Pi in All Its Glory

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As William Schaaf (1998) in *The Nature and History of Pi* remarked, "Probably no symbol in mathematics has evoked as much mystery, romanticism, misconception and human interest as the number π."

Humans lived for millions of years before the significance of π was grasped. Circles surrounded them in many forms other than the wheel, including the pupil of the eye and heavenly bodies like the sun and moon. But it was only after the appearance of organized society, approximately 2000 BC, that a relationship between the diameter of a circle and its area was recognized such that

circumference : diameter = constant for all circles.

An Egyptian scribe named Ahmes, circa 1650 BC, showed in the Rhind Papyrus that the ratio of the circumference to the radius equals 256/81 or 3.160493827—Ahmes's value was off by less than 1 per cent from the true value of pi. However this value did not become known because a thousand years later the Babylonians and early Hebrews simply used 3 for pi. In the Bible, both 1 Kings 7:23 and 2 Chronicles 4:2 contain the following verse: "Also he made a molten sea of ten cubits from brim to brim [the diameter], round in compass, ... and a line of thirty cubits did compass it round about."

In the fourth century BC, Antiphon and Bryson of Heraclea attempted to find the area of a circle using the principle of exhaustion. They took a hexagon, found its area and then continued to double its sides and double them again until the polygon almost became a circle. Antiphon first estimated the area of a circle by inscribing the polygon in a circle and then calculating the area as each successive polygon came closer to being a circle. Bryson calculated the area of two polygons, one inscribed in a circle and one circumscribed around a circle. The area of a circle would then fall between the areas of the two polygons.

Two hundred years later, Archimedes of Syracuse (287–212 BC) was the first mathematician to produce a method of calculating pi to any degree of accuracy. He doubled the sides of two hexagons four times, resulting in two 96-sided polygons. Using polygons inscribed and circumscribed in a circle, he obtained for pi the bounds

3 10/71 < pi < 3 1/7

or in decimal notation, 3.140845... < pi < 3.142857..., less than three ten-thousandths from the true value. This method of computing pi by using regular inscribed and circumscribed polygons is known as the classical method.

The next person of importance to take the pi challenge was the astronomer Claudius Ptolemy (AD 87–165) who used a 192-sided polygon. In his text *Megale Syntaxis tes Astronomias*, he stated that pi was 3° 8' 30" in the sexagesimal system, or 3 + 8/60 + 30/3,600 which is 3.14166667. His value of pi was within 0.003 per cent of the correct value.

The Chinese were considerably more advanced in arithmetical calculations than their western counterparts, because in AD 264 Lui Hui calculated the value of pi to be between 3.141024 and 3.142704 using the same method as Antiphon and Bryson. In the fifth century, Tsu Ch'ung-Chih and his son, Tsu Keng-Chih, used polygons with 24,576 sides (they began with a hexagon and doubled the sides 12 times) and determined that pi was approximately 355/133 which equals 3.1415929. This is only 8 millionths of 1 per cent from the real value of pi, a value not found in the western world until the 16th century.

About AD 530, the great Indian mathematician Aryabhata came up with an equation that he used to calculate the perimeter of a 384-sided polygon, finding it to be $\sqrt{9.8684} \approx 3.1414$.

Brahmagupta (598–670), another famous Indian mathematician, said that the value of pi was $\sqrt{10}$. First he calculated the perimeter of inscribed polygons with 12, 24, 48 and 96 sides and he got $\sqrt{9.65}$, $\sqrt{9.81}$, $\sqrt{9.86}$, $\sqrt{9.87}$. Then he thought that as the polygons approached the circle, the perimeter and therefore pi, would approach $\sqrt{10}$. Of course, he was quite wrong. He didn't see that his square roots were converging to a number significantly less than the square root of 10. In fact, the square of pi is just over 9.8696. Nevertheless, this was the value he expounded, and many mathematicians throughout the middle ages used it.

