with the class, the students were capable of doing the last ones by themselves, provided that I gave them a hint as to the required form. More importantly, students always seem impressed by their ability to use what turns out to be rather straightforward algebraic tools to determine some rather impressive identities. Also, a word from the voice of experience: even though the overhead projector was my favourite mode of presentation in class, I always did this lesson on a 20-foot whiteboard. It made it easier to follow the derivations, look back and record the list of formulae as we went. The calculator regressions are interesting and are best done concurrently with the algebraic derivation, but they are insufficient by themselves. Try obtaining these regressions using Excel, a program capable of doing up to degree 6 polynomial regressions. The result given by Excel for $\sum_{i=1}^{n} i^5$ is shown below. In spite of the given value $R^2 = 1$, the result holds only to approximately the fourth term when compared to exact values determined from

$$\sum_{i=1}^{n} i^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}.$$

n	n^5	partial sum	using derivation	regression eqn
1	1	1	1	0.99994004
2	32	33	33	33.00268032
3	243	276	276	276.0272211
4	1024	1300	1300	1300.145563
5	3125	4425	4425	4425.542505
6	7776	12201	12201	12202.59965
7	16807	29008	29008	29012.00339



Darryl Smith is in his third year of retirement after 34 years with the Edmonton Catholic School District, 30 of which were spent at Austin O'Brien High School. His one regret is that technology use did not arrive in the mathematics classroom until the last third of his career. During the past two years, he has had the privilege of working with many excellent teachers from the Edmonton Catholic School District in workshop settings and relishes these opportunities to implement calculator technology into mathematics education.

A New Approach to Partial Fractions

David E. Dobbs

Algebra courses give instruction for adding or subtracting two rational functions, or ratios of polynomials. One way to do this is to find a common denominator, rewrite each rational function using this common denominator, add or subtract the numerators and simplify the result. In integral calculus, it is often necessary to reverse this process; that is, to rewrite a given rational function as a sum of several simpler rational functions, or partial fractions (see, for instance, Stewart 2001, 405–7). It is possible to describe and justify a method for carrying out this rewriting at the level of a typical precalculus course (see, for instance, Dobbs and Peterson 1993, 197–204). Because this method is designed to be generally applicable, it tends to be rather time-consuming. Moreover, this method consists of several steps, requires the user to introduce a considerable amount of notation (variables that must be solved for in systems of equations) and often taxes students' memories and patience. Although some shortcuts are known, they can fail to lead to a complete solution, such as when the denominator of the given rational function in lowest terms has a multiple root. One currently fashionable way to address this situation is to use computer algebra systems (CAS) to find partial-fraction decompositions. This article will introduce a new paper-and-pencil algorithm for obtaining partial-fraction decompositions. The next two sections will introduce this new approach as an iterative method based on a few easy-to-remember strategies.

Like other paper-and-pencil algorithms for obtaining partial-fraction decompositions, this method is quickest when applied to decomposing rational functions whose denominators are expressed as products of linear factors. This new method has the following advantages. When used on complex calculus problems and differential equations, technology is not required to combat impatience. The new method does not require the user to introduce any notation. Data whose denominators have multiple roots do not pose any additional difficulty. Like traditional approaches to partial-fraction decomposition, the new method can reinforce several topics in the typical precalculus course, including arithmetic operations on rational functions and factorization of quadratic polynomials. In addition, the new method can reinforce the binomial theorem and, if covered, mathematical induction. Moreover, its treatment of irreducible quadratic factors can reinforce arithmetic with complex numbers.

The New Method When Denominators Have Only Linear Factors

The task is to decompose a rational function that is in lowest terms. In this section, we consider the case where the denominator of the given rational function has only linear factors. Notice that a linear factor $ax+b$ can be rewritten as $a(x-c)$, where $c=-\frac{b}{a}$. Thus, there is no harm in assuming that the linear factors of the denominator are of the form $x-d$ for various constants d.

One paradigm will be enough to point the way. Our experience in algebra courses with the addition and subtraction of rational functions tells us that

$$\frac{A}{x-a}+\frac{B}{x-b}=\frac{(A+B)x-(Ab+Ba)}{(x-a)(x-b)}.$$

Therefore, to rewrite

$$\frac{1}{(x-a)(x-b)} \text{ as}$$

$$\frac{A}{x-a}+\frac{B}{x-b},$$

$(A+B)x-(Ab+Ba)$ must equal 1. In particular, the coefficient of x must be 1; that is, $A+B=0$ or, equivalently, $B=-A$. Although this analysis has the flavour of the traditional method for partial-fraction decomposition, all that needs to be remembered from the analysis thus far is that, if $a\neq b$, then

$$\frac{1}{(x-a)(x-b)}$$

can be rewritten as

$$\frac{A}{x-a}-\frac{A}{x-b}$$

for the constant A, which can be determined using the rules for the addition and subtraction of rational functions. For instance, to decompose

$$\frac{1}{(x-2)(x+3)}, \text{ use}$$

$$\frac{A}{x-2}-\frac{A}{x+3},$$

where $A(x+3)-A(x-2)=1$; that is, $5A=1$, or $A=\frac{1}{5}$. The upshot is the partial-fraction decomposition

$$\frac{1}{(x-2)(x+3)}=\frac{1/5}{x-2}-\frac{1/5}{x+3}.$$

The above method is memorable, and can be summarized with the following general formula: If $a \neq b$, then

$$\frac{1}{(x-a)(x-b)}=\frac{1/(-b+a)}{x-a}\ \frac{1/(-b+a)}{x-b}. \quad (1.1)$$

Let us next consider the case that the given rational function has a constant numerator and denominator with more than two linear factors. If the denominator does not have a multiple root (that is, if the denominator does not have a repeated linear factor), the above reasoning only has to be repeated several times. More precisely, to decompose a rational function of the form

$$\frac{1}{(x-a)g(x)},$$

decompose the simpler rational function $\frac{1}{g(x)}$, multiply the resulting sum of partial fractions by $\frac{1}{x-a}$, expand by using the distributive property and then decompose each of the resulting simpler rational functions. The success of the method depends on reducing the problem to a set of simpler rational functions at each step. The rigorous mathematical explanation is that the method uses mathematical induction on the number of linear factors in the denominator. However, most students are fully convinced (and should be) by the following kind of example.

To obtain the partial-fraction decomposition of

$$\frac{1}{(x-2)(x+1)(x-4)},$$

reason as follows. First, as in the general formula from the first decomposition problem, we have

$$\frac{1}{(x+1)(x-4)}=\frac{-1/5}{x+1}+\frac{1/5}{x-4}.$$

Next, multiplying through by $\frac{1}{x-2}$, we find that

$$\frac{1}{(x-2)(x+1)(x-4)} = \frac{1}{x-2}\left(\frac{-1/5}{x+1}+\frac{1/5}{x-4}\right)$$

$$=-\frac{1}{5}\left(\frac{1}{(x-2)(x+1)}\right)+\frac{1}{5}\left(\frac{1}{(x-2)(x-4)}\right).$$

Finally, by again using the earlier general formula and combining like terms, we obtain

$$-\frac{1}{5}\left(\frac{1/3}{x-2}-\frac{1/3}{x+1}\right)+\frac{1}{5}\left(\frac{-1/2}{x-2}+\frac{1/2}{x-4}\right)$$

$$=\frac{-1/6}{x-2}+\frac{1/15}{x+1}+\frac{1/10}{x-4}.$$

To handle nonconstant numerators, we only need to rewrite each occurrence of x in the numerator as $(x-a)+a$, where $x-a$ is a factor of the denominator, and then to rewrite each occurrence of x^n in the numerator by using the binomial theorem to expand $((x-a)+a)^n$. In effect, we are rewriting the numerator, which is a polynomial in the variable x, as a polynomial in the variable $x-a$. (This part of the method is an excellent preparation activity for calculus, in which students could learn, more generally, to approximate an n–times differentiable function by its nth Taylor polynomial centred at a.) Once again, the following sufficiently complicated example can be given to convince a class that the method works in general.

Let's use the above strategy to find the partial-fraction decomposition of

$$\frac{3x^2-7x+5}{(x-2)(x+1)(x-4)}.$$

The numerator is rewritten as

$$3((x-2)+2)^2-7((x-2)+2)+5=$$

$$3((x-2)^2+4(x-2)+4)-7(x-2)-9=$$

$$3(x-2)^2+5(x-2)+3 \quad .$$

Thus, by again using what is known about addition and subtraction of rational functions and then cancelling common factors of corresponding numerators and denominators, we find that

$$\frac{3x^2-7x+5}{(x-2)(x+1)(x-4)}=\frac{3(x-2)^2+5(x-2)+3}{(x-2)(x+1)(x-4)}=$$

$$\frac{3(x-2)}{(x+1)(x-4)}+\frac{5}{(x+1)(x-4)}+\frac{3}{(x-2)(x+1)(x-4)}.$$

We have already seen how to decompose each of the last two terms on the right-hand side as sums of partial fractions. The remaining term,

$$\frac{3(x-2)}{(x+1)(x-4)},$$

is simpler than the original problem and can now be written as a sum of partial fractions by another application of the above method, as follows:

$$\frac{3(x-2)}{(x+1)(x-4)}=\frac{3x-6}{(x+1)(x-4)}=\frac{3((x+1)-1)-6}{(x+1)(x-4)}$$

$$=\frac{3(x+1)-9}{(x+1)(x-4)}=\frac{3}{x-4}-\frac{9}{(x+1)(x-4)}.$$

How should we proceed if the denominator of the given rational function has a multiple root; that is, if it has a repeated linear factor? It is enough to indicate how to handle rational functions of the form

$$\frac{1}{(x-a)^n g(x)},$$

where $x-a$ is not a factor of $g(x)$ (that is, where a is not a root of $g(x)$), because nonconstant numerators can be handled as above using the "$x=(x-a)+a$" trick. To do so, factor $x-a$ out of the denominator so that the problem is presented as

$$\frac{1}{x-a}\left(\frac{1}{(x-a)^{n-1}g(x)}\right).$$

Notice that the second factor,

$$\frac{1}{(x-a)^{n-1}g(x)},$$

represents a simpler problem (inasmuch as its denominator has a degree that is less than the degree of the original denominator). By mathematical induction on this degree (in practice, by repeatedly factoring linear factors out of the denominator), we can reduce the rational function to a sum of simpler rational functions that are, ultimately, already in partial-fraction form or amenable to being rewritten with the help of the general formula from earlier. The following example can be given to illustrate the method.

Let's indicate how to use the above strategy to produce the partial-fraction decomposition of

$$\frac{1}{(x-2)^2(x+4)^3(x-1)}.$$

Note that the given denominator has degree 6. Factor one of the linear factors out of the denominator, such as $x-2$, so that the problem becomes rewriting

$$\frac{1}{x-2}\left(\frac{1}{g(x)}\right),$$

where $g(x)=(x-2)(x+4)^3(x+1)$. Lurking within $\frac{1}{g(x)}$ is sure to be a simpler problem, which is either already in partial-fraction form or amenable to being treated by the general formula from earlier. By the general formula,

$$\frac{1}{(x-2)(x+1)}=\frac{1/3}{x-2}-\frac{1/3}{x+1}.$$

Therefore, in multiplying by $\frac{1}{(x+4)^3}$, we have

$$\frac{1}{g(x)}=\frac{1/3}{(x-2)(x+4)^3}-\frac{1/3}{(x+1)(x+4)^3},$$

and so the given rational function has been rewritten as

$$\frac{1}{(x-2)g(x)}=\frac{1/3}{(x-2)^2(x+4)^3}-\frac{1/3}{(x+1)(x-2)(x+4)^3}.$$

Notice that both of the terms on the right-hand side have degree 5, and are thus simpler than the original data. By iterating the process once more, we come to a sum of terms, each having denominators with degree 4 at most. Further iterations lead to the desired partial-fraction decomposition. For reasons of space, the details of these calculations will be left to the reader.

When Denominators Have Irreducible Quadratic Factors

In this section, we will look at a new way of finding the partial-fraction decomposition of a rational function in lowest terms whose denominator has at least one irreducible quadratic factor. We begin with a notational simplification, just like at the beginning of the preceding section. Any quadratic polynomial ax^2+bx+c can be rewritten as $a(x^2+dx+e)$, where $d=\frac{b}{a}$ and $e=\frac{c}{a}$. Thus, we can assume that the irreducible quadratic factors of the denominator are of the form x^2+bx+c for various constants b and c.

It would appear at first that, in order to continue using the method introduced in the preceding section, more than one formula analogous to the general formula may need to be developed. This approach is rather similar to the classical method of introducing several variables and solving systems of linear equations. For this reason, this section's problem is best handled by reducing it to that of the preceding section. This can be done, provided that you are prepared to use complex numbers as coefficients of various relevant polynomials. If $g(x)=x^2+bx+c$ is an irreducible factor of the denominator, recall that $g(x)$ can be factored as $(x-r_1)(x-r_2)$, where r_1 and r_2 are the roots of $g(x)$. (If necessary, r_1 and r_2 can be found by using the quadratic formula.) Thus, if the denominator is given as having been factored over the real numbers (as is usual in the classical method for partial-fraction decomposition), we can write that denominator as a product of linear factors, possibly with complex number coefficients. We can then proceed to find a partial-fraction decomposition exactly as in the preceding section.

