# delta-k

JOURNAL OF THE MATHEMATICS COUNCIL OF THE ALBERTA TEACHERS' ASSOCIATION

Volume 42, Number 2 June 2005

# GUIDELINES FOR MANUSCRIPTS

*delta-K* is a professional journal for mathematics teachers in Alberta. It is published to

- promote the professional development of mathematics educators, and
- stimulate thinking, explore new ideas and offer various viewpoints.

Submissions are requested that have a classroom as well as a scholarly focus. They may include

- personal explorations of significant classroom experiences;
- descriptions of innovative classroom and school practices;
- reviews or evaluations of instructional and curricular methods, programs or materials;
- discussions of trends, issues or policies;
- a specific focus on technology in the classroom; and
- a focus on the curriculum, professional and assessment standards of the NCTM.

## Manuscript Guidelines

1. *delta-K* is a refereed journal. Manuscripts submitted to *delta-K* should be original material. Articles currently under consideration by other journals will not be reviewed.
2. All manuscripts should be typewritten, double-spaced and properly referenced. All pages should be numbered.
3. The author's name and full address should be provided on a separate page. If an article has more than one author, the contact author must be clearly identified. Authors should avoid all other references that may reveal the author's identity to the reviewers.
4. All manuscripts should be submitted electronically, using Microsoft Word format.
5. Pictures or illustrations should be clearly labelled and placed where you want them to appear in the article. A caption and photo credit should accompany each photograph.
6. If any student sample work is included, please provide a release letter from the student's parent/guardian allowing publication in the journal.
7. Limit your manuscripts to no more than eight pages double-spaced.
8. A 250–350 word abstract should accompany your manuscript for inclusion on the Mathematics Council's website.
9. Letters to the editor or reviews of curriculum materials are welcome.
10. Send manuscripts and inquiries to the coeditors: A. Craig Loewen, 414 25 Street S, Lethbridge, AB T1J 3P3; fax (403) 329-2412, e-mail loewen@uleth.ca or Gladys Sterenberg, 3807 104 Street NW, Edmonton, AB T6J 2J9; e-mail gladyss@ualberta.ca.

## Submission Deadlines

*delta-K* is published twice a year. Submissions must be received by August 31 for the fall issue and December 15 for the spring issue.

**MCATA Mission Statement**

***Providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics.***

delta-K

Volume 42, Number 2 June 2005

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delta-K *is published by the ATA for the Mathematics Council (MCATA). COEDITORS: A. Craig Loewen, 414 25 Street S, Lethbridge, AB T1J 3P3; fax (403) 329-2412, e-mail loewen@uleth.ca and Gladys Sterenberg, 3807 104 Street NW, Edmonton, AB T6J 2J9; e-mail gladyss@ualberta.ca. EDITORIAL AND PRODUCTION SERVICES: Document Production staff, ATA. Opinions expressed herein are not necessarily those of MCATA or of the ATA. Address correspondence regarding this publication to the editor.* delta-K *is indexed in the Canadian Education Index. ISSN 0319-8367*

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**TEACHING IDEAS**

# EDITORIAL

Conversations that began at the 2004 Mathematics Council of the Alberta Teachers' Association (MCATA) conference continue with this issue. Coeditor Craig Loewen collected and assembled a photographic essay that reminds us of our commitment to professional and personal growth within a community of teachers and other educators. Because the summer months can afford us time to reflect on recent developments in mathematics education, I have included several lengthy articles that offer more in-depth investigations of topics presented at the conference.

I thank all those who have contributed to this issue. Your thoughtfulness when writing and your willingness to stimulate thinking and extend our understandings of new ideas in teaching mathematics are appreciated.

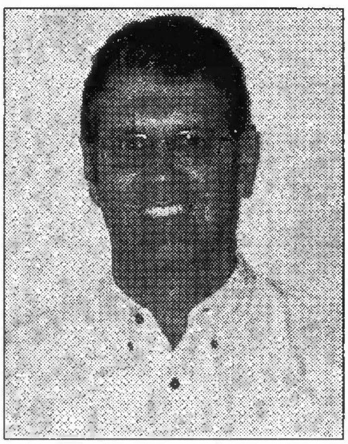
Coediting this issue of *delta-K* has been a challenging and satisfying experience for me. I am very grateful for the mentorship provided by Craig Loewen, the encouragement from Len Bonifacio and other members of the MCATA executive, and the direction provided by Karen Virag, ATA publications supervisor. Many people have provided much encouragement and support as I begin this new endeavour. This has been invaluable as I attempt to continue the tradition of providing high-quality articles pertaining to the professional development of mathematics educators.

On behalf of the MCATA executive, Craig and I are pleased to announce that, as of August 2005, *delta-K* will become a refereed journal. A refereed review process gives a voice to teachers as authorities in the mathematical educators' communities. It is our belief and hope that by establishing *delta-K* as a refereed journal, teachers will participate as authors and reviewers and will engage in ongoing professional conversations about mathematics instruction. We also hope that this process will help build an important mechanism for sharing the many marvellous activities, developments, resources and achievements that are evident across this province.

I invite you to consider making a contribution to *delta-K*. This could be in the form of articles, classroom activities, letters and problems. It is your active participation and your willingness to share your ideas and teaching strategies that makes this journal a relevant and useful resource for mathematics educators.

*Gladys Sterenberg*
*Coeditor*

# From the President's Pen



With the provincial election behind us and a new cabinet in place, I thought it would be appropriate and timely to write about how these changes may affect education and mathematics educators.

We have a new education minister for the first time in five years. Gene Zwozdesky, a former teacher, took over the portfolio from Dr. Lyle Oberg. It was encouraging to hear him speak about wanting to address issues in education, including class size concerns, Grade 12 completion rates, the Learning Commission's recommendations and the stresses that teachers face in doing their jobs. This was very refreshing, and we hope that teachers in all subject areas and all grade levels will see a more cooperative and less adversarial relationship with the provincial ministry.

It is also worth noting that advanced education has again become a separate ministry, after having been combined with basic education only a few years ago. The new minister is Dave Hancock, who is considered to be approachable and very aware of education issues. The reasons for separating the departments are not clear, but as a mathematics educator, I am concerned that this will add distance to the relationship between secondary education stakeholders and postsecondary institutions. More dialogue with postsecondary officials is needed, especially in the areas of acceptance of mathematics courses at the postsecondary level and the use of technology in mathematics education.

Despite these reservations, I feel positive about the changes in general and I think mathematics educators have reason to believe that our working relationship with the provincial ministries will improve. Let's see what the new school year brings. Thank you.

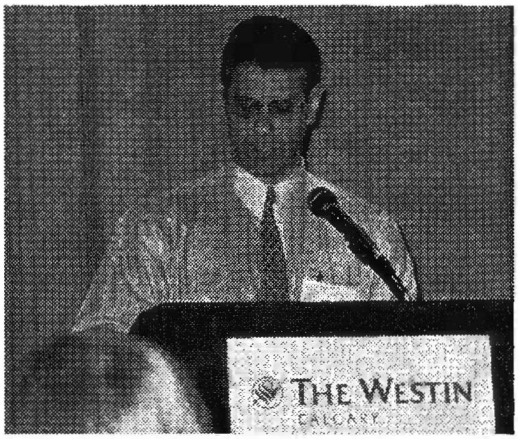
*Len Bonifacio*

# MCATA Annual Conference 2004

The 2004 MCATA annual conference in Calgary was a great success. We did expect a larger registration, but with over 450 delegates, we did well. We scheduled around 70 sessions for teachers, and offered both 60-minute lectures and 90-minute workshops to choose from. Thank you for all the supportive comments. We are looking forward to the 2005 conference in Edmonton. As the information becomes finalized, it will be posted on the MCATA website at www.mathteachers.ab.ca.

*Sandra Unrau, 2004 Conference Director*

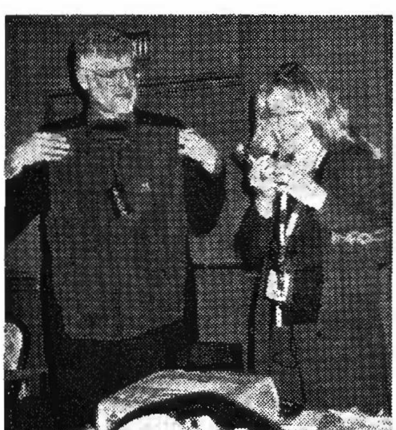
# MCATA Annual Conference 2004
## A Photographic Memory



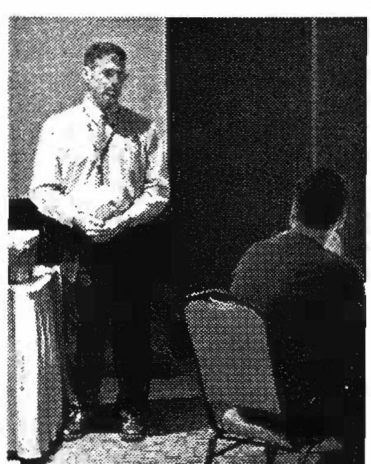

*President Len Bonifacio gets the conference under way.*

*The registration desk.*

*Keynote speaker Dr. Ivars Peterson "The Jungles of Randomness."*



*Past president Sandra Unrau thanks the speaker.*

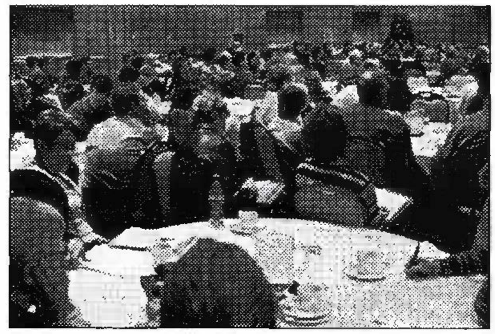


*President Len Bonifacio leads the annual general meeting.*

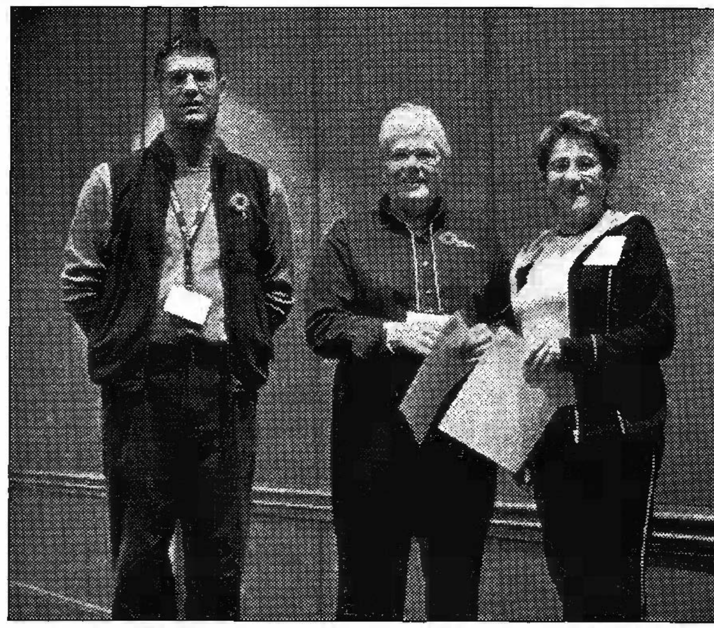
*Keynote speaker Dr. Brent Davis.*

Photos and layout
by A. Craig Loewen,
coeditor, *delta-K*

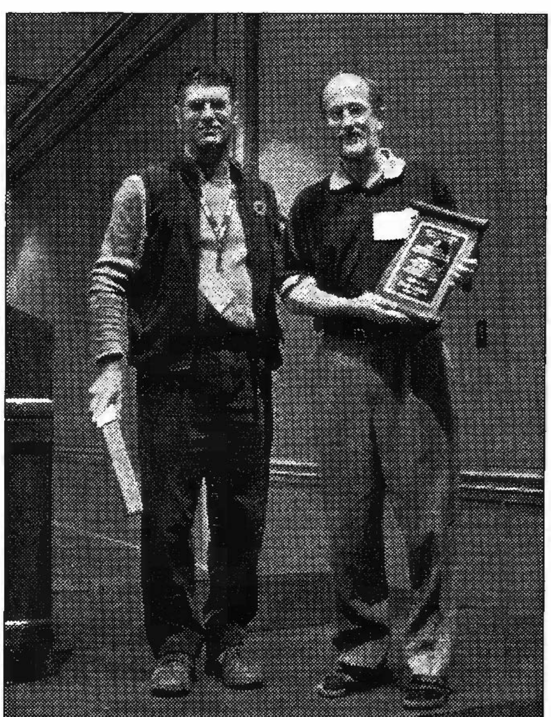
Dr. Art Jorgensen presents the award named in his honour to Lisa Hawk-Meeker.



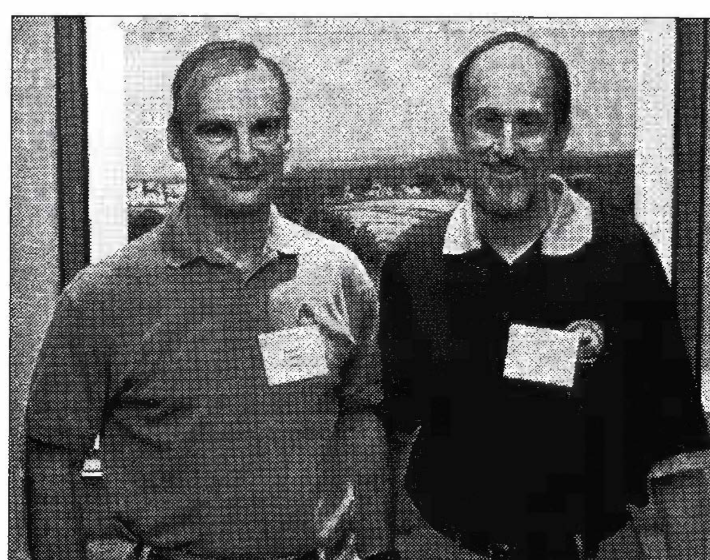
Dr. Art Jorgensen



Two new Friends of MCATA: Helen McIntyre (centre) and Carol Klass (right).



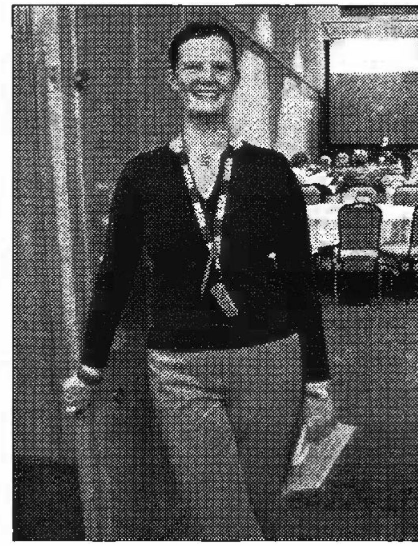
Len Bonifacio presents Percy Zalasky with Math Educator of the Year Award.



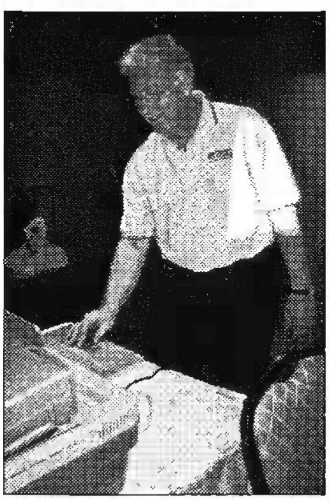
Percy Zalasky (right) with nominator Darryl Smith.