Since the middle of the first millennium, many other mathematicians came up with values of pi, but none of them was more accurate than the early Greek, Chinese and Indian calculations. In fact, it was not until the late 16th century that another significant step was taken.

In 1579, a French lawyer and mathematician, François Viète (1540–1603), used the Archimedean method of inscribed and circumscribed polygons to determine that $3.1415926535 < \pi < 3.1415926537$.

To achieve this, he doubled the sides of two hexagons 16 times and got two 393 216-sided polygons.

In 1593, he broke down his polygons into triangles and found that the ratio of perimeters between one regular polygon and a second polygon with twice the number of sides equalled the cosine θ. With this identity in hand, he used the half angle formula and found a way to describe π as an infinite product:

$$\frac{2}{\Pi} = \sqrt{\frac{1}{2}} \times \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}} \times \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}} \times \ldots$$

This was probably the first time anyone had used an infinite product to describe anything, and it was one of the first steps in the evolution of mathematics toward advanced trigonometry and calculus. However, even though the equation was a breakthrough, it was of little use when it came to actually calculating π because it was very complicated to perform the square root calculations.

Adriaen van Roomen (1561–1615), also known as Adrianus Romanus, a Dutch mathematician, calculated π correct to 15 decimal places by using an inscribed polygon that had over 100 million sides. Ludolfvan Ceulen (1540–1610), a German mathematician, calculated π to 20 decimal places, using the same classical method, but using polygons that had more than 32 billion sides. When he died in 1610, he had calculated 35 digits of π. In Germany today, π is still sometimes referred to as the Ludolfian number in his honour.

After van Ceulen, mathematicians came up with new ideas to calculate π more efficiently. In 1655, John Wallis (1616–1703) discovered a formula that, to this day, bears his name:

$$\frac{\Pi}{2} = \frac{2\times2\times4\times4\times6\times6\times8\ldots}{1\times3\times3\times5\times5\times7\times7\ldots}$$

Like Viete's, Wallis's equation is an infinite product, but it is different in that it only involves simple operations with no need for messy square roots. He reasoned that the first computed term would be higher than $\frac{\pi}{2}$, the second computed term would be lower than $\frac{\pi}{2}$. The third term would also be higher but closer than the first term. The fourth term would also be lower but closer than the second term and so on. This number would slowly converge to $\frac{\pi}{2}$.

In 1675, the Scottish mathematician James Gregory (1638–1675) obtained the extremely elegant infinite series:

$$\arctan x = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}-\frac{x^{11}}{11}+\ldots \quad -1\leq x\leq 1.$$

Three years later, the German Gottfried Willhelm Leibnitz (1646–1716) inserted $x = 1$ into the series to get:

$$\frac{\pi}{4} = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\ldots$$

This method of convergence was too slow to be put into practical use. It took more than 300 terms to even obtain π correct to two decimal places. (But although it took so very many terms, it was still faster than the old inscribing/circumscribing polygon method.)

Isaac Newton (1642–1727) improved on this tedious method using:

$$\arcsin x = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\cdot3}{2\cdot4}\cdot\frac{x^5}{5} + \ldots$$

Substituting $x = \frac{1}{2}$, giving $\arcsin \frac{1}{2} = \frac{\pi}{6}$, this series yields:

$$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{2}\cdot\frac{1}{3\cdot2^3} + \frac{1\cdot3}{2\cdot4}\cdot\frac{1}{5\cdot2^5} + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{1}{7\cdot2^7} + \ldots$$

In this series, calculating just four terms would yield $\pi = 3.1416$.

In 1706, John Machin (1680–1752) used the difference between two arctangents to find 100 digits of pi. He used

$$\frac{\pi}{4} = 4\cdot\arctan\frac{1}{5} - \arctan\frac{1}{239}.$$

This formula turned out to be quite useful, because $\arctan \frac{1}{5}$ is easy to calculate using Gregory's formula and $\arctan \frac{1}{239}$ converges very quickly.