The price to be paid in using this method is that you must be willing to do a considerable amount of arithmetic with complex numbers. Moreover, it is easy to lose heart during the process as to whether the final result will simplify and have only real-number coefficients. Be assured that this will always work out, in view of the general theorem guaranteeing partial-fraction decompositions over any field (see, for instance, Dobbs and Hanks 1992, 31–33). While keeping track of the various factors that arise, remember that the nonreal complex roots of any real polynomial $g(x)$ arise as pairs of complex numbers that have the same multiplicity as roots of $g(x)$ (see, for instance, Dobbs and Hanks 1992, 54).

Be warned that this method of handling irreducible quadratic factors can be time-consuming. However, as promised, it requires only the method of the preceding section coupled with the above formula for factoring quadratic polynomials and the ability to perform the arithmetic operations on complex numbers. We close with an example that illustrates the new method outlined in this section.

Use the above method to find the partial-fraction decomposition of

$$\frac{x^2-7x-9}{(x^2+4)(x-3)}.$$

Using the "$x=(x-a)+a$" trick, we can rewrite the numerator as

$$((x-3)+3)^2-7((x-3)+3)-9=(x-3)^2-(x-3)-21.$$

Therefore, by distributivity and cancelling,

$$\frac{x^2-7x-9}{(x^2+4)(x-3)}.$$

The first and second terms on the right-hand side are already in partial-fraction form. So, all we have left to do is find the partial-fraction decomposition of

$$\frac{1}{(x^2+4)(x-3)}.$$

Because the roots of x^2+4 are $\pm 2i$, the denominator can be factored as $(x+2i)(x-2i)(x-3)$. Next, by using the method of the preceding section and arithmetic with complex numbers, we find

$$\frac{1}{(x^2+4)(x-3)}=\frac{1}{(x+2i)(x-2i)(x-3)}=$$

$$\frac{1}{x-3}\left(\frac{1}{(x+2i)(x-2i)}\right)=\frac{1}{x-3}\left(\frac{i/4}{x+2i}-\frac{i/4}{x-2i}\right)=$$

$$\left(\frac{i}{4}\right)\left(\frac{1}{(x-3)(x+2i)}-\frac{1}{(x-3)(x-2i)}\right)=$$

$$\left(\frac{i}{4}\right)\left[\left(\frac{1}{2i+3}\right)\left(\frac{1}{x-3}-\frac{1}{x+2i}\right)\right]-\left(\frac{i}{4}\right)\left[\left(\frac{1}{3-2i}\right)\left(\frac{1}{x-3}-\frac{1}{x-2i}\right)\right]=$$

$$\left(\frac{i}{4}\right)\left(\frac{1}{2i+3}-\frac{1}{3-2i}\right)\frac{1}{x-3}-\left(\frac{i}{4}\right)\left(\frac{1}{2i+3}\right)\frac{1}{x+2i}+$$

$$\left(\frac{i}{4}\right)\left(\frac{1}{3-2i}\right)\frac{1}{x-2i}=\frac{1/13}{x-3}-\left(\frac{2+3i}{52}\right)\frac{1}{x+2i}+\left(\frac{-2+3i}{52}\right)\frac{1}{x-2i}.$$

Then, by adding the terms that involve nonreal numbers (the same way that any rational functions are added), the final form of the decomposition is

$$\frac{1}{(x^2+4)(x-3)}=\frac{1/13}{x-3}+\frac{-\left(\frac{2+3i}{52}\right)(x-2i)+\left(\frac{-2+3i}{52}\right)(x+2i)}{(x+2i)(x-2i)}=$$

$$\frac{\frac{1}{13}}{x-3}+\frac{-\frac{1}{13}x-\frac{3}{13}}{x^2+4}.$$

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A Related-Rates Problem

Ronald L. Persky

You try to avoid it, but occasionally it happens. That is, you construct a test problem so that the students can use a particular principle demonstrated in class. Then, a student, having no idea how to solve the problem, puts numbers together by happenstance and produces the correct answer. Consider the following problem:

> Two roads are perpendicular. Car A is 6 miles from the intersection and is heading toward it at 39 miles per hour. Car B is 8 miles from the intersection and is heading toward it at 52 miles per hour. At this instant, how fast is the distance between them changing?

One student wrote, $\sqrt{39^2+52^2}$, which is 65 and happens to be correct.

This is a related-rates problem, and the standard procedure is to write $x^2+y^2=z^2$ and take the derivative with respect to time, that is,

$$2x\frac{dx}{dt}+2y\frac{dy}{dt}=2z\frac{dz}{dt}.$$

After solving for $\frac{dz}{dt}$ and substituting values, we can easily obtain the answer.

What made the student's answer work is that $\frac{52}{39}=\frac{8}{6}$ or, in general,

$$\frac{\frac{dx}{dt}}{\frac{dy}{dt}}=\frac{x}{y}. \qquad (1)$$

The intent of this article is to show that, in this problem, the equation below is true given equation (1):

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=\left(\frac{dz}{dt}\right)^2. \qquad (2)$$

First we establish two companion ratios. Starting with

$$x\frac{dx}{dt}+y\frac{dy}{dt}=z\frac{dz}{dt},$$

divide by

$$y\frac{dy}{dt}$$

to produce

$$\frac{x\frac{dx}{dt}}{y\frac{dy}{dt}}+1=\frac{z\frac{dz}{dt}}{y\frac{dy}{dt}}.$$

From equation (1), we can write

$$\frac{y\frac{dx}{dt}}{x\frac{dy}{dt}}=1$$

and use this in place of 1 above. This gives

$$\frac{x\frac{dx}{dt}}{y\frac{dy}{dt}}+\frac{y\frac{dx}{dt}}{x\frac{dy}{dt}}=\frac{z\frac{dz}{dt}}{y\frac{dy}{dt}}.$$

Combine the left side,

$$\frac{x^2\frac{dx}{dt}+y^2\frac{dx}{dt}}{xy\frac{dy}{dt}}=\frac{z\frac{dz}{dt}}{y\frac{dy}{dt}},$$

or

$$\frac{z^2\frac{dx}{dt}}{xy\frac{dy}{dt}}=\frac{z\frac{dz}{dt}}{y\frac{dy}{dt}},$$

which easily gives the first ratio

$$\frac{z}{x}=\frac{\frac{dz}{dt}}{\frac{dx}{dt}}. \qquad (3)$$

Again start with

$$x\frac{dx}{dt}+y\frac{dy}{dt}=z\frac{dz}{dt}.$$

Divide by

$$x\frac{dx}{dt}$$

to produce

$$1+\frac{y\dfrac{dy}{dt}}{x\dfrac{dx}{dt}}=\frac{z\dfrac{dz}{dt}}{x\dfrac{dx}{dt}}.$$

From equation (1), we can also write

$$1=\frac{x\dfrac{dy}{dt}}{y\dfrac{dx}{dt}}$$

and use this in place of 1 above to get

$$\frac{x\dfrac{dy}{dt}}{y\dfrac{dx}{dt}}+\frac{y\dfrac{dy}{dt}}{x\dfrac{dx}{dt}}=\frac{z\dfrac{dz}{dt}}{x\dfrac{dx}{dt}}.$$

Combine the left side,

$$\frac{(x^2+y^2)\dfrac{dy}{dt}}{xy\dfrac{dx}{dt}}=\frac{z\dfrac{dz}{dt}}{x\dfrac{dx}{dt}}$$

or

$$\frac{z^2\dfrac{dy}{dt}}{xy\dfrac{dx}{dt}}=\frac{z}{x}\frac{\dfrac{dz}{dt}}{\dfrac{dx}{dt}}$$

which gives the second ratio

$$\frac{z}{y}=\frac{\dfrac{dz}{dt}}{\dfrac{dy}{dt}}. \qquad (4)$$

Now to establish equation (2). Start again with

$$x\frac{dx}{dt}+y\frac{dy}{dt}=z\frac{dz}{dt}$$

using equation (1),

$$\left(\frac{\dfrac{dx}{dt}y}{\dfrac{dy}{dt}}\right)\frac{dx}{dt}+y\frac{dy}{dt}=z\frac{dz}{dt},$$

multiply by

$$\frac{\dfrac{dy}{dt}}{y},$$

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=\left(\frac{\dfrac{dy}{dt}z}{y}\right)\frac{dz}{dt}.$$

From equation (4),

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=\left(\frac{dz}{dt}\right)^2.$$

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Programming and Problem Solving with the TI-83 Plus: The Structured Search

A. Craig Loewen

Problem solving has always been one of the most difficult areas of the mathematics curriculum to teach well. Despite 25 years of research devoted to improving problem-solving skills, many students still struggle with how to start a problem and how to be flexible (that is, switching from an ineffective strategy to a new one). Our students do not seem to have a wide range of problem-solving strategies at their disposal and instead focus on such strategies as guess-and-test when no algorithm seems immediately available. Guess-and-test seems to have become even more popular in an age when calculators and computers aid in making quick, accurate computations.

Now, don't get me wrong. I'm not speaking against guess-and-test as a strategy. It is a legitimate, recognized and viable strategy, but in some ways it seems a bit inelegant and it is often inefficient.

There are, of course, many different problem-solving strategies, and we know that more sophisticated solvers combine these strategies. For example, the elimination strategy works well only if you are able to generate an effective and comprehensive list, and drawing a picture usually helps identify an appropriate formula. In general, all problem-solving strategies become more powerful when they are blended with other strategies. Add to that the power of the minicomputer. When we can harness the ability of the hand-held computer to quickly make repetitive computations, we may further enhance several of these strategies. The computer can be used to quickly generate lists, check conditions in the problem and complete exhaustive searches. All this can be done much more quickly than we can do on our own. In general, with a few simple programming skills (which are easily mastered by a high school student), we can blend these strategies and tools to create something we could call a *structured search.*

The Structured Search

In a structured search, we would set up a loop that effectively generates a list of possible values that can then be tested against the conditions of the problem. The TI-83 Plus gives us at least two efficient ways to create this kind of loop. This simple program uses the "for"-loop and writes the values 1–5 on the calculator screen:

```
PROGRAM:ABA
:1→A
:While A<6
:Disp A
:A+1→A
:End
:■
```

This following program, which uses a "while"-loop, does the same thing:

```
PROGRAM:ABA
:For(A,1,5)
:Disp A
:End
:■
```

Although the differences between the two types of loops are quite small, the "for"-loop is a little more elegant and much easier to enter. The "for"-loop also forces a fixed number of repetitions of the commands within the loop, whereas the second type of loop is more flexible, continuing indefinitely until a condition is met.

To complete the structured search, we also need to know how to enter simple logical checks, which are called "if"-statments. Let's modify our "for"-loop program above to do the following:

- Consider all the values from 1 through 19, using the command For(A,1,19)
- Print on the screen all of the values of A that are divisible by 3, using the command If A/3 = int(A/3):Disp A

Here is the program:

```
                         3
                         6
                         9
                        12
                        15
                        18
                      Done
▮
```

Here is the screen display when the program is run:

```
PROGRAM:ABA
:For(A,1,20)
:If A/3=int(A/3)
:Disp A
:End
:▮
```

Knowledge of the other mathematical functions available on the TI-83 Plus, together with these simple commands (and a little exploration), will provide us with another means of tackling a variety of problems.

Cryptarithms

A common type of problem appearing in many magazines and newspapers (and even on the Internet) is cryptarithms. A cryptarithm is an arithmetic statement where the digits have been replaced by letters. Here is a classic cryptarithm:

ABCD
× 4
CDBA

In this problem, each of the letters A through D needs to be replaced by a single digit to create a four-digit number that, when multiplied by 4, produces the same four digits in reverse order. Note that both As must be the same digit, which is true for all four letters.

It is a lot of fun to solve this problem by hand, but it is fun to solve it with a structured search, too. We need to set up four loops (one for each variable, A, B, C and D), use them to build the number ABCD, multiply that value by 4 and see what happens.

- Is it possible that there are no solutions to this problem? Is it possible there is more than one solution?
- How would you approach this problem if you were solving it without programming? What is a reasonable strategy?
- How long would you estimate it would take to solve this problem by hand?

Here is a structured search that could solve this problem:

```
PROGRAM:ABCD
:For(A,1,9)
:For(B,0,9)
:For(C,0,9)
:For(D,1,9)
:1000*A+100*B+10
*C+D→E
:1000*D+100*C+10
*B+A→F
:If E*4=F:Disp E
:End
:End
:End
:End
:▮
```

- Why do the A and D loops run from 1 to 9, while the B and C loops run from 0 to 9?
- Could this program generate any extraneous solutions? How could you tell?
- How many solutions does the program generate?
- How many values will this program consider in all?
- What does the line A × 1000 + B × 100 + C × 10 + D → E do?
- What are the values that are stored in the variables E and F?
- Why are there four "end"-statements at the bottom of the program?
- Adapt the program above to solve this similar problem: ABCD × 9 = DCBA.