## Some Sessions and Conference Speakers

*Joanne Adomeit and Alice Laird, "You Too Can Problem Solve!"*



*Stephanie Gower-Storey, "Assessment for Learning in Applied Math 30"*



*Dave Walker, "Using the New Math Type 5 Effectively"*

*Betty Morris, "Best Practices in Mathematics"*

*Marian Oberg, "In Search of Meaningful Practice"*

*Garry Bell, "Turning Your Kids on to Calculators"*

*Dr. Marie Hauk and Bryan Quinn, "Math in Outer Space"*

*Dr. Nola Aitken, "Native Ways of Knowing Mathematics"*

# Times Have Changed

*Frank Jenkins*

A few years back I wrote an article for *The Science Teacher* indicating the critical shortage of chemistry, physics and mathematics teachers in the province (and worldwide). The typical situation was that we at the University of Alberta supplied about 75 biology majors and 12 physical science majors (a 6:1 ratio)—whereas the demand in the high schools was for a ratio of 2:3 (biology:chemistry+physics). About four years ago we graduated two chemistry teachers from the University of Alberta. Fortunately we have had a surplus of biology majors to take some of the physical science positions; however, many of these students did not even have a physical science minor.

Compounding the critical shortage in Alberta was the same shortage worldwide. Many of our physical science and mathematics students use a major and/or minor as a ticket to travel and see the world. This is a recruitment tool that sometimes backfires. We hope that some of these students will be back some day.

Today I am happy to report that we have had some success turning this situation around. For example, the numbers of majors plus minors in the physical sciences now exceeds the number of majors plus minors in biological sciences. The word has got out to students about the career opportunities for physical science and mathematics teachers, and we have created chemistry and physics majors and minors to remove some barriers to specialization and honours science students entering our faculty. The numbers of our students in the University of Alberta education system—year 2–5—are:

| Subject | Majors | Minors | Total |
|---|---|---|---|
| Biology | 131 | 117 | 248 |
| Chemistry | 51 | 48 | 99 |
| General | 81 | 39 | 120 |
| Physical | 46 | 33 | 79 |
| Physics | 31 | 28 | 59 |
| Math | 200 | 139 | 339 |

One problem that remains is to communicate this information to department heads, principals, human resource officers and superintendents; that is, the people who are hiring chemistry, physics and mathematics teachers. There are still reports of non-science majors being hired to fill science positions, because the person doing the hiring thinks that there is still a critical shortage of graduating science–education students. On the contrary, we now have specialists to fill these vacant positions. Similar conditions exist for mathematics–education students.

Please help by getting the word out to the people who do the hiring in your district—send them a copy of this article. Talk to them directly.

It's nice to have some good news to spread.

*Frank Jenkins is an adjunct associate professor and director of the Imperial Oil Centre for Mathematics, Science and Technology Education in the Department of Secondary Education at the University of Alberta.*

# Emergent Insights into Mathematical Intelligence from Cognitive Science[1]

*Brent Davis*

In this article, I point to a handful of recent developments in cognitive science in an attempt to highlight how they might contribute to a rethinking of the nature of mathematical intelligence. In the process, I also offer some preliminary speculations on what these developments might mean for the teaching of mathematics.

I must begin with a disclaimer: Cognitive science is a burgeoning field. It is really only a half-century old, and it has just taken off in the last decade, spurred along by the invention of technologies that enable researchers to peer into brains in real time. Some surprising observations have been made—ones that have compelled researchers to question and reject an array of deeply entrenched assumptions about how people learn, how brains work, what thinking is and what intelligence is all about.

Cognitive science isn't actually a field. The phrase is an umbrella term that stretches across certain research in artificial intelligence, linguistics, cultural studies, philosophy, experimental psychology, neurology, neurophysiology, ecology, cybernetics and complexity science—to mention a handful of the more prominent areas. In brief, the emergence of cognitive science as a domain of research might be taken as recognition that investigations into such phenomena as learning and intelligence require a transdisciplinary approach. None of the above-mentioned fields on its own has the capacity to answer the big questions about human cognition.

With regard to education, this move toward transdisciplinarity is a significant development. For most of the past century, educators relied almost exclusively on psychology for their formal definitions of intelligence, the tools to measure it and advice on how to nurture it. As it turns out, much of that advice was good, despite some troublesome assumptions. But much of it was also a bit problematic. In particular, the reliance on psychology has contributed to some deeply ingrained and unfixable dichotomies—between, for example, skills-based and understanding-oriented instruction, or between teacher-centred and learner-centred instruction. Most of what we've borrowed from psychology compels us to take one side or the other, or to live with some uncomfortable compromise.

But, as John Dewey (1910) noted a century ago, we never solve such radical splits. We simply get over them. So none of what I present here should be taken as an argument for or against, for example, skills-based or student-centred instruction. Rather, I'm actually arguing that recent cognitive science provides us with a way of sidestepping these sorts of quagmires and opening spaces for more interesting and productive discussions.

Before going too much further, it's important to be clear about how cognitive science defines intelligence—and let me emphasize that this definition represents a break with popular and psychology-based orthodoxies. For instance, for the cognitive scientist, intelligence is not what IQ tests measure, as might be inferred from the fact that some patently unintelligent machines are able to perform at the genius-level on most IQ tests. As well, an individual's IQ score can vary by as much as 50 points, depending on the time of day, warm-up activities, hunger, thirst and so on.

Cognitive science uses a much broader definition: Intelligence is the capacity to respond to new situations in ways that are not only appropriate, but that open up new spaces of possibility. Intelligence, then, is not merely about getting the right answer to a trick question. It is about coming up with solutions to real problems, with answers that go beyond routine responses and that enable the person to go further than he or she could before taking on the problem.

Intelligence, in these terms, is about breeding new possibilities, opening up new vistas, not about responding to mind-twisters devised by others.

**Point #1: Consciousness is small.**

One solid, rigorously demonstrated conclusion of the research out of 19th- and 20th-century psychology was that human intelligence is greatly constrained by some rather severe biological limitations on consciousness. In particular, a frequently cited factoid is that humans are capable of juggling a maximum of 6 or 7 details in their heads at a time, but can only do that for about 15 seconds before some or all fall away. This 6–7 limitation is especially interesting when considered against the total number of sensory receptors in an average human body, which is estimated to be somewhere in the 10 to 20 million range. (Some researchers contest that the total is in the order of 1010; see [Norretranders 1998].) To drive that point home, fewer than one in every million sensory events (and the number may be closer to one in a billion events) ever rises to consciousness.

This insight is actually an old one, thoroughly demonstrated in the 1800s. It was a key tenet in the emergence of discourses as diverse and incompatible as B. Skinner's behaviourist psychology and Sigmund Freud's psychoanalysis, both of which were under development about a century ago.

A brief demonstration might be useful here. First read the following instruction, then follow it. Close your eyes and imagine two dots, then three, then four, then five, then 20, then 100.

Chances are that your image of three was arranged in a triangle, that your four was a square, your five was either a pentagon or a square with a dot in the middle. You shouldn't have been able to imagine 20 or 100, but you might have invoked a strategy like a grid to think of these quantities in terms of smaller, more readily imagined amounts.

Now repeat the tasks, this time with all of the imagined objects in a single row—no grids, polygons or subgroupings allowed. You will likely max out at five. I know of no one who can imagine six side-by-side, ungrouped objects.

There is some compelling evidence that the capacity to imagine small quantities might actually be built in. It's been established that very young babies can discern between one object and two objects, likely between two and three, and perhaps between higher quantities (see Gopnik, Meltzoff and Kuhl 1999). It also seems that we share that ability with lots of mammals, some birds and a few other species.

The realization that consciousness is so tremendously limited is one of the principles that undergirds the highly parsed structure of modern curricula, especially mathematics curricula, which have been the subject of more psychologically based research than any other topic area. (In fact, math curricula have been the focus of more research than all the other areas combined.) The practice of structuring a lesson around one small topic, such as adding integers, long division or factoring a trinomial, originated in part from the embrace of the factory model of schooling, but the bolt that holds it in place is research into the limitations of consciousness.

In fact, that research is so compelling that I have structured this article around it. My psychologist colleagues tell me that the best I can hope for is that you'll retain at most six or seven bits of information. So I've limited my foci to seven points.

Before moving on to the second of those seven, I want to nod to a few implications of this first point for our efforts to nurture mathematical intelligence. Two implications:

- We have to limit the amount of new information in any given learning event.
- We have to use design learning in ways that help learners focus their attention on what really matters.

We've already mastered the first point. The second one is a little more complex than it might appear.

There is a connection between intelligence and discernment. In fact, intelligence was originally conceived as the capacity to discern what is really important in a situation. As it turns out, there are teaching strategies that can support people's discernment-making abilities—that is, that help them be intelligent.

Anne Watson of Oxford University and her husband John Mason of the Open University in the United Kingdom have done considerable work on this issue. An example based on their work is the following:

Compare the two lists here:

$3 : 3$
$1.7 : 1.7$
$x : x$
$e^{\pi i} : e^{\pi i}$

and

$3 : 3$
$6 / 4$
$2$ to $9$
$\frac{1.2n}{0.36n^2}$

The point Anne intends through this sort of comparison may seem counterintuitive. She argues that the first list might be a better pedagogical tool because it is designed to assist the learner to make a key mathematical discernment. In contrast, the second list obscures the discernment. Too much is going on. Her argument is that if there is not much variety, we generalize. If there is too much variety, we categorize. And for the most part, the intelligent mathematical action is about making the sorts of discernments that enable generalization, not categorization.

The first list sets itself up for questions like, What's the same? What's different? Is it always, sometimes, or never true? Are there examples that don't fit the pattern? In other words, even though it might look like there is less there, it's much easier to strike up a conversation about what is presented—that is, to pinpoint and emphasize what really matters.

The fact that consciousness is limited also points to the need for repetition and practice, which is something that traditional mathematics teaching has done well and that reform teaching has often done less well. Let me underscore this point.

**Point #2: Intelligence relies on the capacity to routinize knowledge and procedures so that consciousness is freed up to work on other tasks.**

Consider this sequence of numbers:

1, 11, 21, 1211, 111 221, 312 211

What comes next?

The following discussion will be more meaningful if you actually try to respond to the question.

When you first take on this sort of problem, your brain activity spikes and continues to do so until you either find a solution or give up on it. If you do in fact come up to a solution, your brain very quickly works to routinize things by delegating the task to a subregions or clusters of subregions while the rest of the brain returns to its usual near-resting state.

The realization of the importance of routinization for intelligence is quite a recent development. Or, at least, the proof for it is recent. Now that we can watch the brain in action, we can see that brains respond in different ways to novel situations. When presented with an unfamiliar problem or context, all brains begin to fire rapidly. And the whole brain fires when it meets a novel problem, not just parts of it (see Calvin 1996). I'll return to this point later.

The quality that most distinguishes the intelligent brain from the unintelligent brain is that it quickly settles on what's important, routinizes it and assigns it to subconscious processes. So, in terms of the profile, there's an initial spike of whole-brain activity that settles very quickly into lower-level, region-specific activity. By contrast, the unintelligent brain continues at a high level of whole-brain activation, apparently groping for what's important.

The happy thing is that the brain can improve its abilities to make vital discernments. One key is practice. Let me tell you a quick story.

Each week for the past three years, I've been meeting with Krista, an adolescent, about her mathematics. When I first met her, she was in Grade 9 and was unable to see patterns in lists of numbers like

1 4 9 16 25 36 49 64 ...

1 1 2 3 5 8 13 21 ...

It didn't take much probing to discover that a large part of the problem lay in the fact that she couldn't work with even single-digit numbers reliably. Calculations like 6 + 7 and 5 • 3 were problems for her.

This meant that she was failing mathematics badly, and had been doing so since Grade 1. The school board had been testing her annually and she had had at least eight years of focused help with special needs teachers. Yet in Grade 9 she couldn't do things that are routinely expected of children in Grades 2 and 3. I decided to work with her because I thought she might be one of those interesting cases of people with location-specific brain injuries, which I imagined could be a fascinating thing to study from the point of view of an educational researcher. It turned out that I was quite mistaken in this suspicion.

The first year of our association was spent on what I thought of as educating her intuition—a phrase that refers to engagement with processes and situations intended to help one develop a feeling for quantities and manipulations of quantities. For instance, we spent a total of about six hours (a month's work together) figuring out different ways to estimate the number of grains of rice in a bowl. We spent considerably more time on paper-based activities, such as folding, cutting, assembling and dismantling. We did anything I could think of that might be interpreted in terms of basic operations on whole numbers, integers and rationals.

Significantly, I insisted on practice. Krista had daily homework exercises, which included flashcard drill on multiplication facts, writing out explanations of why things seem to work how they work, spending time on non-routine problems and so on. Six months into our work together, the psychometrician who had worked with her for three years was surprised to note that her score on the mathematics portion of the test he used had soared from Grade 2.3 (at the end of Grade 8) to Grade 10.8 (in the last half of Grade 9).

I cite those statistics cautiously. Krista really was not working at a Grade 10.8 level. (I had no access to the test, so I cannot comment on what was really being assessed.) But the numbers do suggest that something important had happened. At the time of this writing, she is enrolled in the Grade 12 applied stream mathematics course. Her average in mathematics is consistently in the 80 per cent-range. That's gratifying, but what is really exciting is that while she's writing an exam she can now tell whether or not she's doing well. Two years ago, she couldn't tell you what sort of grade she might get on a test. If she passed (which was not often), she attributed it to luck. Now she can predict her score with a high degree of accuracy.

I recently asked her about her new capacity to predict her exam results and how she could feel so sure of her predictions. She responded that a few years ago, her brain would "just go crazy in math exams." She couldn't focus, she couldn't remember. Now, in her exact words, her "brain just goes calm" when she realizes she can respond to the questions.

I haven't had a chance to monitor her brain activity, but I'm fairly confident in the assertion that two years ago, in a test situation, her brain was spiking throughout the test, to no avail. Now, it's spiking and settling in—just as an intelligent brain is supposed to do.

As for teaching implications, a central point is one that we all know deeply—if we want to be proficient in an activity, much of it has to be routinized. Be it playing hockey, playing the piano or adding fractions, certain levels of practice are needed not only to develop the basic mechanical competencies but to get a feel for what one is doing.

There is one caveat here. Practice must be contextualized. The brain resists learnings that lack context or that are not anchored in purposeful activity for reasons that I will develop later. But first, I want to make one more point on the role of practice.

**Point #3: Mathematical genius (in fact, any category of genius) is, in general, much more about focus and practice than it is about innate, biologically rooted talents or gifts.**

Rena Upitis of Queen's University often asks audiences to do the following: Think about something you're really, really good at. Now answer two questions: Do you practise it? And did you learn it at school?

You probably said yes to the first and no to the second.

The fact of the matter is that talent and genius are dependent on practice. So long as the basic biology of the brain isn't compromised, an otherwise typical person can obsess his or her way toward genius in some domain of activity because the brain is what neurologist and psychologist Merlin Donald (2001) describes as a "superplastic structure" whose resources can be co-opted and reassigned through dedicated practice. If those resources are focused on mathematics—or golf, or the cello, or plumbing—otherwise ordinary individuals can achieve quite extraordinary feats after years of focused effort.

