Emergent Insights into Mathematical Intelligence from Cognitive Science[1]

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In this article, I point to a handful of recent developments in cognitive science in an attempt to highlight how they might contribute to a rethinking of the nature of mathematical intelligence. In the process, I also offer some preliminary speculations on what these developments might mean for the teaching of mathematics.

I must begin with a disclaimer: Cognitive science is a burgeoning field. It is really only a half-century old, and it has just taken off in the last decade, spurred along by the invention of technologies that enable researchers to peer into brains in real time. Some surprising observations have been made—ones that have compelled researchers to question and reject an array of deeply entrenched assumptions about how people learn, how brains work, what thinking is and what intelligence is all about.

Cognitive science isn't actually a field. The phrase is an umbrella term that stretches across certain research in artificial intelligence, linguistics, cultural studies, philosophy, experimental psychology, neurology, neurophysiology, ecology, cybernetics and complexity science—to mention a handful of the more prominent areas. In brief, the emergence of cognitive science as a domain of research might be taken as recognition that investigations into such phenomena as learning and intelligence require a transdisciplinary approach. None of the above-mentioned fields on its own has the capacity to answer the big questions about human cognition.

With regard to education, this move toward transdisciplinarity is a significant development. For most of the past century, educators relied almost exclusively on psychology for their formal definitions of intelligence, the tools to measure it and advice on how to nurture it. As it turns out, much of that advice was good, despite some troublesome assumptions. But much of it was also a bit problematic. In particular, the reliance on psychology has contributed to some deeply ingrained and unfixable dichotomies—between, for example, skills-based and understanding-oriented instruction, or between teacher-centred and learner-centred instruction. Most of what we've borrowed from psychology compels us to take one side or the other, or to live with some uncomfortable compromise.

But, as John Dewey (1910) noted a century ago, we never solve such radical splits. We simply get over them. So none of what I present here should be taken as an argument for or against, for example, skills-based or student-centred instruction. Rather, I'm actually arguing that recent cognitive science provides us with a way of sidestepping these sorts of quagmires and opening spaces for more interesting and productive discussions.

Before going too much further, it's important to be clear about how cognitive science defines intelligence—and let me emphasize that this definition represents a break with popular and psychology-based orthodoxies. For instance, for the cognitive scientist, intelligence is not what IQ tests measure, as might be inferred from the fact that some patently unintelligent machines are able to perform at the genius-level on most IQ tests. As well, an individual's IQ score can vary by as much as 50 points, depending on the time of day, warm-up activities, hunger, thirst and so on.

Cognitive science uses a much broader definition: Intelligence is the capacity to respond to new situations in ways that are not only appropriate, but that open up new spaces of possibility. Intelligence, then, is not merely about getting the right answer to a trick question. It is about coming up with solutions to real problems, with answers that go beyond routine responses and that enable the person to go further than he or she could before taking on the problem.

Intelligence, in these terms, is about breeding new possibilities, opening up new vistas, not about responding to mind-twisters devised by others.

Point #1: Consciousness is small.

One solid, rigorously demonstrated conclusion of the research out of 19th- and 20th-century psychology was that human intelligence is greatly constrained by some rather severe biological limitations on consciousness. In particular, a frequently cited factoid is that humans are capable of juggling a maximum of 6 or 7 details in their heads at a time, but can only do that for about 15 seconds before some or all fall away. This 6–7 limitation is especially interesting when considered against the total number of sensory receptors in an average human body, which is estimated to be somewhere in the 10 to 20 million range. (Some researchers contest that the total is in the order of 1010; see [Norretranders 1998].) To drive that point home, fewer than one in every million sensory events (and the number may be closer to one in a billion events) ever rises to consciousness.

This insight is actually an old one, thoroughly demonstrated in the 1800s. It was a key tenet in the emergence of discourses as diverse and incompatible as B. Skinner's behaviourist psychology and Sigmund Freud's psychoanalysis, both of which were under development about a century ago.

A brief demonstration might be useful here. First read the following instruction, then follow it. Close your eyes and imagine two dots, then three, then four, then five, then 20, then 100.

Chances are that your image of three was arranged in a triangle, that your four was a square, your five was either a pentagon or a square with a dot in the middle. You shouldn't have been able to imagine 20 or 100, but you might have invoked a strategy like a grid to think of these quantities in terms of smaller, more readily imagined amounts.