Notice how long the calculator takes to run through the entire list of possible solutions. An obvious disadvantage of the structured search is the time required to execute it (although it is still much quicker than doing it by hand). A significant advantage is that the search has considered all possibilities, something we would never be willing to do by hand.

But wait a minute! Does the program have the computer test possibilities that are not reasonable? In other words, is there a way to further limit the number of possibilities considered and thus speed up the program? We already limited A to the values 1 through 9 because 0 cannot appear as the lead digit. However, consider that A must be either 1 or 2. If A is 3 or greater, a digit will be carried into the 10,000s place when A is multiplied by 4—and the product must not involve more than four place values. This one change significantly delimits the number of possibilities we need to consider. We can change the line For(A, 1, 9) to read For(A, 1, 2).

- How many possibilities does this one change eliminate?
- Are there other letters that could be further limited?

Strangely, with this process, we are slowly moving toward greater emphasis on another strategy: applying logical reasoning. Again, we see how strategies become more powerful when blended. If we continue with this process, we may find that logical reasoning leads us through to another, even more elegant solution to the problem. This is the joy of problem solving—identifying a number of ways to attack a problem and implementing a broad range of skills and strategies, moving gracefully between and among them as the solution is built.

Another important idea emerges as we play with this problem. A critical problem-solving skill is being able to identify when a strategy is not appropriate or does not apply. Unless we can significantly limit the number of checks the program needs to make, it could become a very tedious process. For example, when a problem requires five or more variables, it would probably be better to turn to a more powerful computer to engage the search or turn to a different strategy altogether.

Another Example

Let's look at another example of a cryptarithm to explore how we can modify our program to check for solutions:

```
  AB
+ BA
 CDC
```

The first thing we notice is that there are still four variables, but neither A, B nor C can equal zero. We will need to modify our "for"-loops. Obviously, we also need to modify our "if"-statements.

- What would the new "if"-statements look like? How many would we need?
- How many possible solutions could this search generate? How would we check for extraneous solutions?

One possible program looks like this:

```
PROGRAM:ABBA
:For(A,1,9)
:For(B,1,9)
:For(C,1,9)
:For(D,0,9)
:11*A+11*B→E
:101*C+10*D→F
:If E=F:Disp A*1
000+B*100+C*10+D

:End
:End
:End
:End
:
```

When run, the program generates several possible solutions, but be sure to check for ineligible ones; that is, solutions where two letters are assigned the same digit.

Here, another important quality of the problem-solving process is reinforced. Looking back is critical, although it is often overlooked. It is tempting to think that, because the program generated all of these results, they are all viable. This is not the case. Looking back helps confirm which possible solutions are real solutions. It is at the looking-back stage that we are most likely to catch our mistakes (computational or logical) and thus learn from our experience.

Challenges

- How many solutions are there to the equation ABC + CBA = DDD?
- Program your TI-83 Plus to solve the following equation: ABCDE × 4 = EDCBA. Estimate how long you think the calculator will require to generate its results. Use what you know from the ABCD × 4 = DCBA problem. Can you think of ways to limit the search?
- Try to build a routine in your program to eliminate ineligible answers
- Try to generate your own substitution problems that can be solved through a structured search.

A Money Problem

Consider another familiar problem that can be effectively solved using a structured search:

> Uri has 48 coins in his pocket, all nickels and pennies. Altogether, he has exactly $1.72 in change. How many nickels and how many pennies does he have?

We could set up a system of linear equations to solve this problem, but it is also fun to write a simple program as an alternative solution. The program looks like this:

```
PROGRAM:CENTS
:For(N,0,34)
:48-N→P
:If 5*N+P=172:Di
sp N,P
:End
:
```

- How are the dimensions of the N-loop (nickels) determined?
- Why does this program require only one loop?
- How many possible solutions are there?
- Describe the connection between the system of linear equations related to this problem and the program above.

- Assume that Uri also has some dimes. Is there a solution to this problem? How many solutions are there to this problem? Construct a structured search to solve this revised problem.
- Write a money problem of your own that could be solved using a structured search.

Some Challenging Problems

There are many different problems that can be solved using a variation of the structured search. Here are two somewhat more challenging problems with related programs. You may wish to try solving the problems yourself with or without your calculator, or you may find it interesting to work your way through the program, trying to determine the effect of each line.

Find the smallest number that, in each case, produces a remainder that is one less than the divisor when divided by each of the values 2 through 10.

```
PROGRAM:AZ
:ClrHome
:11→N
:0→C
:While C=0
:1→C
:For(A,2,10)
:N-int(N/A)*A→T
:If T≠A-1:Then
:0→C:10→A:N+1→N
:Output(1,1,N)
:End
:End
:End
:Disp N
:█
```

A perfect number is defined as a number that equals the sum of all its factors. For example, the first perfect number is 6 because its factors 1, 2 and 3 have a sum of 6. What are the next two perfect numbers?

```
PROGRAM:PERFECT
:2→C
:ClrHome
:For(A,2,500)
:Output(1,1,A)
:0→T
:For(B,1,√(A))
:If A/B=int(A/B)
:T+B+A/B→T
:End
:If √(A)=int(√(A
)):T-√(A)→T
:If T-A=A:Then:O
utput(C,1,A):C+1
→C:End
:End
:█
```

The topic of perfect numbers has captivated many mathematicians over the centuries. It is worth reading about these numbers and finding other algorithms that have been defined for identifying them more easily.

Cryptarithms and Alphametics

Included below is a collection of cryptarithms and alphametics. An alphametic is a special type of cryptarithm in which the letters used to replace the digits in an equation also form comprehensible words. Sometimes the words themselves form phrases. Here is a familiar alphametic:

  SEND
+ MORE
MONEY

It is not practical to solve the above alphametic with a hand-held computer because it involves eight different letters and thus eight different loops. How many different possibilities would the calculator have to consider in order to solve this problem?

The following puzzles were taken from www.freepuzzles.com.

How many solutions are there for each of the following?

AB + B = BA
C + C + C = DC
WAS × S = ASAW
A × C × AC = CCC

Here are two slightly more challenging puzzles taken from the same website. Each of these problems could also be solved with a structured search.

$A^2 + B^2 + C^2 = D^2 + E^2$

In the equation above, the letters represent consecutive positive integers. Find the corresponding value for each letter.

$(30 + 25)^2 = 3{,}025$

Break the number 3,025 into two parts, 30 and 25. The square of (30 + 25) equals 3,025, as shown. Two more numbers share the same property. Can you find them?

The following website has a huge number of cryptarithms and alphametics, as well as an aid to solve the puzzles: www.tkcs-collins.com/truman/alphamet/alphamet.shtml.

TO + GO = OUT
I + DID = TOO

Conclusion

Doing problems and puzzles like these is a fun and motivating activity that can be easily implemented into the mathematics classroom. The structured search provides another problem-solving tool for effectively approaching these problems.

Uncovering a Test for Divisibility by a Prime: A Journey of Mathematical Discovery

Murray L. Lauber

One of the joys of being a mathematics teacher is the excitement of being a student. Teaching concepts and solving problems with students provides many opportunities to see new relationships between concepts and to discover patterns that we have never noticed before.

This article has grown out of discoveries that I made over a fairly extended period of time teaching an introductory-level university course entitled Higher Arithmetic. Most of the students were in the humanities, and many planned to be elementary school teachers. My discoveries grew out of a particular topic—divisibility—but illustrate the discovery process that mathematics teachers engage in on a regular basis. The topic of divisibility was a part of a section of the course on number theory. In the textbook first employed, the authors presented or suggested ways of developing rules for divisibility by 2, 3, 4, 5, 6, 8 and 9, along with mathematical justification for some of these rules (Meserve 1981, 64–67). These rules and their justifications led me to wonder whether there was a general algorithm for divisibility by a prime. I could have searched for written sources to find the answer but was drawn by the appeal of discovering the rules for myself. Of course, I was far from alone in this kind of experience; the need to discover and the compulsion to generalize are at the heart of the study of mathematics.

There was another way in which I was far from alone. Although it is not often apparent, the process of mathematical discovery is typically somewhat convoluted. Characteristically, the textbook solution of a challenging mathematical problem misrepresents the process by editing out the convolution. The result is a tidied-up version of the solution in which sequential logic and efficient communication trump accurate representation of the process. The need to tidy up a solution is understandable but we should make our students aware that the process leading to a textbook solution is not always so tidy. The reader will benefit by knowing that, in favour of efficient communication, much of the convolution has been edited out of the following description of my journey of discovery.

Some Basic Rules for Divisibility

The following rules, along with the proof of the last one, illustrate how the process of discovery began.

A counting number n is divisible by

- 2 if and only if its last digit is divisible by 2;
- 3 if and only if the sum of the digits is divisible by 3;
- 4 if and only if the number represented by its last two digits is divisible by 4;
- 5 if and only if its last digit is 5 or a 0;
- 6 if and only of it is even and the sum of the digits is divisible by 3 (that is, it is divisible by both 2 and 3);
- 8 if and only if the number represented by its last three digits is divisible by 8;
- 9 if and only if the sum of the digits is divisible by 9.

The rule for divisibility by 7 is more complex and is often left out of such a list. The formulation of that rule was the beginning of my process of discovery. That rule and its proof will be given later. Because the rule for divisibility by 9 was instrumental in suggesting the structure of other rules and their proofs, it seems natural to begin with a proof of that rule. I have labelled it rule 1.

Rule 1: A counting number n is divisible by 9 if and only if the sum of its digits is divisible by 9.

The following proof for a four-digit number is dependent on the closure, commutative and associative properties of addition of counting numbers along with the distributive property of multiplication over addition.

Let $n = d_3d_2d_1d_0$ where d_3, d_2, d_1 and d_0 are its digits.

Then $n = 1{,}000d_3 + 100d_2 + 10d_1 + 1d_0$
$\Rightarrow n = (999 + 1)d_3 + (99 + 1)d_2 + (9 + 1)d_1 + 1d_0$
$\Rightarrow n = (999d_3 + 99d_2 + 9d_1) + (d_3 + d_2 + d_1 + d_0)$ **[1]**

1) To prove the *if* part of the theorem, we need to show that, if the sum of digits of n is divisible by 9 then n is also divisible by 9.
If the sum of n's digits is divisible by 9, then
$d_3 + d_2 + d_1 + d_0 = 9k$ for some $k \in \mathrm{N} = \{1, 2, 3, \ldots\}$
$\Rightarrow n = (999d_3 + 99d_2 + 9d_1) + 9k$ from equation [1]
$\Rightarrow n = 9(111d_3 + 11d_2 + d_1 + k)$
$\Rightarrow n$ is a multiple of 9, that is n is divisible by 9.

2) To prove the *only if* part of the theorem, we need to show that if n is divisible by 9 then the sum of its digits is also divisible by 9. Suppose that n is a multiple of 9, say $n = 9j$ for some $j \in \mathrm{N}$. Then, from equation [1],
$9j = (999d_3 + 99d_2 + 9d_1) + (d_3 + d_2 + d_1 + d_0)$
$\Rightarrow 9j = 9(111d_3 + 11d_2 + 1d_1) + (d_3 + d_2 + d_1 + d_0)$
$\Rightarrow d_3 + d_2 + d_1 + d_0 = 9(111d_3 + 11d_2 + 1d_1) - 9j$
$\Rightarrow d_3 + d_2 + d_1 + d_0 = 9(111d_3 + 11d_2 + 1d_1 - j)$
$\Rightarrow$ the sum of the digits of n is a multiple of 9.

Once this and the other rules had been proven, some persistent exploration that made use of my knowledge of modular arithmetic (a topic that will be explored shortly) led to the following rule for divisibility by 7:

Rule 2: A counting number $n = d_t d_{t-1} d_{t-2} \ldots d_2 d_1 d_0$ is divisible by 7 if and only if the following linear combination of its digits is divisible by 7:

$1d_0 + 3d_1 + 2d_2 + (-1d_3) + (-3d_4) + (-2d_5) + \ldots$
$1d_6 + 3d_7 + 2d_8 + (-1d_9) + (-3d_{10}) + (-2d_{11}) + \ldots$

. . . . . .
. . . . . .
. . . . . .

Note that the linear combination of digits begins with the last digit and that the coefficients of the linear combination repeat every six digits (the means by which these coefficients were determined will be described later).

Example: Determine whether n = 88,580,723 is divisible by 7.