One interesting statistic in this regard is that the rates of mental illness, particularly obsessive compulsive disorders, are several times higher among elite mathematicians, musicians, athletes and other high performers (Richardson 1999). This is not to say that obsession is a good thing; it is merely to underscore that, biologically speaking, most of the super geniuses of the world began life with capacities that were very similar to the ones the rest of us were born with.

I'm not suggesting that people are all born with the same cognitive architectures or that there's no such thing as natural mathematical talent. Clearly, such notions are misguided. The point is that most of the differences that we observe among adults have more to do with habits of mind than with raw horsepower. A person who begins with typical ability but who is obsessive about mathematical concepts can be a much better mathematician than a person with considerable natural ability but no inclination to develop his or her own capacities.

I return to Krista here. Two years ago, she was mathematically inept. She is far from a mathematics genius, but she is now mathematically capable. And just being capable means that her mathematical intelligence has skyrocketed.

The claim here is that one can become more intelligent, and it is an assertion that flies in the face of some deeply engrained beliefs and practices. IQ tests, for instance, are developed around the assumption that something innate is being measured, not something that can be honed through practice. Howard Gardner's theory of multiple intelligences is anchored in the assumption that differences in human capacities for mathematics, interpersonal relations, music and so on are all rooted in variations among inborn brain structures. And we are confronted with tale after tale based on the assumption—and that advances the belief—that mathematical talent is innate. Consider the popular Hollywood films *Good Will Hunting, Little Man Tate, A Beautiful Mind.* The implication in these stories often seems to be that education is supposed to stay out of the way of a genius.

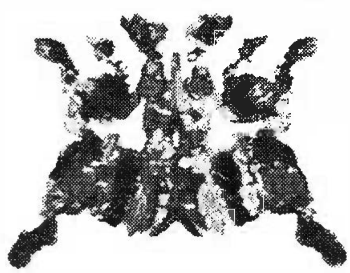
But the fact of the matter is that there are no documented cases, anytime or anywhere, of a full-blown mathematical genius who became that way without extensive practice and some formal education. It simply doesn't happen. By contrast, there is no shortage of evidence to support that assertion that mathematical intelligence is not fixed. We can make ourselves smarter.

Teachers can play an important role here. Emotions like curiosity and pleasure can be infectious. In fact, all emotions are. We humans are prone to being caught up in others' emotional expressions. So it's worthwhile asking yourself what emotions are you expressing in your classroom toward the mathematics? Enthusiasm? Indifference? Amusement? Obsession? (See [Damasio 1994] for a discussion of the relationship between emotion and logical competence.)

On the issue of making ourselves and our students smarter, it turns out that there are critical moments in life for nurturing one's intelligence.

**Point #4: Brains are constantly changing—and they change most rapidly in the first few years of life and during early adolescence.**

What do you see in the inkblot below?



Now, what if I tell you that this is actually a picture of two people squatting back-to-back holding ducks in their laps?

Once you apply this interpretation into the image, you can't help but see what you were told to see.

In other words, I have affected your brain structures by imposing a specific interpretation. That interpretation is compelling because your brain immediately went to work to activate the associations necessary for you to perceive the image as described. That is, your brain is physically different because of my intervention. Every lived experience entails a physical transformation of your brain.

Now consider such common turns-of-phrase as "taking things in," "attaining one's personal potential" and "brain as computer." We have dozens of such expressions, all of which assume and assert a fixed brain architecture—as though the brain were some kind of preset and unchanging receptacle. Nothing could be further from the truth. Some details:

1. Brains account for about 5 per cent of the body's weight, but consume about 20 per cent of the body's energy. In other words, they're incredibly physically active, and when I say physical, I mean physical. Things are actually moving about up there. On an MRI the brain looks vastly more like an anthill than it does a computer.
2. Infant and adolescent brains operate in overdrive, consuming two to three times as much energy as a typical adult brain. The claim has been made that if only a three-year-old could have an adult's knowledge and experience, all of the great problems of science would be solved in short order (Gopnik, Meltzoff and Kuhl 1999). They're geniuses. Super geniuses. And we might expect as much. They have to develop language, put together a theory of how the world works and master the complexities of interpersonal relationships in just a few years. None of us adults can do that.

   Until about five years ago, it was believed that brain activity undergoes a gradual and steady decline from toddlerhood to adulthood. But some recent research has demonstrated that there's a renewed surge of brain activity in early adolescence, especially around junior high age. They're geniuses again.

   Some, including Pinker (1997), theorize that this second surge in brain growth and activity is an evolutionary response to the need to cope with some new and fairly significant distractions. Whether or not that's the case, it would seem to make sense to take advantage of their amplified cognitive powers.
3. One of the differences between intelligent brains and not-so-intelligent brains is the density of neurons. Einstein's brain is pretty normal in size. There are no odd bulgy areas. However Einstein's neurons were more tightly packed and more intricately interconnected than typical brains.

   It turns out that neuronal interconnections can be grown. In fact, whole new neurons can be grown. These things happen in response to experience and need. As Canadian neurologist Donald Hebb (1949) wrote 50 years ago, "Neurons that fire together, wire together." A key here is, once again, contextualized and rich practice.

Considered together, the above points underscore an important conclusion: Your brain, at this moment, is different from the brain that you had when you started reading this article. Every experience you have contributes to the ongoing restructuring of the brain. Put in somewhat different terms, the brain isn't hardware and knowledge isn't data or information. These popular and pervasive ways of talking about learning and knowledge are way, way off.

In terms of implications for teaching, the sorts of things that contribute to increased neural density and interconnectivity are the sorts of things that force learners to think outside the box. Such activities include sustained engagements with mathematical puzzles, attending to the different ways that concepts can be interpreted and doing things that are unfamiliar and nonroutine. In particular, for a learner to develop mathematical intelligence and robust mathematical understandings, she or he has to be aware of how mathematical concepts can be interpreted in different ways. I turn to an example of this presently.

**Point #5: Human thought and learning are mainly associative not rational—that is, analogical, not logical. Mathematical intelligence and creativity are rooted in the capacity to select and blend appropriate associations.**

What is multiplication?

It turns out that this question has at least a dozen distinct responses, all of which are correct. In a recent workshop with a group of K–12 teachers, the following list was generated:

- Repeated addition: $2 \times 3 = 3 + 3$ or $2 + 2 + 2$
- Grouping process: $2 \times 3$ means "2 groups of 3"
- Sequential folds: $2 \times 3$ can refer to the action of folding a page in two and then folding the result in 3
- Many-layered (the literal meaning of *multiply*): $2 \times 3$ means "2 layers, each of which contains 3 layers"
- Grid-generating: $2 \times 3$ gives you 2 rows of 3 or 2 columns of 3
- Dimension-changing: a two-dimensional rectangle of area 6 units$^2$ can be formed when one-dimensional segments of lengths 2 units and 3 units are placed at right angles to one another
- Number-line–stretching or -compressing; $2 \times 3 = 6$ means that "2 corresponds to 6 if a number-line is compressed by a factor of 3"
- Rotating: for example, multiplication by $-1$ means rotate the number line by 180°—which reverses its direction

This list is far from exhaustive. It could easily be extended to include interpretations that are needed to make sense of the multiplication of vectors, matrices and other familiar mathematical objects.

It's important to emphasize that all of these interpretations point to distinct actions. They can be mapped onto one another, but they cannot be reduced to one another. And it's important that they're distinct. The power of mathematical processes like multiplication is not that they can be reduced to a single definition or process, but that they actually consist of clusters of interpretations.

There are some major teaching implications here. For most of the past four centuries, school mathematics has been organized around the assumption that mathematical learning proceeds logically and sequentially, like the construction of a building. Think of some of the metaphors that tend to be used: solid foundations, the basics, a cornerstone of logic, the structure of knowledge, and building and constructing ideas.

There is a popular assumption that the history of mathematics unfolded logically and sequentially as well. Nothing could be further from the truth. The more recent histories of mathematics underscore this point (for example, Mlodinow 2001; Seife 2000). The great leaps in the emergence of mathematical knowledge didn't occur through moments of logical insight, but through the development of new analogies. The concept of multiplication, for instance, has grown over the centuries as new interpretations have been proposed and blended into the existing definition (see Lakoff and Núñez 2000; Mazur 2003).

What does this mean for mathematical intelligence?

Let me preface my answer to that question with a quick visit to the field of artificial intelligence (AI) research. AI started in the 1950s when computers were beginning to outperform their programmers on some difficult mathematical tasks. Based on this early success, computer scientists and science fiction writers began to make confident predictions about the future of machine intelligence, forecasting that electronic intellects would soon dwarf flesh-based intellects.

Fifty years later, we see that they were spectacularly wrong. The reason for the collective error is instructive: They assumed—as did the original IQ-test inventors, many curriculum designers and writers of *Star Trek*—that logic is the root of intelligence. The belief was supported by their own experiences. Like most people, they found logical tasks very difficult.

And there is a reason why they're difficult—it's because our brains are analogical. That is, the root of intelligence is not logic, but the capacity to make new associations among experiences—through storying, analogy, metaphor and other figurative devices. Ours is an intelligence that is capable of logic, but that capacity rides on top of very different sorts of competencies.

There's a rather shocking implication here—our current mathematics curriculum might be stifling mathematical intelligence, not supporting it, an assertion that might be linked to Point #2. Brains resist decontextualized, overly abstract constructs. When the brain meets something new, it works very hard to weave the experience into the web of existing associations. But if the new topic comes without obvious associations, then it can't be learned on any level other than the mechanical. But human brains are notoriously unreliable when it comes to rigidly procedural knowledge.

One of the major implications for teaching is something that we can't do much about at the moment. Mathematics curricula are structured after the model of the logical proof. You begin by developing the premises or basics and proceed by assembling those premises into more sophisticated truths. In terms of the analogical nature of human cognition—and, in fact, in terms of the emergence of mathematical knowledge—this instructional sequence amounts to putting the cart before the horse. Logical justification has always come after the development of a new way of interpreting things.

Speaking of the model of formal logic, did you know that Euclid's five axioms aren't sufficient for his geometry? He missed some necessary axioms because he was thinking analogically, not logically. About a century ago, David Hilbert (1988/1899) identified several others that are needed for Euclid to be logically complete. It took more than two millennia for mathematicians to notice the gap. Why? Because humans are much more analogical than logical.

But, of course, we can't wait for full-scale curriculum restructuring. In the meantime, to nurture your students' mathematical intelligence, I recommend that you work with them to try to uncover the associations that have been built into mathematical concepts. Start with addition. What are some of the ways we interpret adding? (If you want one answer to that question, you might check Lakoff and Núñez 2000.)

Let me re-emphasize that robust understandings and flexible applications of mathematical ideas—that is, the underpinnings of mathematical intelligence—are completely dependent on access to the range of meanings that are knitted together in a concept.

**Point #6: The real power of mathematics arises in cleverly structured symbolic tools, which collect together but conceal the arrays of interpretations and experiences that underlie concepts.**

Close your eyes and imagine $\sqrt{-15}$.

It's not so easy. And yet, as it turns out, $\sqrt{-15}$ is utterly imaginable. Barry Mazur, a Harvard University mathematician, explains how in his 2003 book *Imagining Numbers*. Space prohibits an adequate summary of his discussion, but I can mention that to imagine $\sqrt{-15}$, you have to know that the concept relies on the notion of multiplication-as-rotation. That is, multiplication by a negative means a 180°-rotation and multiplication by two negatives means a 360°-rotation (which takes you back to the starting orientation). One more detail is needed: one might think of a square root as half of a multiplication, as indicated by the exponent of 1/2. If you blend these ideas—as mathematicians did a few centuries ago—you get the root of a negative is a half of a 180°-rotation, which is a 90°-rotation, which generates the complex plane. The roots of −15, then, are the points that are just beneath the +4 and just above the −4 on the i axis of the complex plane.

Lakoff and Núñez (2000) take this sort of thinking even further and demonstrate how it's possible to imagine Euler's formula: $e^{\pi i} + 1 = 0$. Even more significantly, they attempt to impress that this very complex notion is rooted in bodily action, like moving forward, spinning and so on. (See Elizabeth Mowat's article in this issue for a fuller discussion of Lakoff and Núñez.[2])

My point here is not really that such imaginings are doable nor that we should be doing them in our math classes—although I do believe that they are doable and that we should be doing them in our math classes. It is, rather, that knitted into these symbols are an incredible array of experiences and possibilities. They are intelligently designed tools that greatly expand what we are able to do.

To put a finer point on it, tools such as language, mathematical symbols, and calculators aren't just the product of human intelligence—they are bestowers of intelligence. Humans with language are much more intelligent than humans without language. And, although I don't have nearly the raw intelligence of Archimedes or Newton or other mathematical giants of history, I can do things that they didn't even imagine doing because of the tools they helped to build.

Now, by psychologistic definitions of intelligence, you might argue, the fact that I can solve an unsolved differential equation by typing it into Maple does not make me a mathematical genius. And according to measures of IQ, that's true. But going by the cognitive science definition of intelligence (that is, intelligence is the capacity to respond to new situations in ways that are not only appropriate, but that open up new spaces of possibility), intelligence is about an ever-growing horizon of possibility, not the capacity to master what's already been established. What's more, intelligence is obviously not an individual phenomenon. Not only can we make ourselves smarter, we can contribute to the intelligence of others by giving them access to the tools of our intelligence. On this point, it's important to emphasize that we're routinely asking high school students to perform mathematical operations that were accessible only to the geniuses of a few centuries ago.

Now, to be clear, I'm not suggesting that technology on its own makes us smarter. Giving an iMac to a caveman would be a bit of a waste. And we have probably all seen people grab a calculator in order to add 0 or to multiply by 1. Those are decidedly unintelligent acts.

The point is, rather, that intelligence is not some mysterious quantity that's locked in our heads. Intelligence is about appropriate and innovative action, and to be intelligent in mathematics in this day and age requires more than a mastery of the conceptual tools that have been developed by our forebears. Intelligence is greatly enabled by a facility with contemporary tools. That's certainly true among research mathematicians. Our mathematics pedagogy hasn't adapted to take that into account, even though electronic technologies have contributed to dramatic reshaping of the landscape of mathematics research. We have to think about ways of incorporating these technologies to amplify possibilities, not just to brush aside tedious calculations as we cling to a curriculum that hasn't much changed in 400 years.

**Point #7: The clinically based research that supports point #1 is flawed, and the flaws are instructive.**

Most of the consciousness research that was conducted through the 20th century was undertaken in laboratories. And it turns out that if we isolate people in a room without any of the tools we use to extend intelligence, their conscious capacities will turn out to be not just amazingly limited, but amazingly equal, whether they are Nobel Prize laureates with something to prove or six-year-old brats who couldn't care less.

Now, it seems to me that this fact should have prompted curriculum developers to hesitate a little before structuring programs of study around the limitations of consciousness by parsing up concepts into small, 45-minute–lesson-sized concepts. But it didn't. It seems that no one thought to ask what it might be that enables some people, with essentially the same conscious capacities, to achieve such remarkable feats. Inborn ability is certainly part of a factor, but the range of inborn abilities is simply too limited to explain the variations in achievement that we see. Obsession is a huge factor, too, but we all know that obsessing about something doesn't necessarily lead to great insight.