Now repeat the tasks, this time with all of the imagined objects in a single row—no grids, polygons or subgroupings allowed. You will likely max out at five. I know of no one who can imagine six side-by-side, ungrouped objects.

There is some compelling evidence that the capacity to imagine small quantities might actually be built in. It's been established that very young babies can discern between one object and two objects, likely between two and three, and perhaps between higher quantities (see Gopnik, Meltzoff and Kuhl 1999). It also seems that we share that ability with lots of mammals, some birds and a few other species.

The realization that consciousness is so tremendously limited is one of the principles that undergirds the highly parsed structure of modern curricula, especially mathematics curricula, which have been the subject of more psychologically based research than any other topic area. (In fact, math curricula have been the focus of more research than all the other areas combined.) The practice of structuring a lesson around one small topic, such as adding integers, long division or factoring a trinomial, originated in part from the embrace of the factory model of schooling, but the bolt that holds it in place is research into the limitations of consciousness.

In fact, that research is so compelling that I have structured this article around it. My psychologist colleagues tell me that the best I can hope for is that you'll retain at most six or seven bits of information. So I've limited my foci to seven points.

Before moving on to the second of those seven, I want to nod to a few implications of this first point for our efforts to nurture mathematical intelligence. Two implications:

- We have to limit the amount of new information in any given learning event.
- We have to use design learning in ways that help learners focus their attention on what really matters.

We've already mastered the first point. The second one is a little more complex than it might appear.

There is a connection between intelligence and discernment. In fact, intelligence was originally conceived as the capacity to discern what is really important in a situation. As it turns out, there are teaching strategies that can support people's discernment-making abilities—that is, that help them be intelligent.

Anne Watson of Oxford University and her husband John Mason of the Open University in the United Kingdom have done considerable work on this issue. An example based on their work is the following:

Compare the two lists here:

$3 : 3$
$1.7 : 1.7$
$x : x$
$e^{\pi i} : e^{\pi i}$

and

$3 : 3$
$6 / 4$
2 to 9
$\frac{1.2n}{0.36n^2}$

The point Anne intends through this sort of comparison may seem counterintuitive. She argues that the first list might be a better pedagogical tool because it is designed to assist the learner to make a key mathematical discernment. In contrast, the second list obscures the discernment. Too much is going on. Her argument is that if there is not much variety, we generalize. If there is too much variety, we categorize. And for the most part, the intelligent mathematical action is about making the sorts of discernments that enable generalization, not categorization.

The first list sets itself up for questions like, What's the same? What's different? Is it always, sometimes, or never true? Are there examples that don't fit the pattern? In other words, even though it might look like there is less there, it's much easier to strike up a conversation about what is presented—that is, to pinpoint and emphasize what really matters.

The fact that consciousness is limited also points to the need for repetition and practice, which is something that traditional mathematics teaching has done well and that reform teaching has often done less well. Let me underscore this point.

Point #2: Intelligence relies on the capacity to routinize knowledge and procedures so that consciousness is freed up to work on other tasks.

Consider this sequence of numbers:

1, 11, 21, 1211, 111 221, 312 211

What comes next?

The following discussion will be more meaningful if you actually try to respond to the question.

When you first take on this sort of problem, your brain activity spikes and continues to do so until you either find a solution or give up on it. If you do in fact come up to a solution, your brain very quickly works to routinize things by delegating the task to a subregions or clusters of subregions while the rest of the brain returns to its usual near-resting state.

The realization of the importance of routinization for intelligence is quite a recent development. Or, at least, the proof for it is recent. Now that we can watch the brain in action, we can see that brains respond in different ways to novel situations. When presented with an unfamiliar problem or context, all brains begin to fire rapidly. And the whole brain fires when it meets a novel problem, not just parts of it (see Calvin 1996). I'll return to this point later.

The quality that most distinguishes the intelligent brain from the unintelligent brain is that it quickly settles on what's important, routinizes it and assigns it to subconscious processes. So, in terms of the profile, there's an initial spike of whole-brain activity that settles very quickly into lower-level, region-specific activity. By contrast, the unintelligent brain continues at a high level of whole-brain activation, apparently groping for what's important.