Solution: According to Rule 2, n will be divisible by 7 if and only if
$1d_0 + 3d_1 + 2d_2 + (-1d_3) + (-3d_4) + (-2d_5) = 1d_6 + 3\, d_7$ is divisible by 7, that is, if
$(1 \times 3) + (3 \times 2) + (2 \times 7) + (-1 \times 0) + (-3 \times 8) + (-2 \times 5) + (1 \times 8) + (3 \times 8) = 21$ is divisible by 7.
According to the rule, because 21 is divisible by 7, 88,580,723 is also divisible by 7. The reader may use a calculator to verify the above result, but one of the advantages of Rule 2 is that it can be applied to numbers too large to input into your calculator.

The proof for Rule 2 drew on my knowledge of modular arithmetic. What follows is a brief overview of the concepts of modular arithmetic that are needed to discover and prove rules of divisibility.

Modular Arithmetic: A Tool for Exploring Divisibility Rules

Two integers, m and n, are said to be *congruent mod k* (Rosen 2003, 161–63) where k is a particular counting number is they differ by a multiple of k, that is $m - n = jk$ where $j \in \mathrm{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. If this is the case, we write $m \equiv n \pmod{k}$. For example, 7 and 12 are congruent mod 5 because $12 - 7 = 1 \times 5$, a multiple of 5. Perhaps a more intuitive way of looking at this example is to say that 7 and 12 both have the same remainder, 2, when divided by 5. ($7 = 1 \times 5 + 2$ and $12 = 2 \times 5 + 2$). To extend the example, -3 is congruent to both 7 and 12 mod 5 because it also has a remainder of 2 when divided by 5 ($-3 = -1 \times 5 + 2$). In fact, modular arithmetic is often conceptualized as the arithmetic of remainders.

Using this idea, we can generate an infinite family of integers of which all members are congruent to 2 mod 5. That family is the set $\{\ldots, -8, -3, 2, 7, 12, \ldots\}$. It is easy to see that each of the numbers in this family will yield a remainder of 2 when divided by 5. We will say that each of these numbers and a *mod 5 equivalent of 2*.

Congruence mod 5 partitions the integers into five families of integers that are called *equivalence classes*:

$[0] = \{\ldots, -10, -5, 0, 5, 10, \ldots\}$
$[1] = \{\ldots, -9, -4, 1, 6, 11, \ldots\}$
$[2] = \{\ldots, -8, -3, 2, 7, 12, \ldots\}$
$[3] = \{\ldots, -7, -2, 3, 8, 13, \ldots\}$
$[4] = \{\ldots, -6, -1, 4, 9, 14, \ldots\}$

The term equivalence class is used because congruence mod k satisfies the three properties of an *equivalence relation* (Roman 1989, 141–48). More will be said about this shortly. [0] is referred to as the equivalence class associated with 0. Numbers in it are congruent to 0 mod 5. Numbers in [1], the equivalence class associated with 1, are each congruent to 1 mod 5; and so on. Note that the numbers in [0], that is, the numbers congruent to 0 mod 5, are all multiples of 5. In this analysis, $k = 5$, but analogous results can be obtained for any value of k. For example, $k = 12$ results in the "clock arithmetic," where 12 equivalence classes corresponding to the hours on a clock face, a analogy that is sometimes taught in the elementary school curriculum.

Exploration of the rule for divisibility by 7 will make use of congruence mod 7. In mod 7 arithmetic, the equivalence classes are

$[0] = \{\ldots, -14, -7, 0, 7, 14, \ldots\}$
$[1] = \{\ldots, -13, -6, 1, 8, 15, \ldots\}$

. . . .
. . . .
. . . .

$[6] = \{\ldots, -15, -8, -1, 6, 13, \ldots\}$

In mod k arithmetic, we often choose the non-negative numbers 0, 1, 2, . . . , (k-1) to be the representatives of the classes [0], [1], [2], . . ., [k - 1]. However, in developing rules for divisibility by a prime, these are not generally the most appropriate representatives. For example, in the test for divisibility by 7, the formulation of the rule is simpler if we use the numbers -3, -2, -1, 0, 1, 2, 3 as representatives of the classes. As well, as we shall see, the formulation of the rule for divisibility by 11 is far simpler if we use -1 rather than 10 as the representatives of the class [10] = {. . ., -23, -12, -1, 10, 21, 32, . . .}.

In devising the rule for divisibility by a prime p we will use mod p arithmetic with equivalence classes [0], [1], [2], . . . , [p - 1]. We will see that, in general, the formulation of the rule for divisibility by a prime p is simplest if we use the integers between _ 1/2 p and 1/2 p as the representatives of the classes rather than the non-negative integers 0, 1, 2, . . . , (p - 1).

Once useful property of modular arithmetic is that the result will be the same no matter if the remainders from addition or multiplication are determined before or after the operation. To put it more formally, the mod k equivalent of the result of a calculation involving two or more counting numbers is the same if the mod k equivalent of each of the counting numbers is used in the calculation. For example, consider the product 28 × 6:

$28 \times 6 = 168 = 33 \times 5 + 3$

$\Rightarrow 28 \times 6 = 3(\text{mod } 5)$

Now find the mod 5 equivalents before multiplying:

$28 = 5 \times 5 + 3 \Rightarrow 28 \Rightarrow 3(\text{mod } 5)$, and

$6 = 1 \times 5 + 1 \Rightarrow 6 \equiv 1(\text{mod } 5)$

Then $3 \times 1 = 3 \equiv 3(\text{mod } 5)$, the same as the result above.

This example illustrates one of the properties of modular arithmetic. The following theorems describe the properties of congruence mod k that are useful in exploring and proving rules for divisibility.

Theorem 1: Suppose that a, b and c integers are that k is a particular counting number. Then the congruence mod k is:

a) reflexive: $a \equiv a(\text{mod } k)$

b) symmetric: $a \equiv b(\text{mod } k) \Rightarrow b \equiv a(\text{mod } k)$

c) transitive: $a \equiv b(\text{mod } k)$ and $b \equiv c(\text{mod } k) \Rightarrow a \equiv c(\text{mod } k)$;

A relation that is reflexive, symmetric and transitive is called an *equivalence relation.* Congruence mod k is a equivalent relation.

Proof of c): Suppose that $a \equiv b(\text{mod } k)$ and $b \equiv c(\text{mod } k)$

Then $a - b = ik$ and $b - c = jk$ for some $i, j \in Z$

$\Rightarrow a - c = (i + j)k,\ i + j \in Z$

$\Rightarrow a \equiv c(\text{mod } k)$

Theorem 1 c) says that if a and b differ by a multiple of k, and b and c differ by a multiple of k, then a and c will differ by a multiple of k. Or, to put it another way, a, b and c will each yield the same remainder when divided by k.

Theorem 2: Suppose $m, n \in Z$ with $m \equiv r_m(\text{mod } k)$ and $n \equiv r_n(\text{mod } k)$, r_m and $r_n \in Z$ with $0 \leq r_m < k$ and $0 \leq r_n < k$*. Then

a) $[m + n] \equiv [r_m + r_n](\text{mod } k)$, and

b) $[m \times n] \equiv [r_m \times r_n](\text{mod } k)$

Theorem 2 a) says that in computing a sum of two integers, the modular arithmetic can be done either before or after finding the sum; the result will be the same. Theorem 2 b) says the same thing about products. Theorem 2 a) can be extended to a sum with any number of items. Similarly, theorem 2 b) can be extended to a product with any number of factors, including a power, as in the corollary below. Taken together, theorems 2 a) and b) imply that in a calculation involving any combination of sums and products of integers, such as a polynomial, the modular arithmetic may be done either before or after doing the calculation. That is, the remainders may be found either before or after doing the calculations (see theorem 3 below). These results are important in an exploring and proving rules for divisibility. The proof of 2 a) is straightforward and is left to the reader.

Proof of 2 b): $m \equiv r_m(\text{mod } k)$ and $n \equiv r_n(\text{mod } k) \Rightarrow m = q_m k + r_m$ and $n = q_n k + r_n$, $q_m, q_n \in Z$. Then,

$$m \times n = (q_m k + r_m)(q_n k + r_n) = q_m q_n k^2 + q_m r_n k + q_n r_m k + r_m r_n$$

$$= (q_m q_n k + q_m r_n + q_n r_m)k + r_m r_n$$

$$\equiv r_m r_n(\text{mod } k)$$

$$\Rightarrow m \times n \equiv r_m r_n(\text{mod } k)$$

The symbols q_m and q_n are appropriate because they represent quotients, Equally appropriate are r_m and r_n, which represent remainders.

Corollary to theorem 2 b): $a \equiv b(\text{mod } k), \Rightarrow a^n \equiv c^n(\text{mod } k)$ where n is any counting number

Proof: $a \equiv b\ (\text{mod } k) \Rightarrow a \times a \equiv b \times b(\text{mod } k)$

$\Rightarrow a^2 \equiv b^2(\text{mod } k)$

$\Rightarrow a^2 \times a \equiv b^2 \times b(\text{mod } k)$

$\Rightarrow a^3 \equiv b^3(\text{mod } k)$

$\Rightarrow a^3 \times a \equiv b^3 \times b(\text{mod } k)$

$\Rightarrow a^4 \equiv b^4(\text{mod } k)$

. . . . .

. . . . .

. . . . .

Theorems 2 a) and b) and the above corollary lead to a more general theorem that has already been alluded to:

Theorem 3: Let $p(x)$ be a polynomial with integer coefficients and k be a counting number. Then, for integers a and b, $a \equiv b(\text{mod } k) \Rightarrow p(a) \equiv p(b)(\text{mod } k)$.

Theorem 3 says that in evaluating a polynomial mod k it does not matter which member of an equivalence class is used; the result will be the same for all members of the class. The theorem is proved formally in many texts (including Stark 1984, 61–65). The proof formalizes the following argument: the integers a and b are members of the same equivalence class and thus have the same remainder, r, when divided by k. In evaluating $p(a)$ and $p(b)$, x is replaced by a and b, respectively, in $p(x)$. Each evaluation consists of calculating sums and products. Thus, according to theorem 2, the remainders for each of $p(a)$ and $p(b)$ may be found either before or after calculating the sums and products. If the remainder r is found first, the result of the evaluation in both cases is $p(r)$. Thus, both $p(a)$ and $p(b)$ will be congruent to $p(r)$ and therefore congruent to each other.

Developing and Proving New Divisibility Rules

With these concepts from modular arithmetic, it is possible to prove the Rule 2 concerning divisibility by 7. The following is a proof for a 12-digit number n. It can easily be extended to numbers with more digits.

Let $n = d_{11}d_{10}d_9 \ldots d_2d_1d_0$

$\Rightarrow n = (d_{11} \times 10^{11}) + (d_{10} \times 10^{10}) + (d_9 \times 10^9) + (d_8 \times 10^8) + (d_7 \times 10^7) + (d_6 \times 10^6) + (d_5 \times 10^5) + (d_4 \times 10^4) + (d_3 \times 10^3) + (d_2 \times 10^2) + (d_1 \times 10^1) + (d_0 \times 10^0)$ **[2]**

Now $10^0 = 1 \equiv 1(\text{mod } 7)$,

$10^1 = 10 \equiv 3(\text{mod } 7)$ since $10 = 1 \times 7 + 3$,

$10^2 = 100 \equiv 2(\text{mod } 7)$ since $100 = 14 \times 7 + 2$,

$10^3 = 1{,}000 \equiv 6 \equiv -1(\text{mod } 7)$ since $1{,}000 = 143 \times 7 + (-1)$,

$10^4 = 10{,}000 \equiv -3(\text{mod } 7)$ since $10{,}000 = 1{,}429 \times 7 + (-3)$,

$10^5 = 100{,}000 \equiv -2(\text{mod } 7)$ since $100{,}000 = 14{,}286 \times 7 + (-2)$,

$10^6 = 1{,}000{,}000 \equiv 1(\text{mod } 7)$ since $1{,}000{,}000 = 142{,}858 \times 7 + 1$,

$10^7 = 10{,}000{,}000 \equiv 3(\text{mod } 7)$ since $10{,}000{,}000 = 1{,}000{,}000 \times 10 \equiv 3 \times 1(\text{mod } 7)$

[Theorem 2 b)]

It should be clear that this list repeats beginning at 10^6. Because n consists of sums of products in equation [2], we can apply theorem 2 to find the mod 7 equivalent of n by replacing the powers of 10 by their mod 7 equivalents. This will obtain

$\Rightarrow n = [(d_{11} \times -2) + (d_{10} \times -3) + (d_9 \times -1) + (d_8 \times 2) + (d_7 \times 3) + (d_6 \times 1) + (d_5 \times -2) + (d_4 \times -3) + (d_3 \times -1) + (d_2 \times 2) + (d_1 \times 3) + (d_0 \times 1)](\text{mod}7)$**

$\Rightarrow n = [(1d_0 + 3d_1 + 2d_2 + (-1d_3) + (-3d_4) + (-2d_5) + 1d_6 + 3d_7 + 2d_8 + (-1d_9) + (-3d_{10}) + (-2d_{11})](\text{mod}7)$ **[3]**

It is clear that n will be divisible by 7 if and only if $n \equiv 0(\text{mod } 7)$. Because n has a mod 7 equivalent that is equal to the linear combination of the digits in equation [3], it will be divisible by 7 if and only if that linear combination is divisible by 7.