A major clue into the difference between ordinary and extraordinary performances has emerged over the past few decades, as we've developed the technical abilities to study humans in contexts that are a bit more natural than the laboratory setting. Some surprising things have been shown. One of them is that humans have the capacity to "couple their consciousnesses" (Donald 2001); that is, to link their minds, to coordinate the rhythms and cycles of their brains' activities. In the process, they can form grander cognitive unities. One common sort of coupled consciousnesses is a "conversation."

It turns out that, in the context of a conversation, humans are able to collectively juggle not 7 ideas, nor 7 + 7 ideas, but more in the order of 7 × 7 ideas. And some of those ideas can endure not for 10 or 15 seconds, but for minutes and hours.

This point is critical to the production of mathematical knowledge. The image of the focused and still mathematician labouring alone in a locked chamber is not at all representative of how research mathematicians work. There may be moments when they're on their own, but like anyone in any domain who is concerned with the development of new insights, they surround themselves with others and others' ideas. No mathematician is an island.

Elaine Simmt, also of the University of Alberta, and I have been trying to understand the sorts of collective structures that support the work of mathematicians. Drawing from complexity science (see, for example, Kelly 1994), we have identified a handful of conditions that are common to such intelligent collectives (see Davis and Simmt 2003). This thinking is still in its infancy, but I can report briefly on what is involved in prompting the emergence of an intelligent collective in the classroom—a collective that, in turn, supports the development of each individual's mathematical intelligence.

Over the past 20 years, complexity scientists have been labouring to identify the sorts of conditions that enable the emergence of complex systems—how, for example, ants interact to form anthills, species couple together within ecosystems, cells knit themselves into organs, and organs into individuals, and individuals into societies and so on. Among the necessary conditions for these happenings, the following six seem to have a particular relevance to the work of the mathematics teacher:

- Internal Diversity—Internal diversity refers to the pool of possibilities that a system has to choose from when it's faced with a novel circumstance. It is the basis of the collective's intelligence. A system in which all of the components are expected to do the same thing at the same time will not be an intelligent one.
- Internal Redundancy—That being said, it's important that the agents in a system have enough in common to be able to work together, whether talking cells, birds, people or social systems. Redundancy is also necessary for a robust system. If one agent fails, another can step in.

  Some redundancies among participants in a collective have a lot to do with actions and competencies that are automatized. This is where traditional mathematics teaching has focused. The only way that a system's diversity can be a source of intelligence is if its agents are sufficiently alike for the bit of diversity to be embraced and elaborated.

- Neighbour Interactions—This condition might seem ridiculously obvious. Of course the agents in a system need to interact if that system is to become a system.

  But in the context of the classroom, the agents that need to interact aren't necessarily people. They can also be ideas or interpretations. As already mentioned, mathematics knowledge emerges as new ideas are blended with old ones. These blendings open up spaces for more powerful notions. So the phrase *neighbour interactions* doesn't refer to pod seating or group work, but to ensuring there is a sufficient density of diverse thought represented for the possibility of new ideas, as in the example of the varied interpretations of multiplication.

- Liberating Constraints—Consider these three tasks:
  1) Write down all that you know about three fourths.
  2) Write down two fractions equal to three fourths.
  3) Write down three things that you know about three fourths.

  In most classroom contexts, the first of these is much too broad to generate much that is interesting. The second suffers from being much too narrow, but has the same result—it likely won't generate much that's interesting either. But the third, like Baby Bear's porridge and bed, might be just right. It's open enough to allow for diverse possibilities, but sufficiently constrained to ensure that ideas won't be too diverse to prevent them from working together. (Of course, whether it is suitable depends on the collective.)

  Complex systems have to maintain this delicate balance between so much structure that they lock into place and so little structure that they decay into chaos. And the tasks that you set will determine whether or not intelligent—once again, appropriate and innovative—action can emerge.

- Organized Randomness—With a complex system, there's always a bit of randomness. Some of that randomness is ignored by the system—which is to say, it doesn't really affect what the system does. Other bits of randomness come to be really important—the unexpected observation, the sudden insight, the fact that this student's father is a painter and he knows the world doesn't work the way the question about ratios says it should work. Really intelligent systems, it seems, take advantage of more of these random events, and they're able to do so because they have strategies to organize such events.

- Decentralized Control—One of the big changes at Microsoft, Apple, IBM, Hewlett Packard and other locations of cutting-edge knowledge production has been an abandonment of the top-down model of centralized management in favour of a more distributed sort of control. Intelligent collective action can't be orchestrated into existence, at either the individual or the collective level. Space to negotiate the parameters and possibilities is needed.

All this being said, we have a long way to go before we'll be able to give much more direct advice on how to nurture mathematical intelligence. However, we can be quite specific about the opposite—on how to militate against the emergence of intelligent action. For instance, if diversity (among interpretations and among people) is suppressed, if ideas aren't plentiful and not permitted to bump against one another, if tasks are too open or too narrow, if control of the outcomes is strictly in the hands of the teacher, then chances are that intelligence will be stifled—intelligence of not just the collective, but of the individuals in the collective.

## Notes

1. Some of the research data reported in this article are drawn from studies supported by the Social Sciences and Humanities Research Council of Canada. The article itself is a modest revision of a presentation made at the NCTM Regional Conference in Edmonton on November 22, 2003.

2. The reviews of *Where Mathematics Comes From* have been varied, especially with regard to the issue of whether Lakoff and Núñez actually succeed in explaining the bodily basis of Euler's formula. Nevertheless, most reviewers have acknowledged that their discussion of the analogical substrate of our logical abilities is compelling and has significant implications for the teaching of mathematics.

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# Embodied Mathematics and Education

*Elizabeth M. Mowat*

Does understanding mathematics involve nothing more than learning symbols, axioms and theorems? For George Lakoff and Rafael Núñez (2000), authors of *Where Mathematics Comes From: How the Embodied Mind Brings Mathematics into Being*, understanding mathematics means comprehending how mechanisms of the brain and mind enable people to reason mathematically. They use results of research in cognitive science to explore how mathematical ideas are possible and why they make sense.

Lakoff and Núñez suggest that the teaching of mathematics may be enhanced by an understanding of this cognitive perspective of embodied mathematics. In this article, I will attempt to show how ideas put forth in *Where Mathematics Comes From* may be helpful in mathematics education. The article is divided into three parts. The first section describes some aspects of embodied mathematics, based on ordinary human cognitive and bodily mechanisms, which are presented in *Where Mathematics Comes From*. The second section reviews ways in which the theory of embodied mathematics explains sources of student difficulties and the third section discusses how teachers can use these ideas in designing effective instruction for their classrooms.

## Embodied Mathematics

*Where Mathematics Comes From* can be considered a study of the nature of mathematical intuition. The authors claim that automatic, unconscious understanding is developed and refined through activities and experiences in the real world. Lakoff and Núñez provide empirical evidence that this intuitive understanding is neither vague nor ill-defined, but is precise and rigorous enough to form a foundation for mathematical thought.

They assert that mathematics exists by virtue of the embodied mind. Cognitive structures used in mathematical thinking are based on physical sensations and activities. The brain receives input exclusively from the rest of the body. Therefore, what the body is like and how it functions in the world determine the form and content of thought. The mind emerges from distinctive characteristics of the human brain and body; it is embodied. "The detailed natures of our bodies, our brains and our everyday functioning in the world structures human concepts and human reason. This includes mathematical concepts and mathematical reasoning" (Lakoff and Núñez 2000, 5).

Lakoff and Núñez base their assertions of this thesis on the empirical findings of scientists from a wide variety of disciplines: developmental psychology, cognitive neuroscience, neuropsychology, cognitive linguistics and cognitive psychology. Convergent evidence from these fields is used to support and structure the theory presented in *Where Mathematics Comes From*. This book shows how mathematics is embodied through innate arithmetic abilities, the mind's cognitive mechanisms and its basis in bodily experience through grounding metaphors.

### Innate Arithmetic Abilities

Lakoff and Núñez argue that humans are born with certain arithmetic capacities. The very notion of *number* is engraved on our brains. Highly specialized sets of neural circuits enable us to subitize; that is, instantly and accurately recognize very small numbers of objects. At an early age, people possess an understanding of limited addition and subtraction, capacities needed for simple counting and numerosity, which is the ability to make rough consistent estimates for larger numbers. Areas of the brain involved in these activities are thought to be located in the inferior parietal cortex which links vision, hearing and touch.

### Cognitive Mechanisms

Knowing which parts of the brain are activated when people use these very limited innate capacities does not explain where normal arithmetic and more sophisticated mathematics come from. Lakoff and Núñez explain that mathematical thinking engages the same conceptual structures used by humans in other kinds of sense making. These cognitive mechanisms, used automatically and unconsciously in reasoning, are referential systems that assist people in understanding and employing concepts.

Abstract reasoning using cognitive mechanisms is grounded in basic bodily experiences. For example, balance is part of everyday life for all humans. We first encounter balance as babies wobbling across the floor. Over the years, balance becomes such an intrinsic part of our lives that we are hardly aware of it, but it is extremely important for our coherent perception of the world (Johnson 1987). This type of universal body-based experience becomes a cognitive mechanism that can be used to reason about many things like cheque books, relationships or solving equations. Lakoff and Núñez discuss three cognitive mechanisms that are particularly important: the image schema, the conceptual metaphor and the conceptual blend.

### *The Image Schema*

Image schemas, for qualities like balance, straightness or verticality, represent the spatial logic inherent in physical situations. Image schemas are not just mental pictures, but are general and flexible patterns developed through sensori-motor experiences that make our perceptions of the world meaningful.

The container image schema is of particular importance in mathematics. Because our experiences with physical containers involve sight, touch, language and reasoning, the container image schema utilizes the corresponding regions of the brain. Lakoff and Núñez use the image of a set as a "cognitive" container to represent the container image schema (see Figure 1).

The logic of the physical container is projected onto the cognitive container, which can be used to reason about nonspatial situations (Johnson 1987, 34). Normal language use illustrates how common this is. Statements often refer to components of the container: its boundary (he's on the *brink* of disaster), its exterior (she's *out* of her league) and its interior (he's always getting *into* trouble). Modes of reasoning developed through experience with ordinary containers are an essential part of the image schema. Figure 2 shows how the container image schema can link physical experience to mathematics.

The power of the image schema is that it can introduce new ideas or extensions that do not arise from the original physical experience. We can imagine two sets overlapping (Figure 3) even though two physical containers cannot intersect in this way.

**Figure 1**
**Container Image Schema**

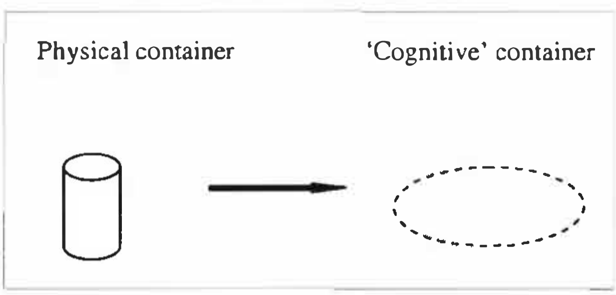



**Figure 3**
**Concept of intersecting sets introduced by the abstract container image schema**

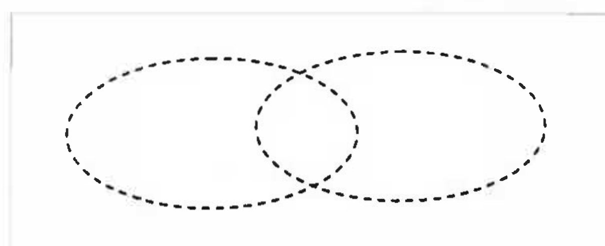
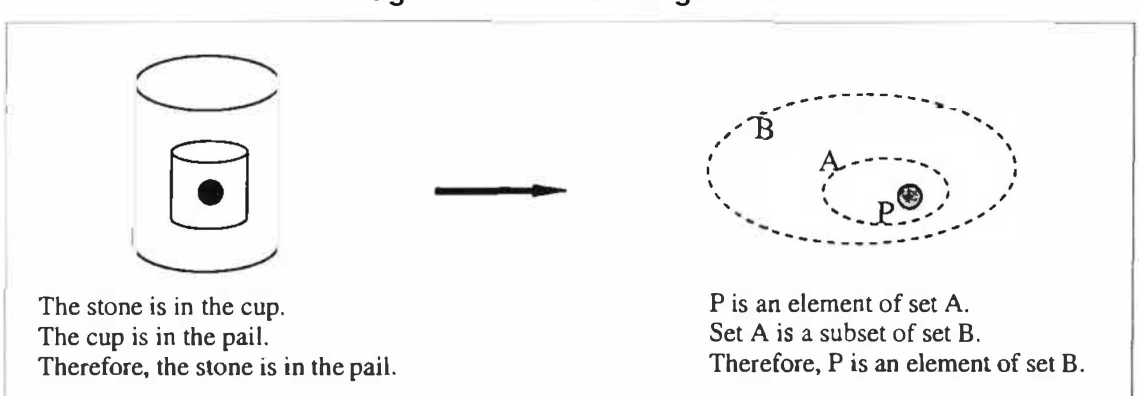


**Figure 2**
**Reasoning from physical experience transferred to abstract mathematics through the container image schema**



### *The Conceptual Metaphor*

Image schemas are linked together by another important cognitive mechanism, the conceptual metaphor. Conceptual metaphors are the basic means by which conceptual thought is made possible. An essential part of all types of human understanding, conceptual metaphors enable people to think about an unfamiliar, abstract concept as if it were familiar and concrete. Many conceptual metaphors arise initially from the everyday experiences of children. A child, held in his mother's arms, feels both love and warmth. Associating affection with cuddling leads to the metaphor of affection as warmth. Evidence of the existence of the metaphor is seen in everyday language. We say "they warmed up to each other" or "she gave him an icy stare." Experiences in the source domain of warmth are mapped onto relationships in the target domain of affection.

Conceptual metaphors are not just linguistic devices, but empirically observable mechanisms of the mind. The simultaneous activation of two different areas of the brain establishes new neural connections between them and generates a single complex experience. Because the inferential structure inherent in these experiences is preserved, the abstract concept of affection can be understood in terms of the concrete experience of warmth.

Conceptual metaphors can also introduce new elements or extensions in the target domain. The statement, "I had to work hard to get that question" is evidence of the metaphor of learning as a job. Subtle aspects of this metaphor, like those set out in Figure 4, are absorbed and unconsciously influence thinking.

**Figure 4**
**Implications of the metaphor of learning as a job**

Learning is work.
Learning is routine.
Learning is difficult.
I deserve some compensation for learning.

Learning is not play.
Learning is not fun.

### *The Conceptual Blend*

Two conceptual metaphors can be combined through a conceptual blend. Lakoff and Núñez offer this example: the unit circle is a conceptual blend of a circle in the Euclidean plane and a Cartesian plane with coordinate axes (see Figure 5)[1]. In the Euclidean plane, a circle consists of all points in the plane a fixed distance, called the radius, from a fixed point, called the centre. The two-dimensional Cartesian plane is defined by two axes set at right angles to each other. The horizontal or x-axis and the vertical or y-axis intersect at a point called the origin, O. By using a unit length on each axis and forming a grid, the position of any point on the Cartesian plane can be described using (x, y) coordinates. The unit circle conceptual blend combines characteristics of both of these metaphors.

In the unit circle conceptual blend, new connections are formed between the neural structures related to the two original types of geometric planes. Thus the blend possesses characteristics of both of the original domains. A circle is still composed of points a set distance from the centre. But now this centre is at the origin, the radius has a length of one unit and coordinates are used to describe points on the circle. Moreover, new concepts or extensions arise. The unit circle blend has properties related to trigonometry that are not part of either of the original metaphors (see Figure 5).

