Developing Algorithms for Fluency and Understanding: A Historical Perspective

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Recent reforms in mathematics have called for a decreased emphasis on pencil-and-paper computations (National Council of Teachers of Mathematics 2000; Alberta Learning 1997). However, a tension between fluency and mathematical understanding exists in many elementary classrooms as teachers grapple with the place of algorithms in the curriculum. A review of the historical development of addition and subtraction algorithms suggests that this tension is not a recent phenomenon. Moreover, many of the standard algorithms used in Canadian classrooms today do not represent the most efficient or pedagogically sound approach to adding and subtracting (Carroll and Porter 1998). By considering the historical development of algorithms, perhaps we can reframe the link between fluency and understanding.

Many of the algorithms used today can be traced to the work by Islamic mathematicians in the eighth century. Of particular importance was Muhammad ibn-Musa al-Khwarizmi, who presented a variety of algorithms in his book on addition and subtraction. In fact, our word algorithm is derived from his name. His book demonstrated the advantages of using a numeration system involving place value and a base of ten. Interestingly, the development of the Hindu-Arabic numeration system coincided with the creation of algorithms and increased mathematical activity in these societies. Barnett (1998, 76) suggests:

> Not only did each new success give rise to the possibility of yet more success in the future, but the newly developed algorithms also allowed mathematicians to concentrate their creative energy on more complicated problems without having to think about the earlier "steps." This suggests that the search for algorithms is—in a very real sense—the driving force of mathematical development.

Prior to the adoption of the Hindu-Arabic numeration system, western societies did not use algorithms for adding and subtracting. Mathematical development stalled there during the European Dark Ages. Finally, in the 12th century, Fibonacci introduced notations and algorithms of al-Khwarizmi to Western Europe. The "new math" created much controversy as mathematicians argued with abacus users about potential benefits and advantages of algorithms over calculating devices.

New algorithms that combined calculating devices and algorithms for adding and subtracting emerged in the 15th and 16th centuries. One such example is the use of reckoning on lines by merchants to determine customer purchases (Mason 1998). This algorithm uses manipulatives to reinforce ideas of place value, regrouping and trading equals for equals, thus appearing to deepen mathematical understandings.

Although new algorithms continued to emerge, the debate over their importance remained unresolved. This debate continued throughout much of the 16th century and is still reflected today in current discussions on the use of calculators and paper-and-pencil algorithms. Although controversial, the debate about the importance of algorithms sparked a resurgence of mathematical development. Indeed, the mathematical creations of Descartes, Fermat, Newton and Leibniz in the 17th century are still evident in today's lessons on analytic geometry and calculus. The connection between the creation of algorithms and mathematical development appears to be strong.

In the 19th and 20th centuries, algorithms for adding and subtracting became increasingly abstract. Emphasis was placed on memorizing the steps of the procedure and connections to manipulatives were diminished. Place value identifiers were dropped. Mathematics became disconnected from the physical world and focused on axiomatic structures of mathematicians (Jones and Coxford 1970). Gaining fluency through the use of algorithms seemed to become the primary goal of mathematics.

However, the tension between mathematical fluency and understanding continued. As arithmetic became an elementary school subject at the end of the 19th century, educators became increasingly concerned with student understanding. Jones and Coxford (1970, 32) write

> Mental discipline as a viable goal of education, and drill as a procedure, were retained along with other newer goals and processes for more than

thirty years after 1894, but the three step process of 'state a rule, give an example, practice' was yielding to inductive, reasoning, and discovery-teaching processes.

In the first textbook approved for use in Alberta, Kirkland and Scott (1895) address the problem of teaching rules by differentiating their approach from conventional textbooks: "The rule is given as a convenient summary of the methods employed in the solutions of the examples which precede it. The aim has been to lead the pupil to derive his own methods of operation" (p. iv). Algorithms for addition are presented with examples, place-value labels have been included, and teachers are encouraged to demonstrate regrouping using "bundles of splints bound together with India rubber bands" (p. 13). Several algorithms for subtraction are presented, including decomposition and equal additions. The student is not told which one to use and, presumably, alternative algorithms are acceptable.

Method A

368
+ 453
700
110
11
821

Begin with the hundreds, then the tens, then the ones. Record each sum. Add them together. This method is sometimes called partial sums (Reys et al. 2004).

Method B

300 + 60 + 8
+ 400 + 50 + 3
700 + 110 + 11
800 + 20 + 1
821

Write each number in expanded form. Add the hundreds, tens and ones. Regroup each to get the expanded form of the answer. Write the answer in standard form (Nova Scotia Department of Education 2002).