The happy thing is that the brain can improve its abilities to make vital discernments. One key is practice. Let me tell you a quick story.

Each week for the past three years, I've been meeting with Krista, an adolescent, about her mathematics. When I first met her, she was in Grade 9 and was unable to see patterns in lists of numbers like

1 4 9 16 25 36 49 64 ...

1 1 2 3 5 8 13 21 ...

It didn't take much probing to discover that a large part of the problem lay in the fact that she couldn't work with even single-digit numbers reliably. Calculations like $6 + 7$ and $5 \cdot 3$ were problems for her.

This meant that she was failing mathematics badly, and had been doing so since Grade 1. The school board had been testing her annually and she had had at least eight years of focused help with special needs teachers. Yet in Grade 9 she couldn't do things that are routinely expected of children in Grades 2 and 3. I decided to work with her because I thought she might be one of those interesting cases of people with location-specific brain injuries, which I imagined could be a fascinating thing to study from the point of view of an educational researcher. It turned out that I was quite mistaken in this suspicion.

The first year of our association was spent on what I thought of as educating her intuition—a phrase that refers to engagement with processes and situations intended to help one develop a feeling for quantities and manipulations of quantities. For instance, we spent a total of about six hours (a month's work together) figuring out different ways to estimate the number of grains of rice in a bowl. We spent considerably more time on paper-based activities, such as folding, cutting, assembling and dismantling. We did anything I could think of that might be interpreted in terms of basic operations on whole numbers, integers and rationals.

Significantly, I insisted on practice. Krista had daily homework exercises, which included flashcard drill on multiplication facts, writing out explanations of why things seem to work how they work, spending time on non-routine problems and so on. Six months into our work together, the psychometrician who had worked with her for three years was surprised to note that her score on the mathematics portion of the test he used had soared from Grade 2.3 (at the end of Grade 8) to Grade 10.8 (in the last half of Grade 9).

I cite those statistics cautiously. Krista really was not working at a Grade 10.8 level. (I had no access to the test, so I cannot comment on what was really being assessed.) But the numbers do suggest that something important had happened. At the time of this writing, she is enrolled in the Grade 12 applied stream mathematics course. Her average in mathematics is consistently in the 80 per cent-range. That's gratifying, but what is really exciting is that while she's writing an exam she can now tell whether or not she's doing well. Two years ago, she couldn't tell you what sort of grade she might get on a test. If she passed (which was not often), she attributed it to luck. Now she can predict her score with a high degree of accuracy.

I recently asked her about her new capacity to predict her exam results and how she could feel so sure of her predictions. She responded that a few years ago, her brain would "just go crazy in math exams." She couldn't focus, she couldn't remember. Now, in her exact words, her "brain just goes calm" when she realizes she can respond to the questions.

I haven't had a chance to monitor her brain activity, but I'm fairly confident in the assertion that two years ago, in a test situation, her brain was spiking throughout the test, to no avail. Now, it's spiking and settling in—just as an intelligent brain is supposed to do.

As for teaching implications, a central point is one that we all know deeply—if we want to be proficient in an activity, much of it has to be routinized. Be it playing hockey, playing the piano or adding fractions, certain levels of practice are needed not only to develop the basic mechanical competencies but to get a feel for what one is doing.

There is one caveat here. Practice must be contextualized. The brain resists learnings that lack context or that are not anchored in purposeful activity for reasons that I will develop later. But first, I want to make one more point on the role of practice.

Point #3: Mathematical genius (in fact, any category of genius) is, in general, much more about focus and practice than it is about innate, biologically rooted talents or gifts.

Rena Upitis of Queen's University often asks audiences to do the following: Think about something you're really, really good at. Now answer two questions: Do you practise it? And did you learn it at school?

You probably said yes to the first and no to the second.

The fact of the matter is that talent and genius are dependent on practice. So long as the basic biology of the brain isn't compromised, an otherwise typical person can obsess his or her way toward genius in some domain of activity because the brain is what neurologist and psychologist Merlin Donald (2001) describes as a "superplastic structure" whose resources can be co-opted and reassigned through dedicated practice. If those resources are focused on mathematics—or golf, or the cello, or plumbing—otherwise ordinary individuals can achieve quite extraordinary feats after years of focused effort.