This concludes the proof of rule 2 for a 12-digit number. This proof could be extended to a number with any length of digits. It should be clear why the mod 7 equivalents of the powers of 10 were chosen to be between -3 and 3 inclusive rather than between 0 and 6 inclusive.

Having discovered the rule for divisibility by 7, I was prepared to move on to other rules. But, as I reflected on the process the following facts struck me as having more than passing significance:

- $10^0 \equiv 1(\text{mod } 7)$
- $10^3 = 1{,}000 \equiv -1(\text{mod } 7)$
- $10^6 = 1{,}000{,}000 \equiv 1(\text{mod } 7)$
- $10^9 = 1{,}000{,}000{,}000 \equiv -1(\text{mod } 7)$

When I recognized that $1{,}000{,}000 = 1{,}000^2$, $1{,}000{,}000{,}000 = 1{,}000^3$ and so on, it occurred to me that a rule with a simpler formulation could be constructed if n were first written in base 1,000. For example, consider again $n = 88{,}580{,}723$. Then

$n = 88 \times 1{,}000^2 + 580 \times 1{,}000^1 + 723 \times 1{,}000^0$

$\Rightarrow n \equiv [88 \times 1 + 580 \times -1 + 723 \times 1](\text{mod } 7)$

$\Rightarrow n \equiv [-231](\text{mod } 7) = [-33 \times 7 + 0](\text{mod } 7) \equiv 0(\text{mod } 7)$

Thus n is divisible by 7.

The above observations and example lead to another formulation of the rule for divisibility by 7.

Rule 2a: A number n is divisible by 7 if and only if, when it is expressed in base 1,000, the alternating sum of its digits beginning with the last digit is divisible by 7. Note that alternating sum is used here to mean that the signs of the digits are alternated between positive and negative. In base 1,000 a digit is typically a 3-digit base 10 number.***

The next step was to develop a rule for divisibility by 11. The exploration process was analogous to that used in developing the rule for divisibility by 7.

First, the mod 11 equivalents of the powers of 10 were found using theorem 1:

$10^0 = 1 \equiv 1 \pmod{11}$,
$10^1 = 10 \equiv -1 \pmod{11}$ since $10 = 1 \times 11 + (-1)$,
$10^2 \equiv [(-1)^2] \pmod{11} \equiv 1 \pmod{11}$,
$10^3 = 1{,}000 \equiv [(-1)^3] \pmod{11} \equiv -1 \pmod{11}$,
$10^4 = 10{,}000 \equiv [(-1)^4] \pmod{11} \equiv 1 \pmod{11}$,
$10^5 = 10{,}000 \equiv [(-1)^5] \pmod{11} \equiv -1 \pmod{11}$

It is clear that the mod 11 equivalents of the powers of 10 alternate in the pattern 1, -1, 1, -1, . . . providing that -1 is used as the representative of the class {. . . , -23, -12, -1, 10, 21, . . .}.

The exploration described above leads to a simple rule for testing divisibility by 11:

Rule 3: A counting number n is divisible by 11 if and only if the alternating sum of its digits is divisible by 11.

As alluded to earlier, the simplicity of rule 3 is dependent on using -1 as the representative of the class {. . ., -23, -12, -1, 10, 21, . . .}. Because the above explanation of the development of the rule also contains the basic elements of its proof, no formal proof is included here. The following is an example of its application.

Determine whether n = 576,213,489,573 is divisible by 11.

Solution: $3 - 7 + 5 - 9 + 8 - 4 + 3 - 1 + 2 - 6 + 7 - 5 = -4 \equiv 7 \pmod{11}$. According to rule 3, n is not divisible by 11.

The techniques used in exploring rules for divisibility by 7 and 11 can be applied toward finding a rule for divisibility by 13, 17 or any prime. Much of that exploration is left for the reader. The next section deals, rather, with a theorem that describes a general algorithm for determining divisibility by any prime p.

A General Algorithm for Testing Divisibility by a Prime

Theorem 4: Suppose that $n = d_t d_{t-1} d_{t-2} \ldots d_2 d_1 d_0$ is a counting number and p a prime with $n > p$. Then n is divisible by p if and only if $m = c_t d_t + c_{t-1} d_{t-1} + c_{t-2} d_{t-2} + \ldots + c_1 d_1 + c_0 d_0$ is divisible by p where each $0 \leq i \leq t$, $c_i \equiv 10^i \pmod{p}$ and $-1/2p < c_i < 1/2p$.

Proof: It will be sufficient to prove than $n \equiv m \pmod{p}$

$n = (d_t \times 10^t) + (d_{t-1} \times 10^{t-1}) + \ldots + (d_i \times 10^i) + \ldots + (d_1 \times 10^1) + (d_0 \times 10^0)$

Note than $c_i \equiv 10^i \pmod{p} \Rightarrow 10^i \equiv c_i \pmod{p}$ [Theorem 1.b]

$\Rightarrow d_i \times 10^i \equiv d_i \times c_i \pmod{p}$ [Theorem 2.b]

$\Rightarrow (d_t \times c_1^t) + (d_{t-1} \times c_1^{t-1}) + \ldots + (d_1 \times c_1^i) + \ldots + (d_1 \times c_1^1) + (d_0 \times c_1^0) \equiv (c_t d_t + c_{t-1} d_{t-1} + c_{t-2} d_{t-2} + \ldots + c_1 d_1 + c_0 d_0) \pmod{p}$ [Theorem 2.a]

$\Rightarrow n \equiv m \pmod{p}$

As a final exercise in this exploration, apply theorem 4 to find a rule for divisibility by 13. In developing the rule, use $10^i \equiv c_1^i \pmod{p}$ to find the mod 13 equivalents of the powers of 10.

$10^0 = 1 \equiv 1 \pmod{13}$,
$10^1 = 10 \equiv -3 \pmod{13}$,
$10^2 \equiv (-3)^2 \pmod{13} \equiv 9 \pmod{13} \equiv -4, \text{mod } 13)$
$10^3 \equiv [(-3)^2 \times (-3)] \pmod{13} \equiv [(-4) \times (-3)] \pmod{13} \equiv 12 \pmod{13} \equiv -1 \pmod{13}$,
$10^4 \equiv [(-3)^3 \times (-3)] \pmod{13} \equiv [(-1) \times (-3)] \pmod{13} \equiv 3 \pmod{13}$,
$10^5 \equiv [(-3)^4 \times (-3)] \pmod{13} \equiv [(3) \times (-3)] \pmod{13} \equiv -9 \pmod{13} \equiv 4 \pmod{13}$,
$10^6 \equiv [(-3)^5 (-3)] \pmod{13} \equiv [(4) \times (-3)] \pmod{13} \equiv -12 \pmod{13} \equiv 1 \pmod{13}$,

. . . .
. . . .
. . . .

Now, suppose $n = d_t d_{t-1} d_{t-2} \ldots d_{11} d_{10} d_9 d_8 d_7 d_6 d_5 d_4 d_3 d_2 d_1 d_0$

$\Rightarrow n = (d_t \times 10^t) + (d_{t-1} \times 10^{t-1}) + (d_{t-2} \times 10^{t-2}) + \ldots + (d_{11} \times 10^{11}) + (d_{10} \times 10^{10}) + (d_9 \times 10^9) + (d_8 \times 10^8) + (d_7 \times 10^7) + (d_6 \times 10^6) + (d_5 \times 10^5) + (d_4 \times 10^4) + (d_3 \times 10^3) + (d_2 \times 10^2) + (d_1 \times 10^1) + (d_0 \times 10^0)$

Then, using the above mod 13 equivalents for the powers of 10, the following is obtained:

Rule 3: A counting number n is divisible by 13 if and only if m is divisible by 13 where

$m = [1d_0 = (-3d_1) + (-4d_2) + (-1d_3) + 3d_4 + 4d_5 + 1d_6 + (-3d_7) + (-4d_8) + (-1d_9) + 3d_{10} + 4\, d_{11} + \ldots]$

Example: Determine whether $n = 889{,}594{,}829{,}357$ is divisible by 13.

Solution: n will be divisible by 13 if and only if m is divisible by 13 where

$m = 1 \times 7 + (-3 \times 5) + (-4 \times 3) + (-1 \times 9) + 3 \times 2 + 4 \times 8 + 1 \times 4 + (-3 \times 9) + (-4 \times 5) + (=1 \times 9) + 3 \times 8 + 4 \times 8 = 13 \equiv 0 \pmod{13}$

$m \equiv 0 \pmod{p} \Rightarrow m$ is divisible by 13
$\Rightarrow n$ is divisible by 13

Conclusion

The process of exploration described in this article began with some textbook rules for divisibility by 2, 3, 4, 5, 6, 8 and 9. Those rules lead to a search for rules for divisibility by other numbers like 7 and 11. The focus was on primes because it seemed that once the rules for divisibility by primes was uncovered, divisibility by a composite number could be tested using a combination of the rules for divisibility by primes. Uncovering the rules for divisibility by 7 and 11 was expedited by calling on the concepts of modular arithmetic. These concepts enabled the culmination of the exploration process, namely the formulation of a general algorithm for determining divisibility by a prime.

This process of exploration was particularly satisfying for a number of reasons:

1. There was the prospect of exploring many rules (because there are many primes that might be of interest) with the possibility of observing some general patterns.
2. The process naturally used the concepts of modular arithmetic and demonstrated a property that the concepts have in common with most mathematical concepts—their ability to expand our native brainpower.
3. The process satisfied a compulsion that has characterized most mathematical exploration over the past couple of centuries—the need to generalize. It lead to the determination of a general algorithm for testing divisibility by a prime.
4. The culmination in a general algorithm gave a feeling of completion to the process. Later, the thought hit me in an Archimedes moment that the algorithm could be made perfectly general. After the formulation and proof of the algorithm, it occurred to me that the properties of modular arithmetic that I had applied to the primes were equally applicable to composites. Therefore, the algorithm can be extended to composites and thus to all counting numbers. This even more general algorithm could be used to verify the rules for divisibility by 4, 6, 8 and 9, and to explore the patterns in the rules for divisibility by other composites.

This process of uncovering the rules for divisibility by a prime is illustrative of the many opportunities for mathematical exploration that teachers encounter. By taking advantage of these opportunities, we can sensitize our students to these opportunities and help them become more acquainted with the nature of mathematical discovery.

Notes

* The symbols rm and rn are used because they are the remainders when m and n, respectively, are divided by k. According to the division algorithm for counting numbers, the remainder r, when a counting number n (the dividend) is divided by another counting number d (the divisor), can be made to be a non-negative number less than d. In our case, the divisor is k so the remainder can be made to be less than k. The division algorithm can be extended to the integers Z.

** Alternatively, one could observe that $n = p(10)$, where $p(x) = d_{11}x^{11} + d_{10}x^{10} + d_9x^9 + \ldots + d_2x^2 + d_1x^1 + d_0x^0$, a polynomial. By theorem 3, since $10 \equiv 3 (\text{mod } 7)$, $p(10) \equiv p(3) (\text{mod } 7)$. The mod 7 equivalent of n could be evaluated by using $p(3)$ instead of $p(10)$. The reader can check that $30 \equiv 1 (\text{mod } 7)$, $31 \equiv 3 (\text{mod } 7)$, $32 \equiv 2 (\text{mod } 7)$, $33 \equiv -1 (\text{mod } 7)$, $34 \equiv -3 (\text{mod } 7)$, $35 \equiv -2 (\text{mod } 7)$ and so on. The result would be the same.

*** In a codified base 1,000 system we would need 1,000 different symbols to represent the numbers 0, 1, 2, . . . , 999. In such a theoretical system each digit would be represented by just one symbol.

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A Little on the Lighter Side!—and Beyond?

Werner Liedtke

Over the years, I have concluded some of my courses' final examinations by asking the students to write, for two marks, a couplet, rhyme or pun about mathematics teaching and learning. The option of printing "The End" is also presented, but many students take the opportunity to write something. I am always amazed at their creativity. When the request has not been part of the exam, students have asked, "What, no poem?" or have written one anyway.

I would like to share a few submissions to illustrate what creative activities can result in, and if some of these result in a smile, then they were worthwhile.

Some students will object to being asked to write something but will do so in verse form, just to make sure. Here are three examples:

You son of a gun,
Thinking I'll write you a pun
About arithmetic fun.
Sorry—I've got to run!

Wish I could
Wish I might
Pass this test
And do all right.
Now it's over.
Now it's true.
'Cept this poem
I can't do.

As I don't have enough time
To make a couplet in rhyme,
I just thought I'd let you know
That I'm about ready to crow
And giving a test like this is a crime!

No doubt, many of the pieces that have appeared over the years are good enough to be put into print. At least one student anticipated this when she wrote:

I like to teach the numbers,
I like to teach the signs,
But I find that in arithmetic
Not many words do rhymes.
And let me just remind you,
If you put this in a book,
I want a cut of the profit,
Or I'll label you a crook!