### *Grounding Metaphors*

Grounding metaphors are conceptual metaphors that establish correlations between physical activities of the body and innate arithmetic. In mathematics, the grounding metaphor is the primary tool that enables the extension of innate numerical abilities to arithmetic within the set of natural numbers and ultimately to more sophisticated concepts. Lakoff and Núñez pay special attention to four grounding metaphors:

- Arithmetic is object collection
- Arithmetic is object construction
- The measuring stick metaphor
- Arithmetic is motion along a path.

Human understanding is grounded in already acquired understanding of ordinary actions. A child who puts blocks into piles is establishing neural connections between areas of the brain responsible for the physical action and innate arithmetic. This initiates the metaphor of arithmetic as object collection, whereby numbers are identified with collections of objects. Adding involves putting two collections together, while subtracting involves taking a small collection from a larger one. The natural number system, which includes numbers too large to be subitized (instantly recognized), is formed. Properties of number-collection entities are consistent with those of innate mathematics, but are extended to include new properties. Since the sum of any two collections is another collection, the sum of any two numbers must be another number. Thus, the natural numbers possess the property of closure, which is not part of innate arithmetic.

A similar grounding metaphor is arithmetic as object construction. Children start to form this metaphor when playing by putting things together to construct a new object. Numbers are identified with wholes made up of parts. Imagine a child building a tower out of blocks; a tower five blocks high represents the number five. Addition means adding more parts to the object. Subtracting means taking some of the parts away. This metaphor ties innate arithmetic to the natural numbers, but can be extended farther. A whole object can be broken up into smaller equal parts giving an embodied meaning to the concept of fractions.

Lakoff and Núñez's third grounding metaphor is the measuring stick metaphor in which objects are measured using physical segments. Blocks might be used to measure the size of a new toy or hands to measure the size of a pony. Number-physical segment entities are created. Addition is putting two segments together end to end, and subtraction is taking a smaller segment away from a larger one. This metaphor is similar to arithmetic-as-object-collection and arithmetic-as-object-construction metaphors, but has different extensions. In this metaphor, any physical segment or anything that can be measured can be considered a number. Consequently, some irrational numbers are grounded.

**Figure 6**
**Grounding √2 and $\pi$ using the measuring stick metaphor**

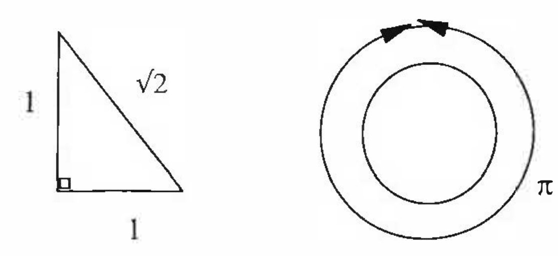
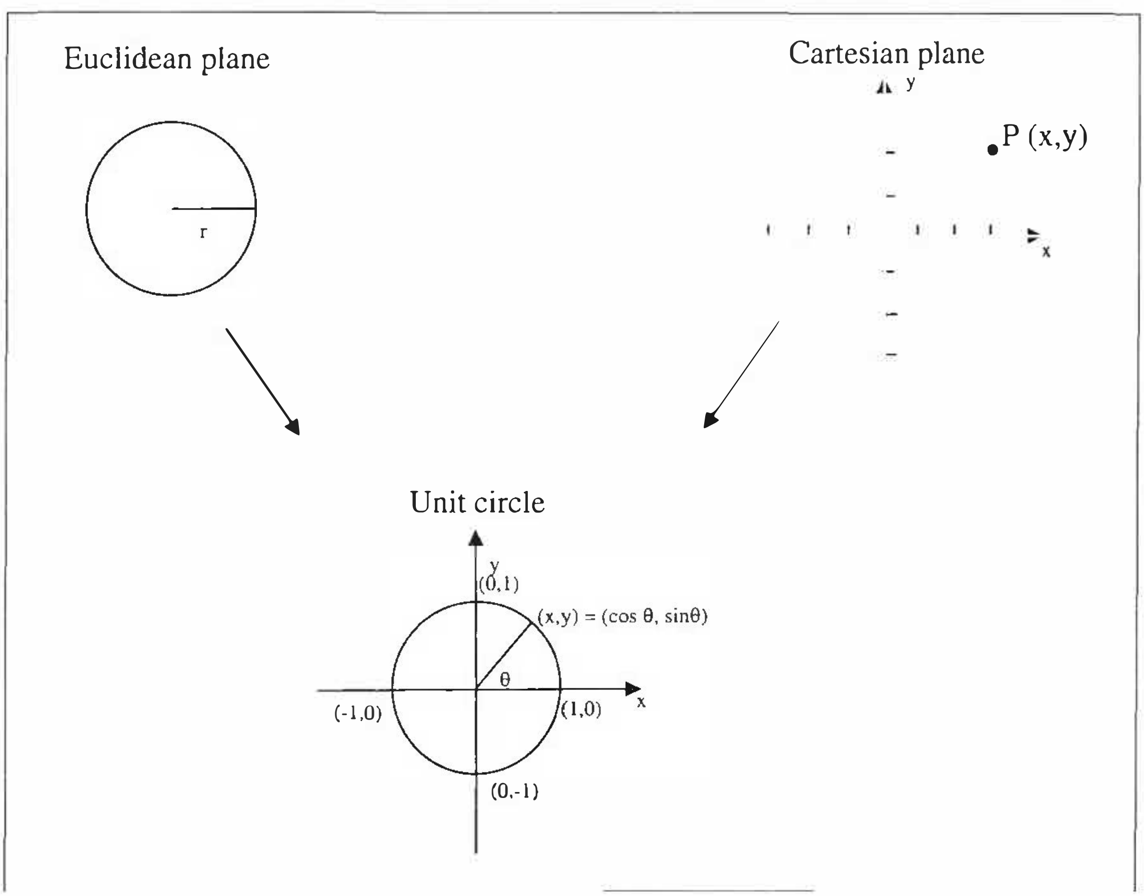



**Figure 5**
**Features of Euclidean and Cartesian geometry combined into the unit circle**



The fourth grounding metaphor is arithmetic as motion along a path. Numbers are point locations on a line. Addition involves moving from a position on the line away from the origin, whereas subtraction involves moving toward the origin. This metaphor, which grounds natural numbers, fractions and irrational numbers, has two unique extensions. Allowing motion on either side of the origin provides a physical basis for integers. Moreover, because a path-number can have any length, the metaphor provides a grounding for the real numbers.

Arithmetic as motion along a path differs significantly from the other grounding metaphors. Lawler and Breck (1998) point out that this metaphor, built on early experiences of crawling and walking, is based on ambulation, involving legs and feet, while the first three grounding metaphors are based on manual manipulation. Moreover, it implies continuous motion rather than discrete objects or segments. The arithmetic-as-motion-along-a-path metaphor is the only grounding metaphor that cannot be combined with subitizing (Chiu 2000). Another unique characteristic is its inherent concept of zero, which is located at the origin of the path.

These four grounding metaphors are not imaginary. Evidence of their existence is found in language and in mathematical constructs of the past. The metaphor of arithmetic as object collection appears in such expressions as "*add* some lettuce *to* the salad" and "*take* a log *from* the woodpile." Arithmetic as object construction is seen in Roman numerals like IX and VII where parts are being added to or subtracted from a whole. The measuring-stick metaphor is shown in units of measurement like cubits, feet and paces. Arithmetic as motion along a path appears in expressions like "6 is *close* to 8" and "*starting* at 20, count to 50."

These four grounding metaphors are not arbitrarily chosen. Of the many grounding metaphors that exist, Lakoff and Núñez found that only these four have physical sources with properties and logic sufficient to form a connection with inborn numerical capacities. "Each of them forms just the right kind of [correlation] with innate arithmetic to give rise to just the right kind of metaphorical mappings so that the inferences of the source domains will map correctly onto arithmetic . . ." (Lakoff and Núñez 2000, 102) and ultimately onto more complex mathematics.

## Where Does Mathematics Come From?

From a rather limited set of inborn skills, mathematics is extended through an ever-growing collection of metaphors. These cognitive mechanisms, which are neurally embodied structures of the mind, abstract patterns of inference from physical experience. Grounding metaphors form correlations between innate arithmetic and physical action to make elementary arithmetic possible. Other conceptual metaphors link arithmetic to more abstract mathematical concepts. Each layer of metaphors carries inferential structure systematically from one domain to another. Complex networks grow as domains that are connected to each other by conceptual blends, and new metaphors involving these blends are formed. Even the most abstract mathematical concept bears traces of its origin in physical perception and motor activity and is, thus, embodied. "The only mathematics that human beings know or can know is a mind-based mathematics, limited and structured by human brains and minds" (Lakoff and Núñez 2000, 4). Hence, the study of embodied mathematics sheds light on difficulties experienced by students in the mathematics classroom.

**Table 1**
**Characteristics of the Four Grounding Metaphors**

| Grounding metaphor | Object collection | Object construction | Measuring stick | Motion along a path |
|---|---|---|---|---|
| Numbers are ... | Collections | Wholes with parts | Physical segments | Points on a line |
| Addition is ... | Adding items | Adding parts | Putting segments together | Moving away from the origin |
| Subtraction is ... | Taking items away | Removing parts | Taking a segment away | Moving toward the origin |
| Number systems | Natural numbers | Fractions | Irrational numbers | Integers, real numbers |
| Physical experiences | Manipulation | Manipulation | Manipulation | Ambulation |
| Properties | Discrete | Discrete | Discrete | Continuous<br>Zero is the origin |

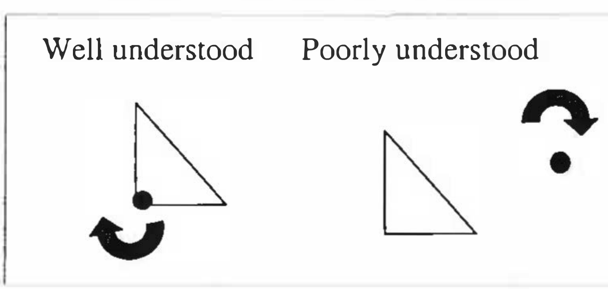
## Understanding Student Difficulties

Because metaphors are the basis of embodied mathematics, the study of their use in the classroom may reveal problems experienced by students in learning mathematics. Metaphors are not usually learned through formal instruction, but arise through informal everyday experiences and develop gradually over time. While conceptual metaphors make mathematics possible and very rich, they can also cause confusion and even apparent paradox if they are not made clear or are taken literally (Núñez 2000). Students may not understand everything that is implicit in a metaphor, what it hides and what it introduces. Research examining how students use and misuse common metaphors has identified some common difficulties experienced by students in their use of metaphors.

### Using an Inappropriate Metaphor

Use of an inappropriate metaphor can cause difficulties for students who are trying to comprehend a mathematical idea. Edwards (2003) found that children and adults studying transformation geometry had difficulty fully understanding the concept of rotation. Rotations of an object about a point that was inside the object were well understood. But all learners, regardless of age, had trouble with situations where the centre of rotation was outside the object (see Figure 7).

**Figure 7**
**Types of rotations about a point**




Edwards realized that learners were using their embodied understandings of turning to make sense of rotations. When students considered babies rolling over or skaters spinning on ice, they thought of themselves as the centre of rotation. Stating that human perception tends to place the body at the centre of the universe, Johnson (1997) clarifies why the metaphor of rotation as turning is used for reasoning about transformations.

For Edwards, this explained why problems in which the centre of rotation was inside the object being rotated were easily understood. Even situations in which a physical link existed between the object being rotated and the centre of rotation were grounded in experiences like playing on a swing and, consequently, were fairly straightforward. But when the centre of rotation and the object being turned are not in physical contact, the questions were harder to deal with. The metaphor rotation as turning was not useful in understanding these types of rotations in transformation geometry.

### Misunderstanding the Source Domain of the Metaphor

The source domains of metaphors provide the foundation for mathematical reasoning. If students do not clearly understand these fundamental patterns of thought, they are unlikely to be able to understand related concepts. "Inadequate understanding of the source domain of a metaphor limits a person's reasoning through that metaphor" (Chiu 2000, 7).

Students may have trouble using a metaphor whose source domain has subtle extensions. For example, difficulties are often experienced in the study of probability, particularly in questions containing the word *or*. These questions often make use of the categories-are-containers metaphor. Consider the following problem: If you draw one card from a deck of 52, what is the probability that it is red or a queen? In this question, learners are dealing with two categories of cards, those that are queens and those that are red. Students see these two categories as mutually exclusive (see Figure 8) when in reality they intersect (see Figure 9).

**Figure 8**
**Students' View of Categories as Physical Containers**

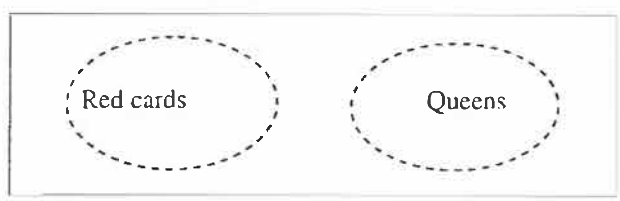
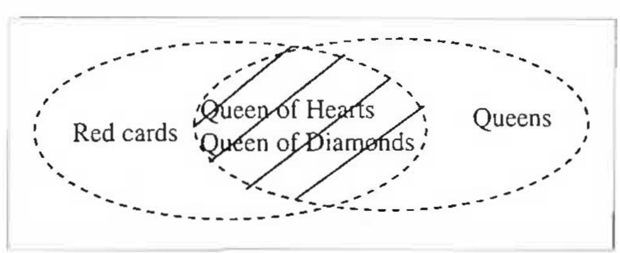



**Figure 9**
**Categories as Containers Using Container Image Schema**



Students' experiences with physical containers limit their thinking about cognitive containers, which are identified in this example with categories. They do not understand the container image schema, which is the source domain of the categories-are-containers metaphor.

This misunderstanding reflects a problem in the thinking processes of students, not in their mastery of the mathematical techniques. In discussion with a colleague, Mr. Michaels, a social studies teacher, found that understanding the container-image schema helped him to understand why his Grade 9 class had difficulty responding to a question about the Russian Revolution. When comparing how Russian people lived under the Czarist and the Communist regimes, students were able to list differences in lifestyles, but could not identify any similarities. Many existed, but their misunderstanding of the source domain of the categories-are-containers metaphor held students back. As in probability, they thought of physical containers (see Figure 10) rather than the cognitive containers of the container image schema (see Figure 11).

## Not Recognizing Limitations of Metaphors

Because metaphors are used unconsciously, learners may fail to recognize their inherent limitations. Tall (2003) found that automatic use of previously mastered metaphors may cause confusion. Young children tended to feel that adding two numbers should always yield a larger sum and that multiplying should lead to a very much larger product. These properties are true for *arithmetic as object collection*. But addition of integers can lead to a smaller sum (2 and –7 makes –5) and multiplication by fractions can lead to a much smaller product ($6 \times 1/12 = 1/2$). Confusion arose in students' minds because they could not realize the limitations of the metaphor they are using. They were held back in their development of arithmetic skills by their reliance on what Tall calls "met-befores."