One interesting statistic in this regard is that the rates of mental illness, particularly obsessive compulsive disorders, are several times higher among elite mathematicians, musicians, athletes and other high performers (Richardson 1999). This is not to say that obsession is a good thing; it is merely to underscore that, biologically speaking, most of the super geniuses of the world began life with capacities that were very similar to the ones the rest of us were born with.

I'm not suggesting that people are all born with the same cognitive architectures or that there's no such thing as natural mathematical talent. Clearly, such notions are misguided. The point is that most of the differences that we observe among adults have more to do with habits of mind than with raw horsepower. A person who begins with typical ability but who is obsessive about mathematical concepts can be a much better mathematician than a person with considerable natural ability but no inclination to develop his or her own capacities.

I return to Krista here. Two years ago, she was mathematically inept. She is far from a mathematics genius, but she is now mathematically capable. And just being capable means that her mathematical intelligence has skyrocketed.

The claim here is that one can become more intelligent, and it is an assertion that flies in the face of some deeply engrained beliefs and practices. IQ tests, for instance, are developed around the assumption that something innate is being measured, not something that can be honed through practice. Howard Gardner's theory of multiple intelligences is anchored in the assumption that differences in human capacities for mathematics, interpersonal relations, music and so on are all rooted in variations among inborn brain structures. And we are confronted with tale after tale based on the assumption—and that advances the belief—that mathematical talent is innate. Consider the popular Hollywood films *Good Will Hunting, Little Man Tate, A Beautiful Mind.* The implication in these stories often seems to be that education is supposed to stay out of the way of a genius.

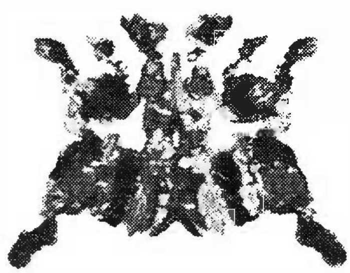
But the fact of the matter is that there are no documented cases, anytime or anywhere, of a full-blown mathematical genius who became that way without extensive practice and some formal education. It simply doesn't happen. By contrast, there is no shortage of evidence to support that assertion that mathematical intelligence is not fixed. We can make ourselves smarter.

Teachers can play an important role here. Emotions like curiosity and pleasure can be infectious. In fact, all emotions are. We humans are prone to being caught up in others' emotional expressions. So it's worthwhile asking yourself what emotions are you expressing in your classroom toward the mathematics? Enthusiasm? Indifference? Amusement? Obsession? (See [Damasio 1994] for a discussion of the relationship between emotion and logical competence.)

On the issue of making ourselves and our students smarter, it turns out that there are critical moments in life for nurturing one's intelligence.

Point #4: Brains are constantly changing—and they change most rapidly in the first few years of life and during early adolescence.

What do you see in the inkblot below?



Now, what if I tell you that this is actually a picture of two people squatting back-to-back holding ducks in their laps?

Once you apply this interpretation into the image, you can't help but see what you were told to see.

In other words, I have affected your brain structures by imposing a specific interpretation. That interpretation is compelling because your brain immediately went to work to activate the associations necessary for you to perceive the image as described. That is, your brain is physically different because of my intervention. Every lived experience entails a physical transformation of your brain.

Now consider such common turns-of-phrase as "taking things in," "attaining one's personal potential" and "brain as computer." We have dozens of such expressions, all of which assume and assert a fixed brain architecture—as though the brain were some kind of preset and unchanging receptacle. Nothing could be further from the truth. Some details:

1. Brains account for about 5 per cent of the body's weight, but consume about 20 per cent of the body's energy. In other words, they're incredibly physically active, and when I say physical, I mean physical. Things are actually moving about up there. On an MRI the brain looks vastly more like an anthill than it does a computer.
2. Infant and adolescent brains operate in overdrive, consuming two to three times as much energy as a typical adult brain. The claim has been made that if only a three-year-old could have an adult's knowledge and experience, all of the great problems of science would be solved in short order (Gopnik, Meltzoff and Kuhl 1999). They're geniuses. Super geniuses. And we might expect as much. They have to develop language, put together a theory of how the world works and master the complexities of interpersonal relationships in just a few years. None of us adults can do that.