There are students who anticipate the request and prepare something ahead of time. This was the case for K. Enders, a distance-education student from Calgary in a course on diagnosis and intervention. She enclosed the following last April:

Number Sense

When I was a child, you see,
Numbers made no sense to me.
It takes a teacher, you see,
to develop NUMERACY.

SUBITIZE !—think fast!—just try?
How many did I see? (Big sigh)
I could not count as they flashed by.
My brain is fried. Oh my, Oh My!

VISUALIZE, now my surprise!
Saw "50" in my mind—three guys!
Count fingers thrice, but toes just twice,
Teacher, am I getting wise??

FLEXIBLE, what does it mean?
To bend, to snap? Oh I'm so keen!
Different numbers make thirteen???
Such combinations I've not seen!

RELATE to other numbers?—NO!
My thinking is so awful slow.
Sixty-six is big I know!
But numbers bigger? Smaller?—OH!

CONNECTING still remains a pain.
It's so hard to engage my brain.
My teachers frowned with great disdain
Relate my math life?—insane!!

I ESTIMATE at sixty-nine,
I'll be doing math just fine,
As in my rocker I decline
And hear, "Well done—now, please resign!"

When I was in your class, you see,
NUMBER SENSE took shape—s-l-o-w-l-y.
Math intervention worked for me.
Thank you—
"Was blind, but now I see!!!!"

I still smile when I think about the student who wrote:

Instead of studying math,
I decided to have me a bath.
The result was—I failed the test,
But of everyone there, I smelled the best!

Then, there are comments that come straight from the affective domain:

I think adding
Is very saddening.
I think subtraction
Is not worth the attraction.
I think multiplication
Is too hard to rhyme with!

Thank God this test is over
I think I am going to die.
I did not know the answers
And I couldn't even lie.

The following is part of a poem made up by K. Koopmans, who was a student in a University of Victoria program in Cranbrook, British Columbia. It was written for a unit for her Grade 2 students:

This is the tale of two young knights,
So noble and so fine.
The first young knight was named Pat Urn,
The second knight: D. Zine

Now, two things must be stated here
To clarify this rhyme,
So please listen most carefully:
I'll state it just one time.

Now, first of all, it must be known,
How to pronounce "D. Zine"
For it does not rhyme with "nineteen":
Instead, it rhymes with "nine."

Now, secondly, no one quite knows
What "D" means in D. Zine,
But he's always called this;
It seems to suit D. fine.

The problem, though, for these two knights,
Was not with just their names,
For folks thought Pat Urn and D. Zine
Were, all in all, the same.

Both knights had flags they flew with pride
Of yellow, green and blue.
They were alike in many ways,
Yet, different in a few.

D. Zine's flag was impressive, yes,
With multicoloured hues,
And random shapes across his flag,
In yellow, green and blue.

D. Zine had stars and squares galore,
Which were spread to and fro,
While shapes that were on Pat Urn's flag
Formed patterns on each row.

Compared with Pat Urn's precise flag,
Of perfect patterned rows,
D. Zine's flag was a mess of shapes
(Pat Urn had told him so!).

And so a Wizard told the town
The answer to their plight.
There was a very simple way
To recognize each knight.

One had to look so very close
At each flag of each knight,
and if the shapes repeated, then:
"It's Pat Urn!" would be right.

But if the shapes did not repeat
In any sort of way,
"This knight must be the great D. Zine!"
Is what you'd have to say.

Thence, both the knights lived happily
Their lives were full and fine,
Because of the great Wizard—
They knew Pat Urns from D. Zines.

Whenever I read the ideas submitted by teachers and teachers-to-be, I cannot help but smile and think how lucky children are to be taught by these people.

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Children's Literature in the Elementary Mathematics Classroom

A. Craig Loewen

Any elementary school teacher can tell you that children love to share in a good story. Although some students are stronger and more avid readers than others, all children seem to enjoy either reading a story or sharing in the experience of having a story read to them. Some of the most intimate and special times I remember sharing with my children involved our rocking chair and a good book, typically one that one of them selected. Times spent with a good book are inherently enjoyable and motivating. We can build on that sense of enjoyment and bring it into our primary mathematics classrooms by using good children's literature as a springboard to fun and challenging mathematics activities. In this article, several children's books will be introduced, accompanied by a few math activities related to each story.

The process whereby I selected the books was really quite simple. I asked my children's teachers what their favourite books were, I asked local librarians for titles of popular books and, most importantly, I asked my children which of our books they enjoyed the most. I have focused on children's books that are very motivating. They have a charming story or intriguing pictures, and some have both. I did not select books that were obviously geared toward teaching mathematics; that is, books in which the story is deliberately secondary to how the main character uses math in his or her life, such as books that tell the story of how a character learns to add or tell time. Although these publications also have their place, I was much more interested in books that emphasized storyline, character, plot, or a surprise or visual appeal—the books we most like to read.

As you peruse the activities that follow, you will notice that the stories serve mainly as context for the mathematical activities, and that the activities may take a variety of forms, including games, problems, manipulative explorations and so on. The books are not selected to teach a concept itself but to stimulate the child's imagination and motivation—to get him or her dreaming about being chased by a monster or eating cookies with Grandma, or imagining how hard it would be to rise above the challenges of a truly rotten day.

Alexander and the Terrible, Horrible, No Good, Very Bad Day by Judith Viorst

Have you ever had a day as bad as Alexander's day? His day was so bad that he was thinking of just packing up and moving to Australia! From the first moment he rises in the morning, everything just starts to go wrong. Getting dressed turns out to be a disaster, breakfast is ruined because there is no toy in his cereal and travelling to school is uncomfortable, but school itself is absolutely awful. Alexander's day just gets worse and worse. He finds out that he is no longer Paul's best friend and discovers that his mother forgot to put the treat in his lunch, and then the dentist finds a cavity in his teeth. I can identify with Alexander. Can you?

Activity 1: Alexander's Game
Objective: Count orally by ones, twos, fives and tens to 100 (Number Concepts, Grade 1).

Part of Alexander's very bad day happens during counting time when he leaves out the number 16. Sometimes leaving out a number is good and even fun!

Play this simple game with a partner. Try skip-counting by twos all the way to 100. If you make it to 100 successfully, start over skip-counting by fives. To increase the challenge, skip-count by threes, starting at 11. In another version of the game, further increase the challenge by snapping your fingers instead of saying the number if it has a four in it (such as 14, 24, 34, 40, 42, 44 and so on). Adapt the games by playing with several players. Any player who makes a mistake drops out of the game. The last person standing wins.

Activity 2: Horrible-Day Race Game
Objective: Explore faces, vertices and edges of 3-D objects (3-D Objects and 2-D Shapes, Grade 2).
Materials: 3-D solids, scrap paper

To play this game, the teacher must first prepare the object cards. Select 8–10 different objects or things that are mentioned in the story—examples include castle, tooth, paper bag and sweater—and write these words on separate index cards.

Players are divided into teams with two to six players, and one player is appointed as the first to draw. Each team needs several sheets of scrap paper and a collection of 3-D solids.

To start the game, each student drawing first for his or her team approaches the teacher, and the teacher reveals the object listed on the first index card to all of them. The players return to their groups and race to draw the object listed on the card. However, players may not draw freehand; instead they must trace around the edges of the faces on their 3-D solids. The player drawing may not speak or try to act out the word. Once another player in the group guesses the name of the object from the drawing, that player goes up to the teacher and whispers the name of the object to the teacher. If correct, the player is shown the object on the next index card. This player returns to the group and takes over the task of drawing for his or her team. In this way, the team races its way through the object cards. The first team to progress through the entire deck of object cards wins.

The True Story of the Three Little Pigs! by Jon Scieszka

We all know the story. You know, the one about the three little pigs and the big bad wolf? This book echoes the familiar tale in which the cruel wolf destroys the homes of the poor piglets and eats them up. But did it ever occur to you that the wolf might really be innocent? Perhaps the entire incident was the fault of the piglets. Maybe they drove him to it. This is the famous story with a new twist, told from the perspective of the wolf—and maybe he has a point!

Activity 3: House of Bricks
Objective: Place objects on a grid, using columns and rows (Transformations, Grade 4).
Materials: Centimetre grid paper

Try building your own house of bricks, where the bricks are locations on a piece of grid paper. Draw your house by shading in squares on your grid paper. Each brick is one square. Make a list of all the ordered pairs representing the squares you shaded in. Try reading your list of ordered pairs to a partner and let him or her try to build a house just like yours from your instructions.

Activity 4: House of Sticks
Objective: Observe and build a given 3-D object (3-D Objects and 2-D Shapes, Grade 1).
Materials: Marshmallows and toothpicks

In the story, the three little pigs each build their own houses—one of straw, one of sticks and one of bricks. Try to build the frame for your own house using only marshmallows and toothpicks. How many marshmallows did you use? How many toothpicks did you use? Can you build a frame for a two-storey house using only 13 marshmallows? How many sticks (toothpicks) are needed for this house? Can you build a frame for a single-storey house using only 16 toothpicks? How many marshmallows are required?

Activity 5: Wolves and Pigs Problem
Objective: Communicate and apply positional language in oral, written or numerical form (Transformations, Grade 2).
Materials: Two-colour markers

There are three wolves and three pigs on the same side of a river and they would like to cross to the other side. They have one boat that can hold only two animals at a time. If, at any time, the wolves outnumber the pigs, the pigs will be eaten! How can all six animals cross the river safely?

The Monster Bed by Jeanne Willis (illustrated by Susan Varley)

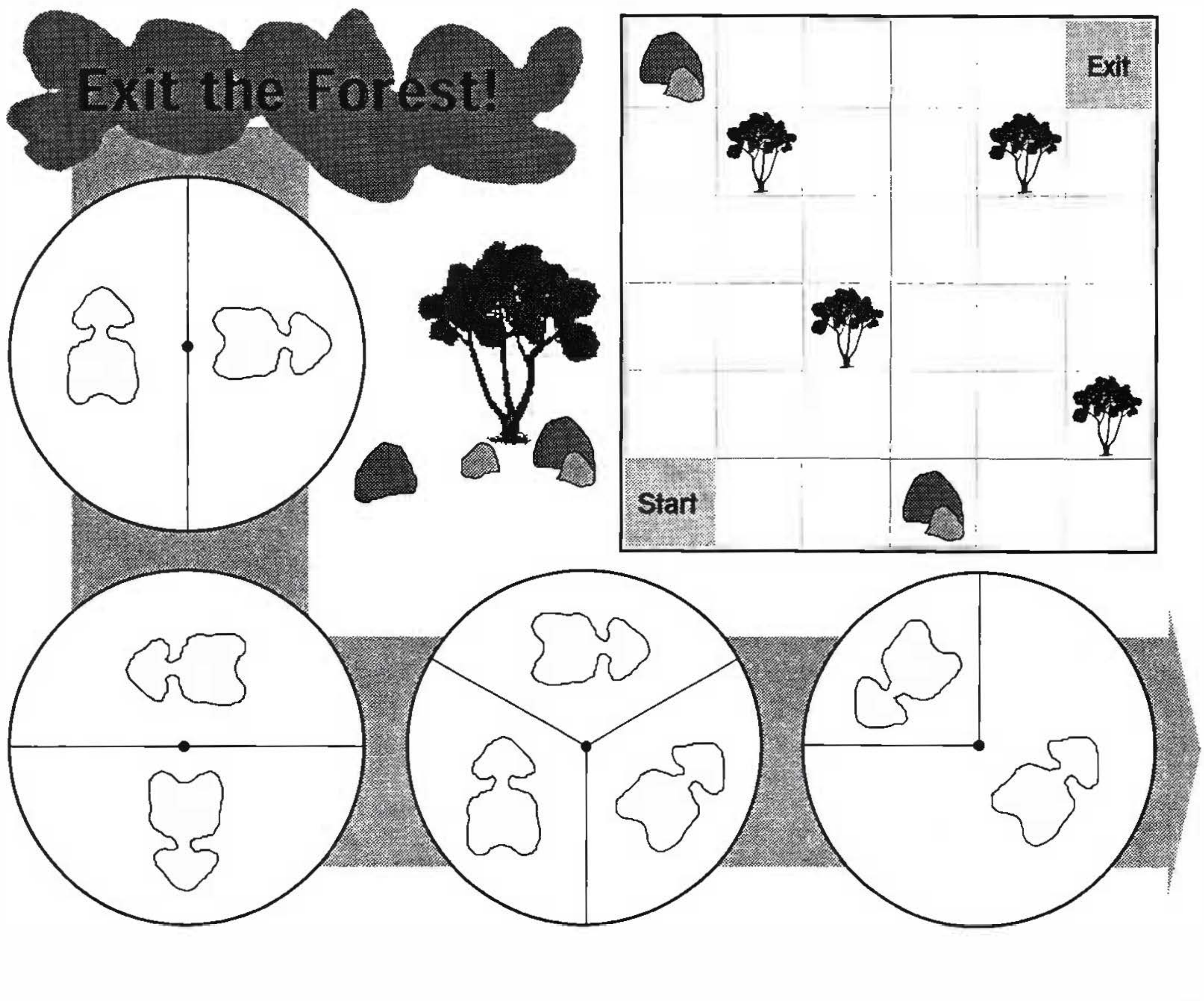
"Never go down to the Withering Wood!"—you never know what you might find or what might happen. The Withering Wood is special, indeed, because it is home to a family of monsters who don't believe in humans. The smallest monster is terribly afraid of humans, but his mother assures him that there is no such thing. Just to be safe, the smallest monster decides to sleep under his bed—after all, no humans would think to look for him there! Along comes a truant (human) boy who has found himself lost in the woods. He discovers the monster's cave and bed and decides to lie down for a little nap. Just as he is about to drift off to sleep, it occurs to the boy to check under the bed for monsters.