**Figure 10**

**Students' View of Categories as Physical Containers**

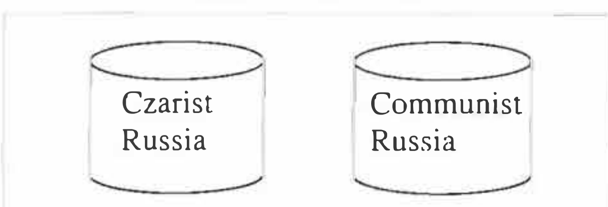
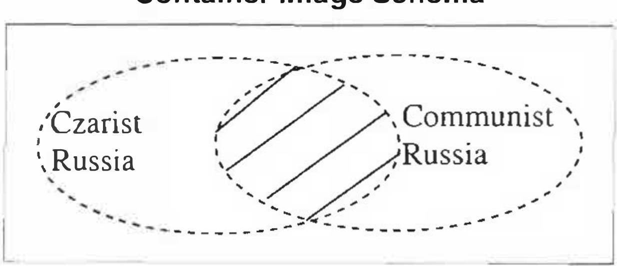



**Figure 11**

**Categories Are Containers Using Container Image Schema**




## Relying Exclusively on a Single Metaphor

In studies of students doing arithmetic with signed numbers, Moses and Cobb (2001) found that children failed to progress because of their reliance on the arithmetic as object collection metaphor. He felt that the arithmetic as motion along a path would be more useful in this situation and developed activities using experiences familiar to students, like riding on the subway, to strengthen this metaphor. With such techniques, he was successful in improving children's understanding of integer arithmetic.

## Using Two Metaphors That Conflict

Núñez, Edwards and Matos (1999) are particularly interested in conflicting metaphors used in the study of continuity of functions. High school students are introduced to "natural" continuity, which is defined as

$$\lim_{x \to a} f(x) = f(a)$$

This notion of continuity, used by Newton and Leibniz, is often described using Euler's idea of "a curve freely leading the hand" (Núñez 1997). Such a perspective is based on motion and uses the metaphor *a line is the motion of a traveller tracing that line*. The line does not move, but to the learner's understanding it does. Expressions commonly used in mathematics reflect this: a function *reaches* its maximum at (1,1); the line *crosses* the x-axis; two curves *meet* at a point; the line *goes through* (2,3); and the limit exists as *x approaches* 2.

At the university level, students are introduced to a new interpretation of continuity. The Cauchy-Weierstrass portrayal of continuity is very different.

$$\lim_{x \to a} f(x) = L$$

if and only if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|f(x) - L| < \varepsilon$.

This definition is based on the metaphor *a line is a set of points*. The idea of continuity here is in terms of preserving closeness: for every $x$ close to a, $f(x)$ is close to $L$.

While both definitions arise from metaphors grounded in experience, they are not compatible. Natural continuity is dynamic, based on properties of motion. The Cauchy-Weierstrass definition is static, based on closeness in containers. Although both definitions are useful, they have very different inferential structures, and this causes difficulties for learners.

Students of calculus are never told that the Cauchy-Weierstrass definition of continuity has a completely different embodied foundation than natural continuity (Núñez 1997; Núñez, Edwards and Matos 1999; Lakoff and Núñez 2001). Indeed, they are often told that it captures the essence of natural continuity. To compound the problem, both techniques talk of a limit as *x approaches a*, even though this terminology is inconsistent with the metaphor that the Cauchy-Weierstrass definition is based on. Because the two metaphors are not integrated into a coherent whole, it is understandable that students have trouble adapting to the Cauchy-Weierstrass method.

## Not Integrating Multiple Metaphors

Metaphors have their own inferential structures and can "lead to different conscious and unconscious beliefs that can cause obstacles to drawing various aspects into a central core concept" (Watson, Spyrou and Tall 2003). Students commonly learn two methods for adding vectors: the parallelogram method and the triangle method, as illustrated in Figure 12. Both methods are based on embodied metaphors and, although the underlying metaphors are different, both techniques are useful in understanding operations with vectors.

The parallelogram method is based on the *vector as a force* metaphor. Situations like two people pulling a sled or having two friends grab your arms and drag you along are within the experience of students. Both result in the sled or the person moving forward as if one force pulls it. It is natural therefore to think of the combination of two forces as a single force acting between the two forces. On the other hand, the triangle method is based on the *vector as a journey* metaphor. The sum of two vectors consists of two successive moves. We first move from A to B and then from B to C. The result is a journey starting at A and ending up at C: $\vec{AB} + \vec{BC} = \vec{AC}$.

Students who are introduced first to addition of vectors using the triangle method may have difficulty understanding general properties of vectors like the commutative law. In a journey where the order of the two components does matter, $\vec{BC} + \vec{AB}$ does not make sense. Consequently, the vector as a journey metaphor is not helpful in making sense of commutativity. In contrast, from the perspective of the vector as a force metaphor lying behind the parallelogram method, the commutative law is obvious. Watson and Tall (2002) found that emphasizing the vector as a force metaphor in this context was of benefit to students.

In turn, the parallelogram method does not easily explain subtraction of vectors. As shown in Figure 13, the difference $\vec{a} - \vec{b}$ lies on the diagonal of the parallelogram. It joins the endpoints of $\vec{a}$ and $\vec{b}$ and ends where the minuend $\vec{a}$ ends (see Figure 13). Nothing in everyday experience corresponds to this force. The vector as a journey metaphor explains subtraction much better. Students can think of $\vec{a} - \vec{b}$ as $\vec{a} + -\vec{b}$ by reversing the direction of the second component of the journey as shown in Figure 14.

**Figure 12**
**Two Approaches to Adding Vectors**

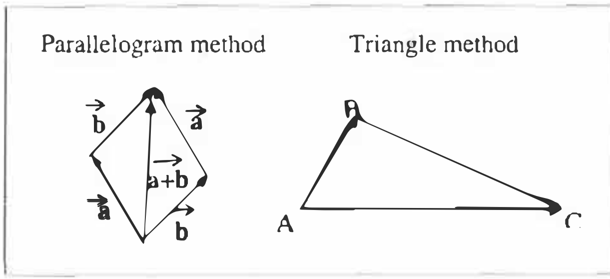



**Figure 13**
**Subtraction of Vectors Using the Parallelogram Method**

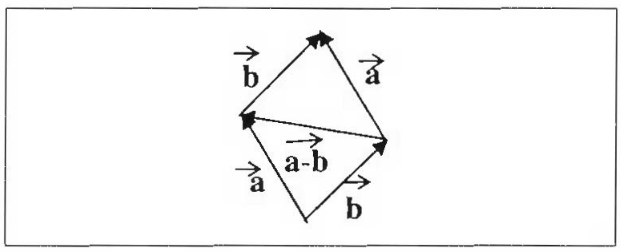
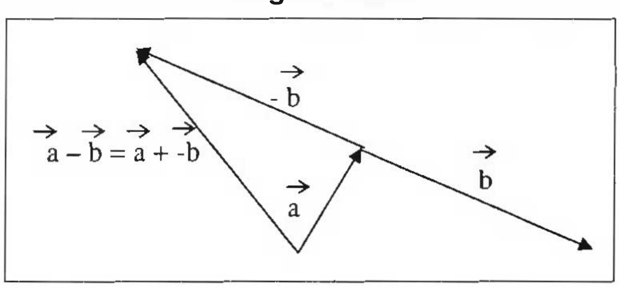



**Figure 14**
**Subtraction of Vectors Using the Triangle Method**



Knowledge of both methods with their underlying metaphors is necessary for a thorough understanding of operations on vectors. Hsu and Oehrtman (2000) found that students became confused when they were not able to integrate multiple metaphors that could be used to structure a mathematical concept.

## Implications for Teaching

"Mathematical concepts on the surface may seem to be neat and well organized, but underneath, in the workings of the brain, all sorts of conflicts and confusions occur" (Tall 2003). Many theorists feel that teachers would find knowledge of cognitive structures inherent in mathematical concepts useful (Núñez 2000; Núñez, Edwards and Matos 1999). With this understanding, they could assist students to better understand mathematical concepts through appropriate use of metaphors.

Activities can be designed to provide initial grounding for conceptual metaphors (Núñez, Edwards and Matos 1999; Tall 2003). For example, working with scales can provide experience with balance thus developing a basis for metaphoric thinking when solving equations. Grounding metaphors that rely on everyday experiences of students, like playing or even taking part in cultural activities, have been found to have a powerful effect on student understanding (Chiu 2000).

Teachers can strongly encourage the use of metaphors in classroom communication. Using metaphors in classroom discussions encourages students to accept metaphoric thought as a normal method in mathematics. Madden (2001) mentions the importance of social interaction in determining the efficacy and usefulness of patterns of metaphoric thought. Communities of learners, like communities of mathematicians, can share and explain the metaphors they use and adopt or correct them as needed. When metaphors are legitimated and spread among students, metaphoric thought is strengthened (Bazzini 2001).

The importance of metaphoric thinking in the history of mathematics can be highlighted. Making students aware of different metaphors used at various times in the development of concepts like calculus will help them understand why conflicting metaphors sometimes appear in mathematics.

Mathematics is traditionally taught as a collection of techniques, skills and attitudes that students must acquire. Pure logic holds a dominant position. "The body has been ignored because reason has been thought to be abstract and transcendent, that is, not tied to any of the bodily aspects of human understanding ... [Our] bodily movements and interactions in various physical domains of experience are experiential in structure ... and that structure can be projected by metaphor on to abstract domains" (Johnson 1987, XV). A better understanding of the hidden, very ordinary origins of complex concepts in mathematics can only result in more effective learning and teaching.

## Notes

1. Figure 5 was my own attempt to illustrate the unit circle conceptual blend. Later, I discovered that it has a remarkable similarity to figures on pages 390–392 in *Where Mathematics Comes From*. Independent development of the diagram illustrates how particular metaphors compel certain interpretations. It is likely that any graphic representation of the unit circle conceptual blend would closely resemble Lakoff and Núñez's images.

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# Developing Algorithms for Fluency and Understanding: A Historical Perspective

*Gladys Sterenberg*

Recent reforms in mathematics have called for a decreased emphasis on pencil-and-paper computations (National Council of Teachers of Mathematics 2000; Alberta Learning 1997). However, a tension between fluency and mathematical understanding exists in many elementary classrooms as teachers grapple with the place of algorithms in the curriculum. A review of the historical development of addition and subtraction algorithms suggests that this tension is not a recent phenomenon. Moreover, many of the standard algorithms used in Canadian classrooms today do not represent the most efficient or pedagogically sound approach to adding and subtracting (Carroll and Porter 1998). By considering the historical development of algorithms, perhaps we can reframe the link between fluency and understanding.

Many of the algorithms used today can be traced to the work by Islamic mathematicians in the eighth century. Of particular importance was Muhammad ibn-Musa al-Khwarizmi, who presented a variety of algorithms in his book on addition and subtraction. In fact, our word algorithm is derived from his name. His book demonstrated the advantages of using a numeration system involving place value and a base of ten. Interestingly, the development of the Hindu-Arabic numeration system coincided with the creation of algorithms and increased mathematical activity in these societies. Barnett (1998, 76) suggests:

> Not only did each new success give rise to the possibility of yet more success in the future, but the newly developed algorithms also allowed mathematicians to concentrate their creative energy on more complicated problems without having to think about the earlier "steps." This suggests that the search for algorithms is—in a very real sense—the driving force of mathematical development.

Prior to the adoption of the Hindu-Arabic numeration system, western societies did not use algorithms for adding and subtracting. Mathematical development stalled there during the European Dark Ages. Finally, in the 12th century, Fibonacci introduced notations and algorithms of al-Khwarizmi to Western Europe. The "new math" created much controversy as mathematicians argued with abacus users about potential benefits and advantages of algorithms over calculating devices.

New algorithms that combined calculating devices and algorithms for adding and subtracting emerged in the 15th and 16th centuries. One such example is the use of reckoning on lines by merchants to determine customer purchases (Mason 1998). This algorithm uses manipulatives to reinforce ideas of place value, regrouping and trading equals for equals, thus appearing to deepen mathematical understandings.

Although new algorithms continued to emerge, the debate over their importance remained unresolved. This debate continued throughout much of the 16th century and is still reflected today in current discussions on the use of calculators and paper-and-pencil algorithms. Although controversial, the debate about the importance of algorithms sparked a resurgence of mathematical development. Indeed, the mathematical creations of Descartes, Fermat, Newton and Leibniz in the 17th century are still evident in today's lessons on analytic geometry and calculus. The connection between the creation of algorithms and mathematical development appears to be strong.

In the 19th and 20th centuries, algorithms for adding and subtracting became increasingly abstract. Emphasis was placed on memorizing the steps of the procedure and connections to manipulatives were diminished. Place value identifiers were dropped. Mathematics became disconnected from the physical world and focused on axiomatic structures of mathematicians (Jones and Coxford 1970). Gaining fluency through the use of algorithms seemed to become the primary goal of mathematics.

However, the tension between mathematical fluency and understanding continued. As arithmetic became an elementary school subject at the end of the 19th century, educators became increasingly concerned with student understanding. Jones and Coxford (1970, 32) write

> Mental discipline as a viable goal of education, and drill as a procedure, were retained along with other newer goals and processes for more than

thirty years after 1894, but the three step process of 'state a rule, give an example, practice' was yielding to inductive, reasoning, and discovery-teaching processes.

In the first textbook approved for use in Alberta, Kirkland and Scott (1895) address the problem of teaching rules by differentiating their approach from conventional textbooks: "The rule is given as a convenient summary of the methods employed in the solutions of the examples which precede it. The aim has been to lead the pupil to derive his own methods of operation" (p. iv). Algorithms for addition are presented with examples, place-value labels have been included, and teachers are encouraged to demonstrate regrouping using "bundles of splints bound together with India rubber bands" (p. 13). Several algorithms for subtraction are presented, including decomposition and equal additions. The student is not told which one to use and, presumably, alternative algorithms are acceptable.

<table>
<tr><td>Method A<br>368<br>+ 453<br>700<br>110<br>11<br>821</td><td>Begin with the hundreds, then the tens, then the ones. Record each sum. Add them together. This method is sometimes called partial sums (Reys et al. 2004).</td></tr>
<tr><td>Method B<br>300 + 60 + 8<br>+ 400 + 50 + 3<br>700 + 110 + 11<br>800 + 20 + 1<br>821</td><td>Write each number in expanded form. Add the hundreds, tens and ones. Regroup each to get the expanded form of the answer. Write the answer in standard form (Nova Scotia Department of Education 2002).</td></tr>
<tr><td>Method C<br>368 = 3 hundreds and 6 tens and 8 ones<br>+ 453 = 4 hundreds and 5 tens and 3 ones<br>7 hundreds and 11 tens and 11 ones<br>7 hundreds and 11 tens and (1 ten and 1 one)<br>7 hundreds and (11 tens and 1 ten) and 1 one<br>7 hundreds and 12 tens and 1 one<br>7 hundreds and (1 hundred and 2 tens) and 1one<br>(7 hundreds and 1 hundred) and 2 tens and 1 one<br>8 hundreds and 2 tens and 1 one<br>821</td><td>Write each number in expanded form using place value names. Add the hundreds, tens and ones. Regroup the ones if necessary. Put the tens together. Write in a simpler way. Regroup the tens if necessary. Put the hundreds together. Write in a simpler way. Write the answer in standard form (Lock 1996).</td></tr>
<tr><td>Method D<table><tr><td>368<br>+ 453<br>7</td><td>368<br>+ 453<br>71<br>8</td><td>368<br>+ 453<br>711<br>82</td><td>368<br>+ 453<br>711<br>821</td></tr></table></td><td>Begin with the hundreds to get 7, then add the tens to get 11. Change the 7 to an 8 because of the additional 100. Add the ones to get 11. Change the 1 to a 2 because of the additional 10 (Nova Scotia Department of Education 2002).</td></tr>
<tr><td>Method E<table><tr><td><br>368<br>453<br>+ 128</td><td>1<br>368<br>453<br>1<br>128<br>9</td><td>11<br>368<br>7<br>453<br>2<br>128<br>49</td><td>1<br>368<br>4<br>453<br>8<br>128<br><br>949</td></tr></table></td><td>Begin by adding the ones of the first two numbers to get 11. Record this number by writing the ones digit and putting the tens digit above the tens column. Add the ones from the third number to this to get 9, recording the new ones number and putting the tens digit above the tens column if necessary. When the ones column is complete, repeat with the tens column and the hundreds column. This is known as Hutchings' low-stress algorithm (Lock 1996)</td></tr>
</table>

Perhaps the tension between fluency and mathematical understanding can be reframed. Instead of viewing them as ends of an either/or dichotomy, historical accounts suggest they are deeply interconnected. Barnett (1988, 77) cautions,

> We have encountered this same difficulty [decreased mathematical understanding] in the past when we allowed the current algorithms we teach to become an end in themselves. Our challenge as educators is to identify what is being learned from the algorithm (whether it be a traditional one or not) besides the ability simply to execute it.