   Until about five years ago, it was believed that brain activity undergoes a gradual and steady decline from toddlerhood to adulthood. But some recent research has demonstrated that there's a renewed surge of brain activity in early adolescence, especially around junior high age. They're geniuses again.

   Some, including Pinker (1997), theorize that this second surge in brain growth and activity is an evolutionary response to the need to cope with some new and fairly significant distractions. Whether or not that's the case, it would seem to make sense to take advantage of their amplified cognitive powers.
3. One of the differences between intelligent brains and not-so-intelligent brains is the density of neurons. Einstein's brain is pretty normal in size. There are no odd bulgy areas. However Einstein's neurons were more tightly packed and more intricately interconnected than typical brains.

   It turns out that neuronal interconnections can be grown. In fact, whole new neurons can be grown. These things happen in response to experience and need. As Canadian neurologist Donald Hebb (1949) wrote 50 years ago, "Neurons that fire together, wire together." A key here is, once again, contextualized and rich practice.

Considered together, the above points underscore an important conclusion: Your brain, at this moment, is different from the brain that you had when you started reading this article. Every experience you have contributes to the ongoing restructuring of the brain. Put in somewhat different terms, the brain isn't hardware and knowledge isn't data or information. These popular and pervasive ways of talking about learning and knowledge are way, way off.

In terms of implications for teaching, the sorts of things that contribute to increased neural density and interconnectivity are the sorts of things that force learners to think outside the box. Such activities include sustained engagements with mathematical puzzles, attending to the different ways that concepts can be interpreted and doing things that are unfamiliar and nonroutine. In particular, for a learner to develop mathematical intelligence and robust mathematical understandings, she or he has to be aware of how mathematical concepts can be interpreted in different ways. I turn to an example of this presently.

Point #5: Human thought and learning are mainly associative not rational—that is, analogical, not logical. Mathematical intelligence and creativity are rooted in the capacity to select and blend appropriate associations.

What is multiplication?

It turns out that this question has at least a dozen distinct responses, all of which are correct. In a recent workshop with a group of K–12 teachers, the following list was generated:

- Repeated addition: $2 \times 3 = 3 + 3$ or $2 + 2 + 2$
- Grouping process: 2×3 means "2 groups of 3"
- Sequential folds: 2×3 can refer to the action of folding a page in two and then folding the result in 3
- Many-layered (the literal meaning of *multiply*): 2×3 means "2 layers, each of which contains 3 layers"
- Grid-generating: 2×3 gives you 2 rows of 3 or 2 columns of 3
- Dimension-changing: a two-dimensional rectangle of area 6 units^2 can be formed when one-dimensional segments of lengths 2 units and 3 units are placed at right angles to one another
- Number-line–stretching or -compressing; $2 \times 3 = 6$ means that "2 corresponds to 6 if a number-line is compressed by a factor of 3"
- Rotating: for example, multiplication by -1 means rotate the number line by 180°—which reverses its direction

This list is far from exhaustive. It could easily be extended to include interpretations that are needed to make sense of the multiplication of vectors, matrices and other familiar mathematical objects.

It's important to emphasize that all of these interpretations point to distinct actions. They can be mapped onto one another, but they cannot be reduced to one another. And it's important that they're distinct. The power of mathematical processes like multiplication is not that they can be reduced to a single definition or process, but that they actually consist of clusters of interpretations.

There are some major teaching implications here. For most of the past four centuries, school mathematics has been organized around the assumption that mathematical learning proceeds logically and sequentially, like the construction of a building. Think of some of the metaphors that tend to be used: solid foundations, the basics, a cornerstone of logic, the structure of knowledge, and building and constructing ideas.

There is a popular assumption that the history of mathematics unfolded logically and sequentially as well. Nothing could be further from the truth. The more recent histories of mathematics underscore this point (for example, Mlodinow 2001; Seife 2000). The great leaps in the emergence of mathematical knowledge didn't occur through moments of logical insight, but through the development of new analogies. The concept of multiplication, for instance, has grown over the centuries as new interpretations have been proposed and blended into the existing definition (see Lakoff and Núñez 2000; Mazur 2003).

What does this mean for mathematical intelligence?