Method C

368 = 3 hundreds and 6 tens and 8 ones
+ 453 = 4 hundreds and 5 tens and 3 ones
7 hundreds and 11 tens and 11 ones
7 hundreds and 11 tens and (1 ten and 1 one)
7 hundreds and (11 tens and 1 ten) and 1 one
7 hundreds and 12 tens and 1 one
7 hundreds and (1 hundred and 2 tens) and 1one
(7 hundreds and 1 hundred) and 2 tens and 1 one
8 hundreds and 2 tens and 1 one
821

Write each number in expanded form using place value names. Add the hundreds, tens and ones. Regroup the ones if necessary. Put the tens together. Write in a simpler way. Regroup the tens if necessary. Put the hundreds together. Write in a simpler way. Write the answer in standard form (Lock 1996).

Method D

368	368	368	368
+ 453	+ 453	+ 453	+ 453
7	71	711	711
	8	82	821

Begin with the hundreds to get 7, then add the tens to get 11. Change the 7 to an 8 because of the additional 100. Add the ones to get 11. Change the 1 to a 2 because of the additional 10 (Nova Scotia Department of Education 2002).

Method E

	1	**11**	1
368	368	368	368
453	453	**7**	**4**
+ 128	**1**	453	453
	128	**2**	**8**
	9	128	128
		49	
			949

Begin by adding the ones of the first two numbers to get 11. Record this number by writing the ones digit and putting the tens digit above the tens column. Add the ones from the third number to this to get 9, recording the new ones number and putting the tens digit above the tens column if necessary. When the ones column is complete, repeat with the tens column and the hundreds column. This is known as Hutchings' low-stress algorithm (Lock 1996)

Perhaps the tension between fluency and mathematical understanding can be reframed. Instead of viewing them as ends of an either/or dichotomy, historical accounts suggest they are deeply interconnected. Barnett (1988, 77) cautions,

> We have encountered this same difficulty [decreased mathematical understanding] in the past when we allowed the current algorithms we teach to become an end in themselves. Our challenge as educators is to identify what is being learned from the algorithm (whether it be a traditional one or not) besides the ability simply to execute it.

Encouraging students to create alternative algorithms for adding and subtracting could strengthen the links between fluency and understanding. By connecting the development of algorithms to mathematical knowing, educators can begin to reconsider how addition and subtraction are taught and learned.

Alternative Algorithms for Adding

Presented below are several different methods for adding. These algorithms show step-by-step procedures for computing sums that have been used at some point in the history of mathematics education. I suggest that you familiarize yourself with the procedure by trying it out. Create new problems for yourself to develop your understanding. Think about why the procedure works.

Alternative Algorithms for Subtracting

Presented below are several different methods for subtracting. These algorithms show step-by-step procedures for computing differences that have been used at some point in the history of mathematics education. Again, I suggest that you familiarize yourself with the procedure by trying it out, creating new problems and thinking about why the procedure works.

Method A 81 – 25 56	Begin with the ones column. Subtract the second digit from the first. If necessary, regroup the tens of the first number and rename the ones by increasing the ones by ten and diminishing the tens by one. Repeat with the tens column. This method is known as the decomposition algorithm and is the standard algorithm used in Canadian classrooms.
Method B 81 – 25 56	Begin with the ones column. Subtract the second digit from the first. If necessary, add 10 to the top number and add 10 to the second number. Do this by increasing the ones by 10 in the first number and the tens by 1 in the second number. Repeat with the tens column. This method is known as the equal additions algorithm. It was taught in North America until the 1940s (Cathcart et al. 2003).
Method C 81 + 5 = 86 – 25 + 5 = 30 56	Add a number to the second number (subtrahend) to make it a multiple of 10. Add this same number to the first number (minuend). Subtract. This is known as subtraction by base complement additions (McCarthy 2002).
Method D 81 + 50 – 25 + 5 + 1 56	Consider how far apart 81 and 25 are. Begin from 25 and add, in steps, the numbers that bring you closer to 81. Record these numbers. When you reach 81, find the sum of these numbers. This will be the answer. This method is known as adding up (Carroll and Porter 1998).
Method E 81 – 25 60 – 4 56	Move left to right. Begin with the largest place value (in this case, 10s). Record the difference between the two numbers in each column. If the first number (subtrahend) is larger, the difference is recorded as a deficit or a negative number. Combine the partial differences (Carroll and Porter 1998).

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