Activity 6: Measure a Monster
Objective: Construct items of specific lengths (Measurement, Grade 2).
Materials: Six-sided die, ruler

Draw your own monster. Start with a piece of scrap paper. Draw an oval approximately five centimetres long in the middle of the paper. This oval will be the body of your monster. To complete your monster, you need to add a head, nose, wings, legs, arms, ears, eyes and anything else you might choose. To add pieces to your monster, first decide what part you wish to draw and then roll the die to determine how long (in centimetres) that part is to be. For example, if you decide to draw the right leg of your monster and you roll a three, you would draw a leg three centimetres in length. You can add more than one set of arms or legs, or none at all depending on your imagination. Whose monster is the scariest?

Activity 7: Exit the Forest
Objective: Compare outcomes as equally likely, more likely or less likely (Chance and Uncertainty, Grade 4).
Materials: Exit the Forest game board, clear plastic overhead spinner

This is a probability game you can play by yourself, cooperatively with another player or competitively with other players as a race to the exit. To play the game, place a small marker (such as a bean or a block) on the "Start" square in the bottom left corner of the game board. Place the clear spinner on any one of the four given spinner mats and twirl the spinner. Move your marker in the direction spun (one space horizontally, vertically or diagonally). If the game is played competitively, the first player to make his or her way around the obstacles to the upper right corner wins. You may not leave the game board or land on any space occupied by an object or another player.



Alphabet City by Stephen T. Johnson

This is actually a picture book of art; no reading is necessary to thoroughly enjoy this work. *Alphabet City* is a collection of watercolour paintings and drawings that show where many of the letters of the alphabet occur in a city setting. The pictures do not show just how letters are used to form words on signs and so on, but also how the actual shapes of the letters themselves are all around us, if only we would notice. *A* is not for apple in *Alphabet City*; the letter *A* is formed by a traffic barricade. *B* is found on some metal stairs, *C* is found in a cathedral window, *D* makes a great border for a flower garden and so on. Can you guess where the letter *Q* can be found? My favourite is the letter *S*. What's yours?

Activity 8: Symmetry City
Objective: Create and verify symmetrical 2-D shapes by drawing lines of symmetry (Transformations, Grade 4).
Materials: Geoboard, elastics, mira board or mirror

As you look at the wonderful pictures in *Alphabet City* you will notice that many of the letters have more than one line of symmetry. Which letters have one line of symmetry? Which letters have two lines of symmetry? Which letters have more than two lines of symmetry? What words can you make that have a line of symmetry running through the entire word by using the letters? You may need to spell some words from top to bottom on your page to see the line of symmetry.

Try making some of the letters of the alphabet with elastics on a geoboard. Use a mirror or your mira board to explore the letters in the book, and use the geoboard to find all the lines of symmetry.

Activity 9: Word Sums
Objective: Apply a variety of estimation and mental-mathematics strategies to addition and subtraction problems (Number Operations, Grade 2).
Materials: Calculator

Have you ever heard of a two-dollar word? Usually we refer to very long or unusual words as two-dollar words. Begin by making a list of all the letters of the alphabet and writing a number next to each letter showing its place in the alphabet. Write 1 next to the *A*, 2 next to the *B* and so on. These numbers represent the values of each letter. Find the value of your name by adding up the values of each letter in your name. Whose name has the greatest sum? Can you find a word with a value of exactly 100? What about 200?

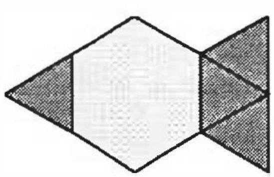
Blue Sea by Robert Kalan

This story is about a little fish that encounters a big fish, which would like to eat him for lunch, except that along comes a bigger fish. These fish are chased by an even larger fish! How should the littlest fish escape? Each fish is eventually trapped until the little fish swims away safely into the blue sea. This story incorporates many mathematical concepts, including simple addition and subtraction, as well as relative sizes and shapes.

Activity 10: Pattern Block Fish
Objective: Use manipulatives and diagrams to demonstrate and describe the processes of addition and subtraction of numbers to 18 (Number Operations, Grade 1).
Materials: Pattern blocks, pencil crayons

This is a simple exploration activity in which students construct models of simple addition equations. In this activity, students select any number of two different colours of blocks and arrange them to resemble the shape of a fish. The students then write an addition sentence to represent the number of blocks of each colour used.

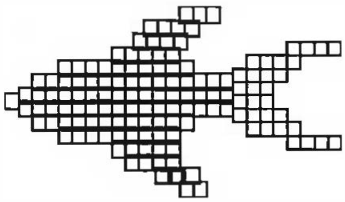
For example, in the fish below, one yellow hexagon and four green triangles were used. The related sentence is $1 + 4 = 5$.



Ask students to trace the blocks on a piece of paper in the shape of their fish and colour the fish according to the colour of the blocks before writing the addition sentence.

Activity 11: Going Fishing
Objective: Estimate, measure, record and compare the area of shapes, using nonstandard units (Measurement, Grade 2).
Materials: Coloured tiles (or paper squares cut approximately one inch on a side)

In this exploration activity, you will use coloured tiles (or paper squares) to create fish shapes. Whose shape has the greatest area (measured in tiles)? Whose shape uses the greatest number of blue tiles? How about red tiles? Create the best fish shape using exactly 24 tiles. If you build your fish out of paper squares and glue them to a larger page, the children can create a giant aquarium bulletin board. Develop estimation questions based on your bulletin board display. For example, how many tiles were used in all to make these fish?



The Doorbell Rang by Pat Hutchins

Don't you just hate it when someone calls or the doorbell rings just as you are sitting down to supper? Or even worse, a caller arrives just as you are about to dive into a plate of fresh-baked cookies! In this story, several children are about to do just that—share a plate of cookies—when the doorbell rings and several friends walk in. After the cookies are redistributed, the kids are again about to start eating when the doorbell rings and more friends arrive. This happens over and over again until . . .

Activity 12: Cookie Countdown
Objective: Apply a variety of estimation and mental-mathematics strategies to addition and subtraction problems (Number Operations, Grade 2).
Materials: Chocolate chips (or other small markers, such as gram blocks)

This activity is both a problem and a game, based on the traditional game of Nimh. Start with 21 cookies (chocolate chips or small markers) placed on the table between two players. Players take turns removing (eating) one or two cookies. The player who eats the last cookie wins. How can you make sure that you win each time?

Activity 13: Cookie Dough
Objective: Calculate products and quotients, using estimation strategies and mental mathematics strategies (Number Operations, Grade 3).
Materials: Chocolate chips (or other small markers), one six-sided die

This is a game for two or more players. Each player starts with 20 chocolate chips. The first player rolls a die and divides the chips into groups according to the value rolled. For example, assume a player rolls a three on his or her turn while owning 20 chips. This player would divide the 20 chips into six sets of three chips with two chips remaining. The remaining chips are given away to the next player, who now rolls the die. Each time, the leftovers are passed to the next player. Players each take several turns. The player with the most chips at the end of the game wins.

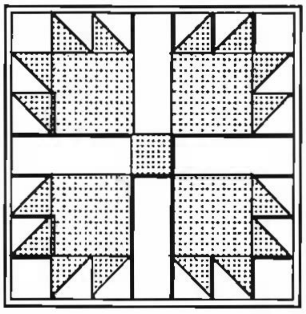
Selina and the Bear Paw Quilt by Barbara Smucker

This story is about a displaced Mennonite family during the American Civil War. Those of the Mennonite faith tried desperately to stay out of the conflict and were therefore mistreated by both the North and the South. As a result, many found that they had to relocate to Upper Canada. This is the story of Selina and how her family is driven apart by war when Selina's grandmother is forced to stay behind, unable to make the journey. To comfort her and to remind her forever of her grandmother, Selina is presented with the beautiful bear paw quilt.

Activity 14: Mirror Quilts
Objective: Recognize motion as a slide (translation), turn (rotation) or flip (reflection) (Transformations, Grade 5).
Materials: Quilt pattern page, crayons

Quilt patterns can be made by applying mathematical transformations to specific shapes or designs. For example, quilts can be designed so that adjacent squares are mirror images of each other (reflections) or so that adjacent squares are turned a quarter turn from one square to the next (rotations). In this activity, students are given a pattern for a quilt (such as the one shown). One student colours the left side of the page using crayons. This page is then passed to another student who colours the right side of the page so that it is a mirror image of the left. For a more challenging activity, ask students to construct quilt patterns or pictures that demonstrate rotations, reflections or dilations.

Please refer to the quilt pattern sheets at the end of this article.



Activity 15: Pattern Block Quilt
Objective: Create, extend and describe patterns, including numerical and non-numerical patterns (Patterns, Grade 2).
Materials: Pattern blocks, paper

In this problem-solving activity, students will attempt to build a quilt design by covering a plain piece of paper entirely with a collection of pattern blocks arranged in a design. The student will write questions about the quilt, such as, How many yellow blocks were used? How many more blue blocks than green blocks were used? If this panel were only one of 12 needed to complete the quilt, how many red blocks would be needed? Students will get a friend to try answering their questions. The friend may then attempt to extend the quilt using the same pattern.

Alexander, Who Used to Be Rich Last Sunday by Judith Viorst

Alexander is a lot like the rest of us: his money is all gone! Have you ever felt rich right after you were paid, and then slowly watched it all disappear as you paid bills, loans and everything (and everyone) else? This is the same thing that happens to Alexander. He starts with a dollar on Sunday but, by the next Sunday, all he has are bus tokens. Alexander's money simply disappears on bad debts, bubble gum and renting snakes. If you had a dollar, what would you spend it on?

Activity 16: A Change of Heart
Objective: Create and recognize that a given value of money can be represented in many different ways (Measurement, Grade 3).
Materials: Coins

Using a collection of coins, students try to solve the following problem: How many different ways are there to make 55¢ using only dimes and nickels? How many more ways are there if you can use pennies, nickels, dimes and quarters?

Activity 17: Dollar 1-2-3
Objective: Estimate, count and record collections of coins and bills up to $10 (Measurement, Grade 3).
Materials: Coins, one six-sided die

This game is played with two or more players. Place a large collection of coins (pennies, nickels, dimes and quarters) between the players. The first player rolls the die and, based on the value rolled, takes that number of coins from the pile in the centre of the table and adds them to his or her collection. For example, if the player rolls a 5, he or she can take three nickels and two quarters, or any other combination of five coins. Play passes to the left.

Players try to build a set of coins with a value of exactly $1. Players can trade in coins—two nickels for a dime, and so on. Players cannot take coins that would take the amount of their collection over $1. For instance, if the player rolls a 3 when his or her collection totals 99¢, the player simply passes the turn. After a player has collected exactly $1, he or she tries to collect another dollar. The first player to build $1, then $2 and finally $3 is the winner.

Conclusion

By simply using the main theme or idea of a story—whether it is money or monsters—it is easy to develop a wide variety of mathematical activities to enhance our instruction in the elementary grades. In the activities above, examples of games, manipulatives, applications and problem-solving activities can be found. Teachers should select books and stories that they find interesting because it is likely that their students will find them interesting, too. Consider asking your students to bring in some of their favourite books. Perhaps they would enjoy the challenge of developing and sharing some activities of their own. By capitalizing on the inherent interest and motivation that these stories hold for students, we can introduce increased variety and excitement into our mathematics lessons.