Let me preface my answer to that question with a quick visit to the field of artificial intelligence (AI) research. AI started in the 1950s when computers were beginning to outperform their programmers on some difficult mathematical tasks. Based on this early success, computer scientists and science fiction writers began to make confident predictions about the future of machine intelligence, forecasting that electronic intellects would soon dwarf flesh-based intellects.

Fifty years later, we see that they were spectacularly wrong. The reason for the collective error is instructive: They assumed—as did the original IQ-test inventors, many curriculum designers and writers of *Star Trek*—that logic is the root of intelligence. The belief was supported by their own experiences. Like most people, they found logical tasks very difficult.

And there is a reason why they're difficult—it's because our brains are analogical. That is, the root of intelligence is not logic, but the capacity to make new associations among experiences—through storying, analogy, metaphor and other figurative devices. Ours is an intelligence that is capable of logic, but that capacity rides on top of very different sorts of competencies.

There's a rather shocking implication here—our current mathematics curriculum might be stifling mathematical intelligence, not supporting it, an assertion that might be linked to Point #2. Brains resist decontextualized, overly abstract constructs. When the brain meets something new, it works very hard to weave the experience into the web of existing associations. But if the new topic comes without obvious associations, then it can't be learned on any level other than the mechanical. But human brains are notoriously unreliable when it comes to rigidly procedural knowledge.

One of the major implications for teaching is something that we can't do much about at the moment. Mathematics curricula are structured after the model of the logical proof. You begin by developing the premises or basics and proceed by assembling those premises into more sophisticated truths. In terms of the analogical nature of human cognition—and, in fact, in terms of the emergence of mathematical knowledge—this instructional sequence amounts to putting the cart before the horse. Logical justification has always come after the development of a new way of interpreting things.

Speaking of the model of formal logic, did you know that Euclid's five axioms aren't sufficient for his geometry? He missed some necessary axioms because he was thinking analogically, not logically. About a century ago, David Hilbert (1988/1899) identified several others that are needed for Euclid to be logically complete. It took more than two millennia for mathematicians to notice the gap. Why? Because humans are much more analogical than logical.

But, of course, we can't wait for full-scale curriculum restructuring. In the meantime, to nurture your students' mathematical intelligence, I recommend that you work with them to try to uncover the associations that have been built into mathematical concepts. Start with addition. What are some of the ways we interpret adding? (If you want one answer to that question, you might check Lakoff and Núñez 2000.)

Let me re-emphasize that robust understandings and flexible applications of mathematical ideas—that is, the underpinnings of mathematical intelligence—are completely dependent on access to the range of meanings that are knitted together in a concept.

Point #6: The real power of mathematics arises in cleverly structured symbolic tools, which collect together but conceal the arrays of interpretations and experiences that underlie concepts.

Close your eyes and imagine $\sqrt{-15}$.

It's not so easy. And yet, as it turns out, $\sqrt{-15}$ is utterly imaginable. Barry Mazur, a Harvard University mathematician, explains how in his 2003 book *Imagining Numbers*. Space prohibits an adequate summary of his discussion, but I can mention that to imagine $\sqrt{-15}$, you have to know that the concept relies on the notion of multiplication-as-rotation. That is, multiplication by a negative means a 180°-rotation and multiplication by two negatives means a 360°-rotation (which takes you back to the starting orientation). One more detail is needed: one might think of a square root as half of a multiplication, as indicated by the exponent of 1/2. If you blend these ideas—as mathematicians did a few centuries ago—you get the root of a negative is a half of a 180°-rotation, which is a 90°-rotation, which generates the complex plane. The roots of –15, then, are the points that are just beneath the +4 and just above the –4 on the i axis of the complex plane.

Lakoff and Núñez (2000) take this sort of thinking even further and demonstrate how it's possible to imagine Euler's formula: $e^{\pi i} + 1 = 0$. Even more significantly, they attempt to impress that this very complex notion is rooted in bodily action, like moving forward, spinning and so on. (See Elizabeth Mowat's article in this issue for a fuller discussion of Lakoff and Núñez.[2])

My point here is not really that such imaginings are doable nor that we should be doing them in our math classes—although I do believe that they are doable and that we should be doing them in our math classes. It is, rather, that knitted into these symbols are an incredible array of experiences and possibilities. They are intelligently designed tools that greatly expand what we are able to do.