Encouraging students to create alternative algorithms for adding and subtracting could strengthen the links between fluency and understanding. By connecting the development of algorithms to mathematical knowing, educators can begin to reconsider how addition and subtraction are taught and learned.

## Alternative Algorithms for Adding

Presented below are several different methods for adding. These algorithms show step-by-step procedures for computing sums that have been used at some point in the history of mathematics education. I suggest that you familiarize yourself with the procedure by trying it out. Create new problems for yourself to develop your understanding. Think about why the procedure works.

## Alternative Algorithms for Subtracting

Presented below are several different methods for subtracting. These algorithms show step-by-step procedures for computing differences that have been used at some point in the history of mathematics education. Again, I suggest that you familiarize yourself with the procedure by trying it out, creating new problems and thinking about why the procedure works.

| | |
|---|---|
| **Method A**<br>81<br>– 25<br>56 | Begin with the ones column. Subtract the second digit from the first. If necessary, regroup the tens of the first number and rename the ones by increasing the ones by ten and diminishing the tens by one. Repeat with the tens column. This method is known as the decomposition algorithm and is the standard algorithm used in Canadian classrooms. |
| **Method B**<br>81<br>– 25<br>56 | Begin with the ones column. Subtract the second digit from the first. If necessary, add 10 to the top number and add 10 to the second number. Do this by increasing the ones by 10 in the first number and the tens by 1 in the second number. Repeat with the tens column. This method is known as the equal additions algorithm. It was taught in North America until the 1940s (Cathcart et al. 2003). |
| **Method C**<br>81 + 5 = 86<br>– 25 + 5 = 30<br>56 | Add a number to the second number (subtrahend) to make it a multiple of 10. Add this same number to the first number (minuend). Subtract. This is known as subtraction by base complement additions (McCarthy 2002). |
| **Method D**<br>81 + 50<br>– 25 + 5<br>+ 1<br>56 | Consider how far apart 81 and 25 are. Begin from 25 and add, in steps, the numbers that bring you closer to 81. Record these numbers. When you reach 81, find the sum of these numbers. This will be the answer. This method is known as adding up (Carroll and Porter 1998). |
| **Method E**<br>81<br>– 25<br>60<br>– 4<br>56 | Move left to right. Begin with the largest place value (in this case, 10s). Record the difference between the two numbers in each column. If the first number (subtrahend) is larger, the difference is recorded as a deficit or a negative number. Combine the partial differences (Carroll and Porter 1998). |

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# Pi in All Its Glory

*Sandra M. Pulver*

As William Schaaf (1998) in *The Nature and History of Pi* remarked, "Probably no symbol in mathematics has evoked as much mystery, romanticism, misconception and human interest as the number $\pi$."

Humans lived for millions of years before the significance of $\pi$ was grasped. Circles surrounded them in many forms other than the wheel, including the pupil of the eye and heavenly bodies like the sun and moon. But it was only after the appearance of organized society, approximately 2000 BC, that a relationship between the diameter of a circle and its area was recognized such that

circumference : diameter = constant for all circles.

An Egyptian scribe named Ahmes, circa 1650 BC, showed in the Rhind Papyrus that the ratio of the circumference to the radius equals 256/81 or 3.160493827—Ahmes's value was off by less than 1 per cent from the true value of pi. However this value did not become known because a thousand years later the Babylonians and early Hebrews simply used 3 for pi. In the Bible, both 1 Kings 7:23 and 2 Chronicles 4:2 contain the following verse: "Also he made a molten sea of ten cubits from brim to brim [the diameter], round in compass, ... and a line of thirty cubits did compass it round about."

In the fourth century BC, Antiphon and Bryson of Heraclea attempted to find the area of a circle using the principle of exhaustion. They took a hexagon, found its area and then continued to double its sides and double them again until the polygon almost became a circle. Antiphon first estimated the area of a circle by inscribing the polygon in a circle and then calculating the area as each successive polygon came closer to being a circle. Bryson calculated the area of two polygons, one inscribed in a circle and one circumscribed around a circle. The area of a circle would then fall between the areas of the two polygons.

Two hundred years later, Archimedes of Syracuse (287–212 BC) was the first mathematician to produce a method of calculating pi to any degree of accuracy. He doubled the sides of two hexagons four times, resulting in two 96-sided polygons. Using polygons inscribed and circumscribed in a circle, he obtained for pi the bounds

3 10/71 < pi < 3 1/7

or in decimal notation, 3.140845... < pi < 3.142857..., less than three ten-thousandths from the true value. This method of computing pi by using regular inscribed and circumscribed polygons is known as the classical method.

The next person of importance to take the pi challenge was the astronomer Claudius Ptolemy (AD 87–165) who used a 192-sided polygon. In his text *Megale Syntaxis tes Astronomias*, he stated that pi was 3° 8' 30" in the sexagesimal system, or 3 + 8/60 + 30/3,600 which is 3.14166667. His value of pi was within 0.003 per cent of the correct value.

The Chinese were considerably more advanced in arithmetical calculations than their western counterparts, because in AD 264 Lui Hui calculated the value of pi to be between 3.141024 and 3.142704 using the same method as Antiphon and Bryson. In the fifth century, Tsu Ch'ung-Chih and his son, Tsu Keng-Chih, used polygons with 24,576 sides (they began with a hexagon and doubled the sides 12 times) and determined that pi was approximately 355/133 which equals 3.1415929. This is only 8 millionths of 1 per cent from the real value of pi, a value not found in the western world until the 16th century.

About AD 530, the great Indian mathematician Aryabhata came up with an equation that he used to calculate the perimeter of a 384-sided polygon, finding it to be $\sqrt{9.8684} \approx 3.1414$.

Brahmagupta (598–670), another famous Indian mathematician, said that the value of pi was $\sqrt{10}$. First he calculated the perimeter of inscribed polygons with 12, 24, 48 and 96 sides and he got $\sqrt{9.65}$, $\sqrt{9.81}$, $\sqrt{9.86}$, $\sqrt{9.87}$. Then he thought that as the polygons approached the circle, the perimeter and therefore pi, would approach $\sqrt{10}$. Of course, he was quite wrong. He didn't see that his square roots were converging to a number significantly less than the square root of 10. In fact, the square of pi is just over 9.8696. Nevertheless, this was the value he expounded, and many mathematicians throughout the middle ages used it.

Since the middle of the first millennium, many other mathematicians came up with values of pi, but none of them was more accurate than the early Greek, Chinese and Indian calculations. In fact, it was not until the late 16th century that another significant step was taken.

In 1579, a French lawyer and mathematician, François Viète (1540–1603), used the Archimedean method of inscribed and circumscribed polygons to determine that $3.1415926535 < \pi < 3.1415926537$.

To achieve this, he doubled the sides of two hexagons 16 times and got two 393 216-sided polygons.

In 1593, he broke down his polygons into triangles and found that the ratio of perimeters between one regular polygon and a second polygon with twice the number of sides equalled the cosine $\theta$. With this identity in hand, he used the half angle formula and found a way to describe $\pi$ as an infinite product:

$$\frac{2}{\Pi} = \sqrt{\frac{1}{2}} \times \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}} \times \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}} \times \ldots$$

This was probably the first time anyone had used an infinite product to describe anything, and it was one of the first steps in the evolution of mathematics toward advanced trigonometry and calculus. However, even though the equation was a breakthrough, it was of little use when it came to actually calculating $\pi$ because it was very complicated to perform the square root calculations.

Adriaen van Roomen (1561–1615), also known as Adrianus Romanus, a Dutch mathematician, calculated $\pi$ correct to 15 decimal places by using an inscribed polygon that had over 100 million sides. Ludolfvan Ceulen (1540–1610), a German mathematician, calculated $\pi$ to 20 decimal places, using the same classical method, but using polygons that had more than 32 billion sides. When he died in 1610, he had calculated 35 digits of $\pi$. In Germany today, $\pi$ is still sometimes referred to as the Ludolfian number in his honour.

After van Ceulen, mathematicians came up with new ideas to calculate $\pi$ more efficiently. In 1655, John Wallis (1616–1703) discovered a formula that, to this day, bears his name:

$$\frac{\Pi}{2} = \frac{2\times2\times4\times4\times6\times6\times8\ldots}{1\times3\times3\times5\times5\times7\times7\ldots}$$

Like Viete's, Wallis's equation is an infinite product, but it is different in that it only involves simple operations with no need for messy square roots. He reasoned that the first computed term would be higher than $\frac{\pi}{2}$, the second computed term would be lower than $\frac{\pi}{2}$. The third term would also be higher but closer than the first term. The fourth term would also be lower but closer than the second term and so on. This number would slowly converge to $\frac{\pi}{2}$.

In 1675, the Scottish mathematician James Gregory (1638–1675) obtained the extremely elegant infinite series:

$$\arctan x = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}-\frac{x^{11}}{11}+\ldots \quad -1\leq x\leq 1.$$

Three years later, the German Gottfried Willhelm Leibnitz (1646–1716) inserted $x = 1$ into the series to get:

$$\frac{\pi}{4} = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\ldots$$

This method of convergence was too slow to be put into practical use. It took more than 300 terms to even obtain $\pi$ correct to two decimal places. (But although it took so very many terms, it was still faster than the old inscribing/circumscribing polygon method.)

Isaac Newton (1642–1727) improved on this tedious method using:

$$\arcsin x = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\cdot3}{2\cdot4}\cdot\frac{x^5}{5} + \ldots$$

Substituting $x = \frac{1}{2}$, giving $\arcsin \frac{1}{2} = \frac{\pi}{6}$, this series yields:

$$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{2}\cdot\frac{1}{3\cdot2^3} + \frac{1\cdot3}{2\cdot4}\cdot\frac{1}{5\cdot2^5} + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{1}{7\cdot2^7} + \ldots$$

In this series, calculating just four terms would yield $\pi = 3.1416$.

In 1706, John Machin (1680–1752) used the difference between two arctangents to find 100 digits of pi. He used

$$\frac{\pi}{4} = 4\cdot\arctan\frac{1}{5} - \arctan\frac{1}{239}.$$

This formula turned out to be quite useful, because $\arctan \frac{1}{5}$ is easy to calculate using Gregory's formula and $\arctan \frac{1}{239}$ converges very quickly.

In the middle of the 18th century, one of the greatest and most prolific mathematicians of all times, Leonhard Euler (1707–1783), found many arctangent formulas and infinite series to calculate pi. These formulas converged more quickly than those that came before. Some of his formulas were

$$\frac{\pi}{4} = 2\arctan\frac{1}{3} + \arctan\frac{1}{7}$$

$$\frac{\pi}{4} = 5\arctan\frac{1}{7} + 2\arctan\frac{3}{79}$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$$

$$\frac{\pi^3}{32} = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots \quad 79$$

$$\frac{\pi}{2} = \frac{3\times5\times7\times11\times13\times17\times19\times23\times\ldots}{2\times6\times6\times10\times14\times18\times18\times22\times\ldots}$$

Euler also developed an equation that some believe to be among the most fascinating of all time:

$$e^{i\pi} + 1 = 0.$$

The irrationality of $\pi$ was proven by Johann Heinrich Lambert (1728–1777) and Adrien-Marie Legendre (1752–1883). Lambert investigated certain continued fractions and proved the following:

If $x$ is a rational number other than zero, then tan $x$ cannot be rational.

From this, it immediately followed that: If tan $x$ is rational, then $x$ must be irrational or zero.

(For if it were not so, the original theorem would be contradicted.) Since $\tan\left(\frac{\pi}{4}\right) = 1$ is rational, $\left(\frac{\pi}{4}\right)$ must be irrational and the irrationality of $\pi$ is established.

Legendre proved the irrationality of $\pi$ more rigourously. He wrote:

> It is probable that the number $\pi$ is not even contained among the algebraic irrationalities, i.e., that it cannot be the root of an algebraic equation with a finite number of terms whose coefficients are rational. But it seems very difficult to prove this strictly.

Legendre was correct on both counts; $\pi$ is not algebraic, but transcendental. The equation

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 = 0,$$

where $n$ is finite and all the coefficients are rational, is called algebraic. Numbers that were not merely irrational but that could not even be roots of an algebraic equation are transcendental. It was not at all obvious that such numbers exist.

With the arrival of the age of computers, came $\pi$ calculated to an ever-increasing number of decimal places. In 1947, D. F. Ferguson calculated 808 decimal places for $\pi$. It took the computer one year to do that. In 1949, ENIAC (Electronic Numerical Integrator and Computer) computed 2,037 decimals of $\pi$ in 70 hours. In 1955, NORC (Naval Ordinance Research Calculator) computed 3,089 decimals in 13 minutes.

In 1959, in Paris, an IBM 704 computed 16,167 decimals of $\pi$. Three years later, John Wrench and Daniel Shanks used an IBM 7090 to find 100,265 decimals. In 1966, in Paris, an IBM 7030 computed 250,000 decimal places of $\pi$. In 1967, a CDC 6600, in Paris, computed 500,000 decimals. In 1973, Jean Guilloud and Martine Bouyer used a CDC 7600, in Paris, to compute one million decimals in less than one day.

In 1983, Y. Tamura and Y. Kanada used a HITAC M-280H to compute 16 million decimals of $\pi$ in less than 30 hours. In 1988, Kanada computed 201,326,000 digits in six hours on a Hitachi S-820. In 1989, the Chudnovsky brothers found 1 billion digits. In 1995, Kanada computed 6 billion digits. In 1996, the Chudnovsky brothers found 8 billion. In 1997, Kanada and Takahashi calculated 51.5 billion digits on a Hitachi SR2201 in just over 29 hours. The current record is over 60 billion digits of $\pi$!

This is not the end of our quest for knowledge of $\pi$. The number pi has been the subject of a great deal of mathematical and popular folklore. It has been worshipped, maligned, misunderstood, overestimated, underestimated and worked on by scholars and everyday laymen. People have dedicated their lives in the quest for pi.