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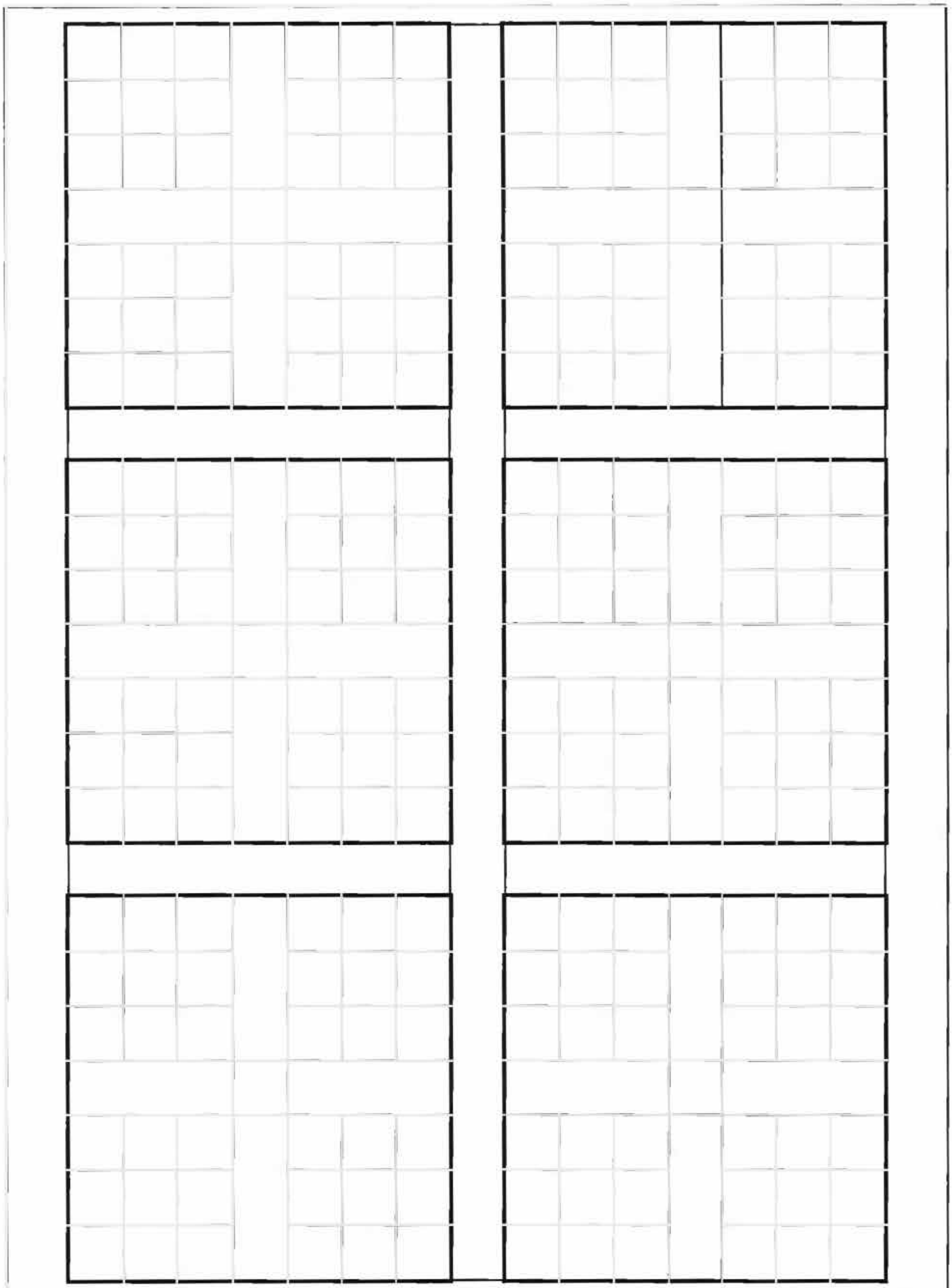
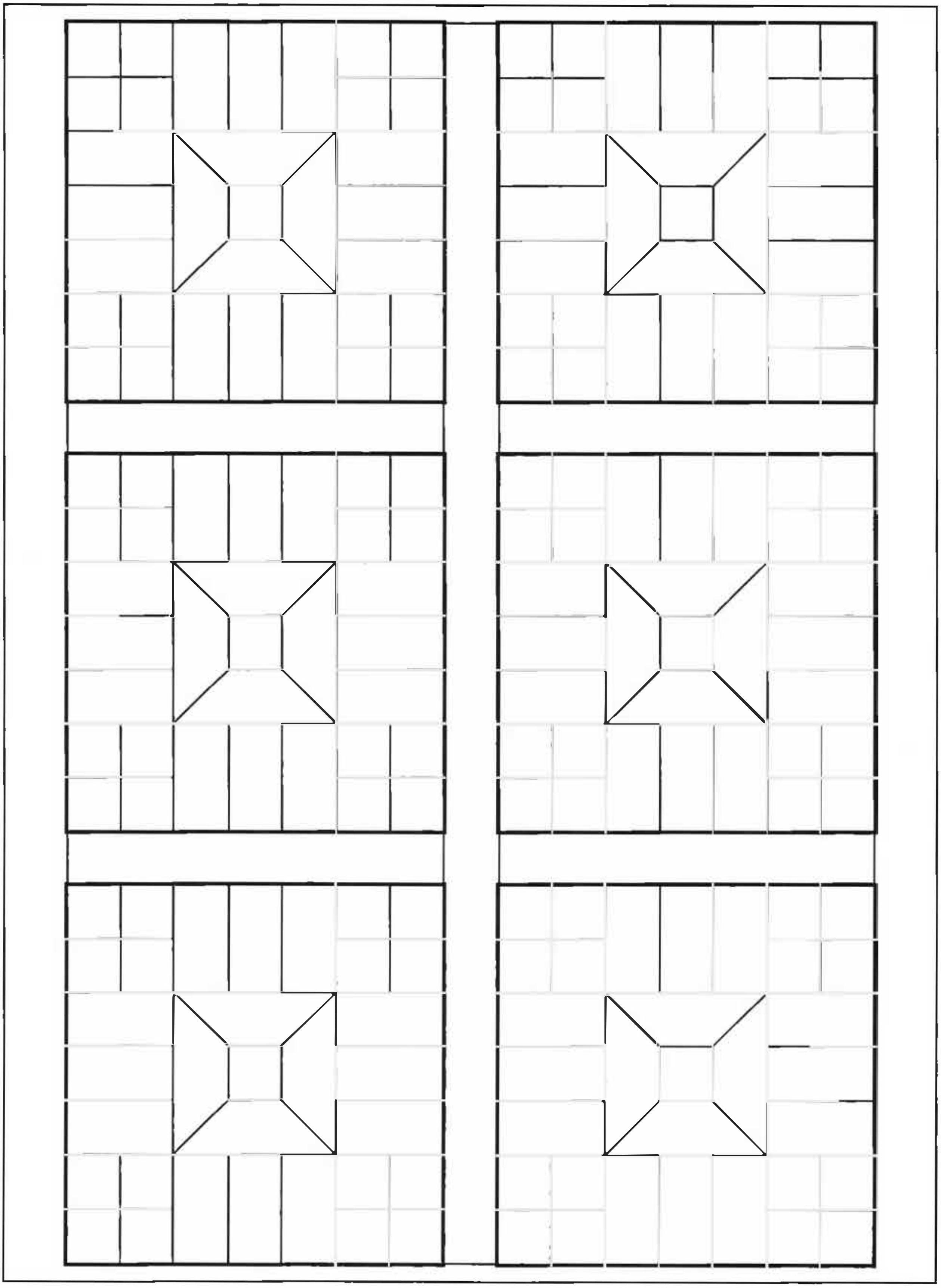
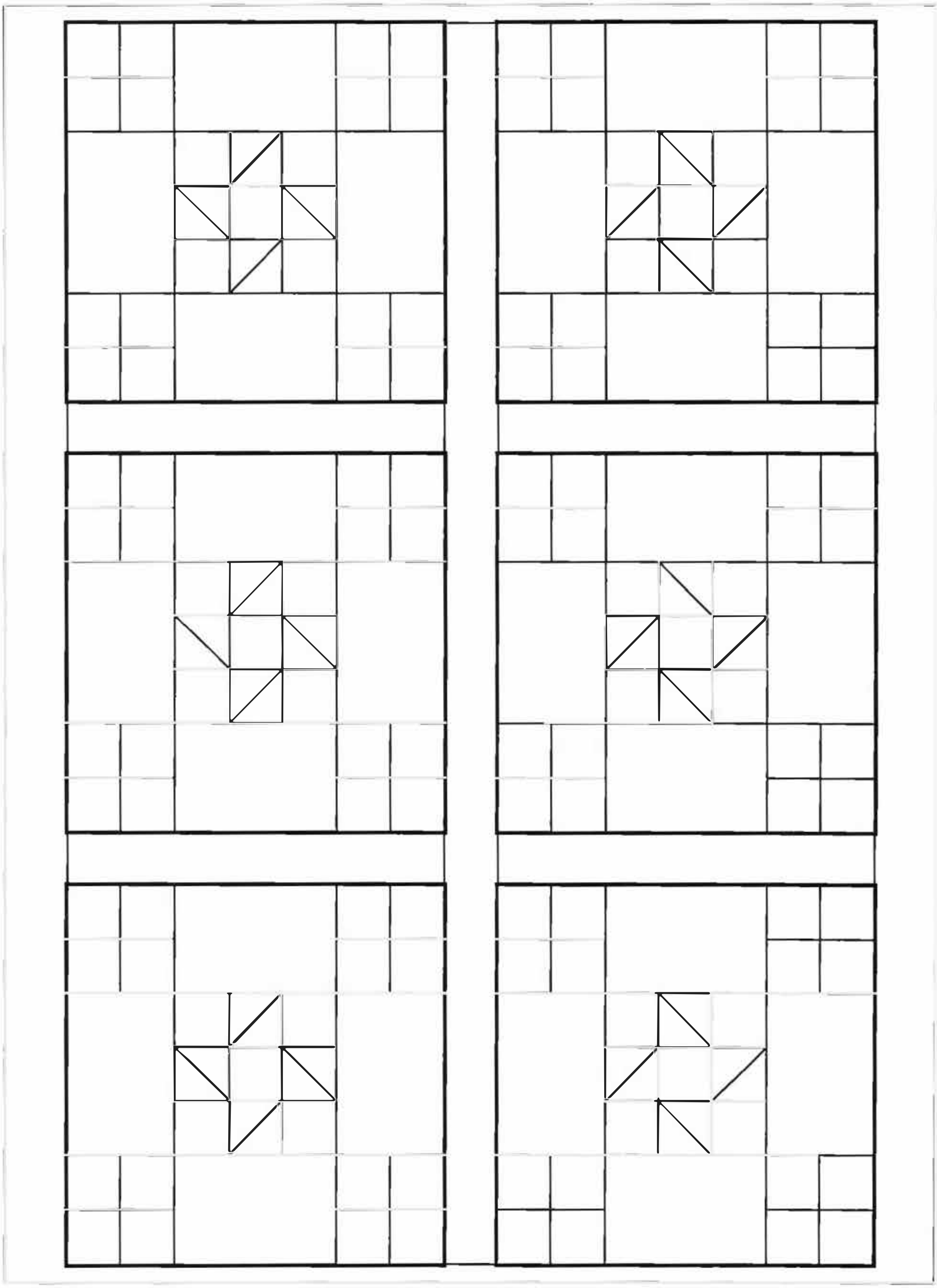






A Page of Problems

A. Craig Loewen, The University of Lethbridge

HIGH SCHOOL

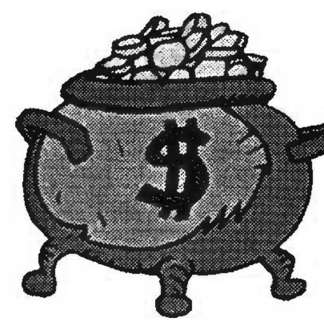
Find the smallest number that, when divided by each of the values 2, 3, 4, 5, 6, 7, 8, 9 and 10, will give, in each case, a remainder that is one less than the divisor.

Hint: restate the problem in your own words.

Source: Kantecki, C, and L. E. Yunker. "Problem Solving for the High School Mathematics Student." *Math Monograph* 7 (1982): 49–60.

JUNIOR HIGH

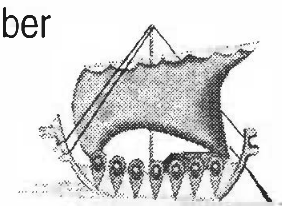
What is the least number of coins (the largest of which is a quarter) necessary to make any amount from 1¢ to 99¢?



Source: *Mathematics Teacher* 82, no. 8 (1989): 626.

ELEMENTARY

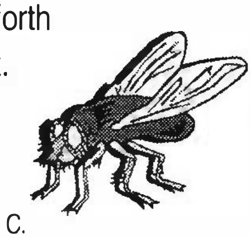
Three cannibals and three missionaries are on the same side of the river. They have one boat that can hold only two people. How can they all cross the river safely knowing that any time the cannibals outnumber the missionaries, the missionaries will be eaten?



Source: Adapted from http://partner.galileo.org/schools/strathmore/math/fair/prob5.html.

HIGH SCHOOL

Two cars each travelling at 60 kph at 2 km apart. They begin travelling toward each other at the same time. A very fast fly flies from the bumper of one car to the bumper of the other at 120 kph. As soon as it touches the bumper it turns around and heads back to the bumper of the first car. It continues back and forth until the two cars meet. How far does the fly travel?



Source: Adapted from Fowler, J. C. www2.spsu.edu/math/stinger/1-50/puz009.htm.

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