To put a finer point on it, tools such as language, mathematical symbols, and calculators aren't just the product of human intelligence—they are bestowers of intelligence. Humans with language are much more intelligent than humans without language. And, although I don't have nearly the raw intelligence of Archimedes or Newton or other mathematical giants of history, I can do things that they didn't even imagine doing because of the tools they helped to build.

Now, by psychologistic definitions of intelligence, you might argue, the fact that I can solve an unsolved differential equation by typing it into Maple does not make me a mathematical genius. And according to measures of IQ, that's true. But going by the cognitive science definition of intelligence (that is, intelligence is the capacity to respond to new situations in ways that are not only appropriate, but that open up new spaces of possibility), intelligence is about an ever-growing horizon of possibility, not the capacity to master what's already been established. What's more, intelligence is obviously not an individual phenomenon. Not only can we make ourselves smarter, we can contribute to the intelligence of others by giving them access to the tools of our intelligence. On this point, it's important to emphasize that we're routinely asking high school students to perform mathematical operations that were accessible only to the geniuses of a few centuries ago.

Now, to be clear, I'm not suggesting that technology on its own makes us smarter. Giving an iMac to a caveman would be a bit of a waste. And we have probably all seen people grab a calculator in order to add 0 or to multiply by 1. Those are decidedly unintelligent acts.

The point is, rather, that intelligence is not some mysterious quantity that's locked in our heads. Intelligence is about appropriate and innovative action, and to be intelligent in mathematics in this day and age requires more than a mastery of the conceptual tools that have been developed by our forebears. Intelligence is greatly enabled by a facility with contemporary tools. That's certainly true among research mathematicians. Our mathematics pedagogy hasn't adapted to take that into account, even though electronic technologies have contributed to dramatic reshaping of the landscape of mathematics research. We have to think about ways of incorporating these technologies to amplify possibilities, not just to brush aside tedious calculations as we cling to a curriculum that hasn't much changed in 400 years.

Point #7: The clinically based research that supports point #1 is flawed, and the flaws are instructive.

Most of the consciousness research that was conducted through the 20th century was undertaken in laboratories. And it turns out that if we isolate people in a room without any of the tools we use to extend intelligence, their conscious capacities will turn out to be not just amazingly limited, but amazingly equal, whether they are Nobel Prize laureates with something to prove or six-year-old brats who couldn't care less.

Now, it seems to me that this fact should have prompted curriculum developers to hesitate a little before structuring programs of study around the limitations of consciousness by parsing up concepts into small, 45-minute–lesson-sized concepts. But it didn't. It seems that no one thought to ask what it might be that enables some people, with essentially the same conscious capacities, to achieve such remarkable feats. Inborn ability is certainly part of a factor, but the range of inborn abilities is simply too limited to explain the variations in achievement that we see. Obsession is a huge factor, too, but we all know that obsessing about something doesn't necessarily lead to great insight.

A major clue into the difference between ordinary and extraordinary performances has emerged over the past few decades, as we've developed the technical abilities to study humans in contexts that are a bit more natural than the laboratory setting. Some surprising things have been shown. One of them is that humans have the capacity to "couple their consciousnesses" (Donald 2001); that is, to link their minds, to coordinate the rhythms and cycles of their brains' activities. In the process, they can form grander cognitive unities. One common sort of coupled consciousnesses is a "conversation."

It turns out that, in the context of a conversation, humans are able to collectively juggle not 7 ideas, nor 7 + 7 ideas, but more in the order of 7 × 7 ideas. And some of those ideas can endure not for 10 or 15 seconds, but for minutes and hours.

This point is critical to the production of mathematical knowledge. The image of the focused and still mathematician labouring alone in a locked chamber is not at all representative of how research mathematicians work. There may be moments when they're on their own, but like anyone in any domain who is concerned with the development of new insights, they surround themselves with others and others' ideas. No mathematician is an island.

Elaine Simmt, also of the University of Alberta, and I have been trying to understand the sorts of collective structures that support the work of mathematicians. Drawing from complexity science (see, for example, Kelly 1994), we have identified a handful of conditions that are common to such intelligent collectives (see Davis and Simmt 2003). This thinking is still in its infancy, but I can report briefly on what is involved in prompting the emergence of an intelligent collective in the classroom—a collective that, in turn, supports the development of each individual's mathematical intelligence.