In the middle of the 18th century, one of the greatest and most prolific mathematicians of all times, Leonhard Euler (1707–1783), found many arctangent formulas and infinite series to calculate pi. These formulas converged more quickly than those that came before. Some of his formulas were

$$\frac{\pi}{4} = 2\arctan\frac{1}{3} + \arctan\frac{1}{7}$$

$$\frac{\pi}{4} = 5\arctan\frac{1}{7} + 2\arctan\frac{3}{79}$$

$$\frac{\pi^2}{6} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots$$

$$\frac{\pi^3}{32} = \frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\ldots \quad 79$$

$$\frac{\pi}{2} = \frac{3\times5\times7\times11\times13\times17\times19\times23\times\ldots}{2\times6\times6\times10\times14\times18\times18\times22\times\ldots}$$

Euler also developed an equation that some believe to be among the most fascinating of all time:

$$e^{i\pi} + 1 = 0.$$

The irrationality of π was proven by Johann Heinrich Lambert (1728–1777) and Adrien-Marie Legendre (1752–1883). Lambert investigated certain continued fractions and proved the following:

If x is a rational number other than zero, then $\tan x$ cannot be rational.

From this, it immediately followed that: If $\tan x$ is rational, then x must be irrational or zero.

(For if it were not so, the original theorem would be contradicted.) Since $\tan\left(\frac{\pi}{4}\right) = 1$ is rational, $\left(\frac{\pi}{4}\right)$ must be irrational and the irrationality of π is established.

Legendre proved the irrationality of π more rigourously. He wrote:

> It is probable that the number π is not even contained among the algebraic irrationalities, i.e., that it cannot be the root of an algebraic equation with a finite number of terms whose coefficients are rational. But it seems very difficult to prove this strictly.

Legendre was correct on both counts; π is not algebraic, but transcendental. The equation

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 = 0,$$

where n is finite and all the coefficients are rational, is called algebraic. Numbers that were not merely irrational but that could not even be roots of an algebraic equation are transcendental. It was not at all obvious that such numbers exist.

With the arrival of the age of computers, came π calculated to an ever-increasing number of decimal places. In 1947, D. F. Ferguson calculated 808 decimal places for π. It took the computer one year to do that. In 1949, ENIAC (Electronic Numerical Integrator and Computer) computed 2,037 decimals of π in 70 hours. In 1955, NORC (Naval Ordinance Research Calculator) computed 3,089 decimals in 13 minutes.

In 1959, in Paris, an IBM 704 computed 16,167 decimals of π. Three years later, John Wrench and Daniel Shanks used an IBM 7090 to find 100,265 decimals. In 1966, in Paris, an IBM 7030 computed 250,000 decimal places of π. In 1967, a CDC 6600, in Paris, computed 500,000 decimals. In 1973, Jean Guilloud and Martine Bouyer used a CDC 7600, in Paris, to compute one million decimals in less than one day.

In 1983, Y. Tamura and Y. Kanada used a HITAC M-280H to compute 16 million decimals of π in less than 30 hours. In 1988, Kanada computed 201,326,000 digits in six hours on a Hitachi S-820. In 1989, the Chudnovsky brothers found 1 billion digits. In 1995, Kanada computed 6 billion digits. In 1996, the Chudnovsky brothers found 8 billion. In 1997, Kanada and Takahashi calculated 51.5 billion digits on a Hitachi SR2201 in just over 29 hours. The current record is over 60 billion digits of π!

This is not the end of our quest for knowledge of π. The number pi has been the subject of a great deal of mathematical and popular folklore. It has been worshipped, maligned, misunderstood, overestimated, underestimated and worked on by scholars and everyday laymen. People have dedicated their lives in the quest for pi.

As David Blatner said in *The Joy of Pi*, "The search for pi is deeply rooted in our irrepressible drive to test our limits."

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