Over the past 20 years, complexity scientists have been labouring to identify the sorts of conditions that enable the emergence of complex systems—how, for example, ants interact to form anthills, species couple together within ecosystems, cells knit themselves into organs, and organs into individuals, and individuals into societies and so on. Among the necessary conditions for these happenings, the following six seem to have a particular relevance to the work of the mathematics teacher:

- Internal Diversity—Internal diversity refers to the pool of possibilities that a system has to choose from when it's faced with a novel circumstance. It is the basis of the collective's intelligence. A system in which all of the components are expected to do the same thing at the same time will not be an intelligent one.
- Internal Redundancy—That being said, it's important that the agents in a system have enough in common to be able to work together, whether talking cells, birds, people or social systems. Redundancy is also necessary for a robust system. If one agent fails, another can step in.

Some redundancies among participants in a collective have a lot to do with actions and competencies that are automatized. This is where traditional mathematics teaching has focused. The only way that a system's diversity can be a source of intelligence is if its agents are sufficiently alike for the bit of diversity to be embraced and elaborated.

- Neighbour Interactions—This condition might seem ridiculously obvious. Of course the agents in a system need to interact if that system is to become a system.

  But in the context of the classroom, the agents that need to interact aren't necessarily people. They can also be ideas or interpretations. As already mentioned, mathematics knowledge emerges as new ideas are blended with old ones. These blendings open up spaces for more powerful notions. So the phrase *neighbour interactions* doesn't refer to pod seating or group work, but to ensuring there is a sufficient density of diverse thought represented for the possibility of new ideas, as in the example of the varied interpretations of multiplication.

- Liberating Constraints—Consider these three tasks:
  1) Write down all that you know about three fourths.
  2) Write down two fractions equal to three fourths.
  3) Write down three things that you know about three fourths.

  In most classroom contexts, the first of these is much too broad to generate much that is interesting. The second suffers from being much too narrow, but has the same result—it likely won't generate much that's interesting either. But the third, like Baby Bear's porridge and bed, might be just right. It's open enough to allow for diverse possibilities, but sufficiently constrained to ensure that ideas won't be too diverse to prevent them from working together. (Of course, whether it is suitable depends on the collective.)

  Complex systems have to maintain this delicate balance between so much structure that they lock into place and so little structure that they decay into chaos. And the tasks that you set will determine whether or not intelligent—once again, appropriate and innovative—action can emerge.

- Organized Randomness—With a complex system, there's always a bit of randomness. Some of that randomness is ignored by the system—which is to say, it doesn't really affect what the system does. Other bits of randomness come to be really important—the unexpected observation, the sudden insight, the fact that this student's father is a painter and he knows the world doesn't work the way the question about ratios says it should work. Really intelligent systems, it seems, take advantage of more of these random events, and they're able to do so because they have strategies to organize such events.

- Decentralized Control—One of the big changes at Microsoft, Apple, IBM, Hewlett Packard and other locations of cutting-edge knowledge production has been an abandonment of the top-down model of centralized management in favour of a more distributed sort of control. Intelligent collective action can't be orchestrated into existence, at either the individual or the collective level. Space to negotiate the parameters and possibilities is needed.

All this being said, we have a long way to go before we'll be able to give much more direct advice on how to nurture mathematical intelligence. However, we can be quite specific about the opposite—on how to militate against the emergence of intelligent action. For instance, if diversity (among interpretations and among people) is suppressed, if ideas aren't plentiful and not permitted to bump against one another, if tasks are too open or too narrow, if control of the outcomes is strictly in the hands of the teacher, then chances are that intelligence will be stifled—intelligence of not just the collective, but of the individuals in the collective.

Notes

1. Some of the research data reported in this article are drawn from studies supported by the Social Sciences and Humanities Research Council of Canada. The article itself is a modest revision of a presentation made at the NCTM Regional Conference in Edmonton on November 22, 2003.

2. The reviews of *Where Mathematics Comes From* have been varied, especially with regard to the issue of whether Lakoff and Núñez actually succeed in explaining the bodily basis of Euler's formula. Nevertheless, most reviewers have acknowledged that their discussion of the analogical substrate of our logical abilities is compelling and has significant implications for the teaching of mathematics.

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