As David Blatner said in *The Joy of Pi*, "The search for pi is deeply rooted in our irrepressible drive to test our limits."

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# Explorations with Simulated Dice: Probability and the TI-83+

*A. Craig Loewen*

The probability strand of the mathematics curriculum is most likely the one strand above others that begs for a great deal of exploration and creativity. Many different, fun activities and games can be implemented across the grade levels. The TI-83+ is a useful tool for conducting those explorations while engaging in a wide variety of creative problem-solving activities.

The TI-83+ calculator has several nice features. First, many different applications are available and can be downloaded free from the Texas Instrument Internet site. One is called probability simulation, and I encourage the reader to explore this piece of software, as it is quite powerful and versatile. This application simulates rolling dice, flipping coins, drawing cards, twirling a spinner and drawing coloured marbles from a defined set with or without replacement. A creative teacher could devise many interesting and fun explorations from this application alone. However, one disadvantage is that it restricts the number and type of dice—the user may select up to three dice only, and all dice must have the same number of sides.

A second positive feature of the TI-83+ is that it is programmable, and this dramatically expands our possibilities—we can use the programmability of the calculator to overcome the limitations of the probability simulation application. It is quite easy to devise simple programs to explore situations with many dice with different numbers of sides. The calculator can store the results of the exploration in tables and lists for later analysis, if desired.

Whether the explorations are quite simple or more complex, the TI-83+ calculator can be a powerful and flexible tool.

## Writing a Simple Program on the TI-83+

The user will need to know a few key commands to program most probability explorations. First, the user will need to know how to select a random number. The TI-83+ makes this easy—there are two possible commands: **rand** and **randInt(v1.v2)**. Both commands can be accessed from the fourth menu (PRB) under the Math key.

Try this: Select the **rand** command and press the enter key repeatedly. The calculator will display 10-digit random values between 0 and 1. Now, return to the PRB menu and this time select **randInt(**. Complete the command by adding two values separated by a comma and enclosed by parentheses. Again, press enter several times to see the result. The calculator will provide a series of random integers in the range specified, including the chosen values.

- How could you use this command to simulate the rolling of a 12-sided die?
- What happens when one (or more) of the values you provide is less than zero?

Now, in order to conduct an experiment we will need a command to have the calculator roll the die several times. The **For** command is very useful here. The following mini-program causes the calculator to roll a six-sided die five times and display the results on the screen.

```
PROGRAM:PROB1
:ClrHome
:For(A,1,5)
:Disp randInt(1,
6)
:End
:▮
```

When the program is run, the following might appear on the screen.

```
                 5
                 2
                 3
                 6
                 1
              Done
▮
```

Assume we want the calculator to keep track of the running total for 100 rolls of a six-sided die and display the average. One possible program follows.

```
PROGRAM:PROB2
:ClrHome
:0→T
:For(A,1,100)
:T+randInt(1,6)→
T
:End
:Disp T/100
:
```

- What value should that average approximate? Why?
- What would the average approximate for 100 rolls of an eight-sided die? Why?
- Run the experiment several times. Find the average of these averages. What does this value represent?

We now have enough simple commands to solve a variety of fun and challenging problems.

## Lucky 11

On the table you have a selection of 4, 6, 8, 10, 12 and 20-sided dice. There are several of each kind of dice. You may select any two of the dice and roll them. Your opponent will do likewise, taking turns. The first person to roll lucky 11 wins. Which two dice should you select?

- Try playing the game with your classmates. Have each person select a different combination of dice. Which combination(s) of dice seem to be the best?

A very basic program can be constructed that rolls two dice 500 times and keeps track of how often the value 11 turns up. Presumably the program that generates the result 11 most frequently is also the combination most likely to generate it first. In the following program, simulating the rolling of two 6-sided dice, T is a counter that is increased each time an 11 appears.

```
PROGRAM:PROB3
:ClrHome
:0→T
:For(A,1,500)
:randInt(1,6)+ra
ndInt(1,6)→R
:If R=11:T+1→T
:End
:Disp T
:
```

- When rolling two 6-sided dice, about how many 11s would you expect to appear in 500 rolls?
- How could this program be modified to simulate rolling two 8-sided dice?
- How many different combinations of dice are possible in this problem? Can any combinations be eliminated at the outset of the problem? Why?
- Are you better off to roll two 6-sided dice or a 4-and 8-sided die together?
- Which combination of dice is optimal for rolling an 11? Can you explain your result theoretically?
- Rewrite the program above such that it will calculate the experimental probability (simulated) of rolling an 11 with any combination of the same two dice.

## The Game of Pig

This game appears in the book *About Teaching Mathematics: A K–8 Resource* by Marilyn Burns (1992, 71). In this game a player rolls two 6-sided dice and sums the results. The player must now make a choice: accept the score or roll again. If the player accepts the score, the points are added to the total collected from previous turns and play passes to the left. If the player rolls again, those points are added to the total from the first roll. The player may continue to roll as many times as he or she wants, but if at any time the player rolls a one on either dice, the player loses all the points collected on that turn and play passes to the left. The points collected thus far on each turn are risked each time a player rolls. If at any point in the game a player rolls double ones, the score is reset to zero and the dice pass to the left. The first player to collect 100 or more points wins. This game is appropriately titled because if you get too greedy, eventually you will get caught. What is the probability of rolling at least one 1 on any given roll?

Again, the rolling of dice within this game can be simulated on your calculator. In this program, we simply have the computer check the results of both dice and if either dice equals one then our counter increases by one.

```
PROGRAM:PROB4
:ClrHome
:0→T
:For(A,1,500)
:randInt(1,6)→B
:randInt(1,6)→C
:If B=1 or C=1:T
+1→T
:End
:Disp T
:
```

- What is the probability of rolling at least one 1 on a turn in this game? Can you prove this result theoretically as well?

- How would it change the game (and your strategy in playing the game) if you used two 8-sided dice instead of two 6-sided dice? Two 4-sided dice?
- How would it change the game (and your strategy) if you used three 6-sided dice instead of two 6-sided dice?
- Challenge: What is the likelihood of completing the whole game on one turn (using 6-sided dice); that is, making it all the way to 100 without rolling a single 1?

## Even the Odds

Even the Odds (Loewen and Firth 1994) is a very simple game played with four regular 6-sided dice and a game board that includes a list of the values 1 through 20. On a turn, a player rolls all four dice and may take the value of any one die, sum any two dice, sum any three dice or sum all four dice. The player crosses the chosen value off of his or her game board. The player must cross off all of the even values before crossing off any of the odd values. The first player to cross off all 20 values wins.

When playing this game, it quickly becomes apparent that the most difficult values to cross off are 19 and 20. In fact, typically, crossing off 20 takes several turns. What are the odds of rolling a sum of 20 in a single roll with four regular 6-sided dice?

This problem is very easy to model with the TI-83+. Here the calculator simulates rolling a die four times and sums the values; if the sum equals 20 the counter is increased by one. We need to ask the calculator to do this several times in order to get a reasonable estimate of the probability.

```
PROGRAM:PROB5
:0→C
:For(A,1,1000)
:randInt(1,6)+ra
ndInt(1,6)+randI
nt(1,6)+randInt(
1,6)→B
:If B=20:C+1→C
:End
:Disp C
:
```

The first time the program was run, I obtained a result of 1, implying the probability of rolling a sum of 20 with a single roll is about 1 in 100. The second time the program was run, the result was 4, implying 4 in 100.

- How many rolls of the four dice are necessary to get a reasonable estimate of this probability?
- How could you modify the program to simulate rolling the dice 1,000 times?

- What is the theoretical probability of rolling a sum of 20 with four dice?

How can we calculate a theoretical probability for this problem? To do this we need to know the number of possible combinations of dice that sum to 20 and the number of possible outcomes.

$$\text{Probability}(20) = \frac{\text{Possible Combinations of 20}}{\text{Total Possible Outcomes}}$$

The number of possible outcomes is easy to calculate as it is simply 6 × 6 × 6 × 6 or 64. There are 1,296 possible outcomes.

There are a variety of ways to determine the possible combinations of 20, but I turned to my calculator again and used a simple routine.

```
PROGRAM:AAA
:0→T
:For(A,1,6)
:For(B,1,6)
:For(C,1,6)
:For(D,1,6)
:If A+B+C+D=20:T
+1→T
:End
:End
:End
:End
:Disp T
:
```

By running the program it was shown that there are 35 combinations that total 20. Using our formula, we can show that the probability of rolling a 20 is

$$\text{Probability}(20) = \frac{35}{1296}$$

The theoretical probability of rolling a sum of 20 is about three times in 100 tries.

- Calculate the probability of crossing off the value 19 with a roll of four dice.
- How can you calculate the probability of crossing off the value 18 on a turn? Remember, this can be done with combinations of three or four dice!
- Challenge: Knowing that you can select any one die, the sum of any two dice, the sum of any three dice or the sum of all four dice, calculate the probability of being able to cross off any given number on a turn.

## The Unusual Die

Two players are trying a simple dice game in which they select one die at the start of the game and roll it five times, summing the values rolled. The player with the highest sum after five rolls wins. There are two different dice from which the player may select.

One is a regular six-sided die; the other die has six sides, but unusual values on its sides. The net for die is shown below. Which die should you select to maximize your chances of winning?

The easiest way to simulate this game is to have the calculator roll both dice five times and compare the sums. If the calculator played the game several times, we could get an indication of the better die to choose.

Note that rolling the unusual die is rather like flipping a coin where one side of the coin has a value of six, the other has a value of one. As an alternative, it may be interesting to try acting out this problem with a die and a coin.

In the program below, the values for the regular die are summed and stored in the variable C, while the values for the unusual die are summed in D. The variable M counts the number of times the regular die wins, while the variable N counts the number of times the unusual die wins. The variable Q records ties. What is the variable T used for in this program?

```
PROGRAM:PROB6
:0→M:0→N:0→Q
:For(A,1,500)
:0→C
:0→■
:For(B,1,5)
:randInt(1,6)+C→
C
:randInt(1,2)→T
:If T=2:6→T
:T+■→D
:End
:If C>D:1+M→M
:If D>C:1+N→N
:If C=D:1+Q→Q
:End
:Disp M,N,Q
:■
```

- Try running the program several times to calculate an average for M, N and Q, or try modifying the program to play the game a larger number of times. Can you draw any conclusions from these simulated experimental probabilities?
- For each die, what is the theoretical average sum after five rolls? What is the theoretical probability of the regular die winning? Of the unusual die winning? Of obtaining a tie?
- Which die is the better die for this game?

## Horse Race

The Horse Race game is played with two or more players. Players each roll a single regular six-sided die simultaneously and repeatedly. A player must roll a one before he or she may roll for the two and so on. The first player to roll all of the values on the die in order wins. On average, how many rolls are required to roll each of the values one through six in order?

Before we construct a program, it is worth considering the given task a little more closely. At first the task feels a bit overwhelming because it is hard to comprehend the number of rolls necessary to finish the race. But, ask yourself this: is there any difference in the likely number of rolls necessary to obtain a two as compared with a one? No! The die does not care what we choose as our target value. In other words, the task is really just the same as racing to be the first to roll a one—six times over! We can simplify our problem, then, by asking this question: what is the average number of rolls necessary to roll a one?

The following program instructs the calculator to keep picking random digits between one and six inclusive until a one is selected, keeping track of the number of selections in the variable T. This program simulates rolling a die until a one appears.

```
PROGRAM:PROB7
:0→T
:0→X
:While X≠1
:randInt(1,6)→X
:1+T→T
:End
:Disp T
:■
```

Of course, when we run the program as it is shown above, we only find the number of rolls until the first one appears. We may be lucky and get it on the first roll, or it may take several rolls. I ran the program several times, and some experiments required in excess of 25 rolls! Sometimes I got it on the first try. We can have the TI-83+ calculate an average for us by modifying the program slightly, placing our **While** loop inside of a **For** loop so as to repeat the experiment several times. The variable Q is used to calculate an average of the number of rolls required in each experiment.

```
PROGRAM:PROB7
:0→Q
:For(A,1,100)
:0→T
:0→X
:While X≠1
:randInt(1,6)→X
:1+T→T
:End
:Q+T→Q
:End
:Disp Q/100
:■
```

- Modify the program above to simulate rolling until you have obtained 500 ones.
- Modify the program above to simulate rolling for a two instead of a one. How does this affect the required number of rolls?
- Calculate the theoretical average number of rolls required to roll each value from one to six in sequence.
- What is the probability of rolling your way straight through to the finish line in six rolls?
- How are the probabilities affected if you must roll all of the even values in sequence before you may roll the odd values, also in sequence?
- What is the average number of rolls needed to complete the horse race with an eight-sided die? with a ten-sided die?

## Conclusion

Unquestionably, there are a few limitations in programming on the TI-83+, primarily among them the calculator's processing speed and data entry functions. However, the TI-83+ is a surprisingly powerful tool in conducting simple probability explorations. With a surprisingly small number of commands, the calculator can easily simulate rolling dice and thus enable the exploration of a wide variety of fun and challenging problem-solving activities.

## References

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# Mathematical Stories for the Junior High Classroom: An Annotated Bibliography

*Gladys Sterenberg*

Stories have always existed. In earliest times, these stories were communicated verbally and pictorially. Over time, additional stories were written, and the study of human culture became centred on fine arts and literature. A fracture between arts and sciences emerged, and mathematics was placed firmly in the latter category.

For much of the 20th century, we settled into a pattern portraying mathematics as tenseless and timeless. We communicated mathematics through graphs, equations, proofs and algorithms. Our texts of mathematics were the products or artifacts of mathematical thinking. We seemed to have forgotten that mathematical texts throughout history included narrative letters, explanations, poetry and word problems—the texts of patterned, storied thought.

The separation of mathematics from the humanities is no longer feasible. Returning to and expanding the notion that mathematics is socially constructed and negotiated, mathematics educators and researchers are promoting new curricula that emphasize the mathematical processes of communication and connections. We are beginning to understand that the development of mathematical concepts occurs in a contextual and relational manner and that this context can provide meaning. When mathematics is placed in a social and cultural context, we can think of mathematics as humanity. Using stories in mathematics classrooms enables us to experience the human dimensions of mathematics.

Considering mathematics as story suggests that using literature in school mathematics humanizes mathematics. By challenging common misconceptions of mathematics as a disconnected set of rules and procedures to be memorized, and of mathematicians as isolated social loners, stories show mathematics as part of human culture. Perhaps this is the most compelling reason for teaching and learning through literature. Pragmatically, using literature integrates learning across curricular areas, thus addressing the issue of limited time resources. Students are interested in stories, and literature provides an alternative way for communicating about mathematics. This bibliography attempts to explore possible ways of using literary resources in middle school mathematics classrooms.

The bibliography is composed of picture books, puzzle books, novels and non-fiction writings organized into five sections: number; patterns and relations; shape and space; statistics and probability; and puzzles and recreational problems. Each book was read and analyzed on the basis of mathematical and literary standards. Particular elements noted in each entry include mathematical concepts, text features, possible teaching suggestions, and the place of the entry in the program of studies. A comprehensive bibliography of books for Grades 1–12 entitled *Once Upon a Mathematical Time* is available at www.ionemaste.ca/homepage/resources.html.

When mathematics is presented vibrantly and creatively, students begin to appreciate and understand mathematical concepts. By linking mathematics and literature, the role mathematics plays in our society can be investigated. It is hoped that teachers will explore the potential of these books to promote mathematical thinking in their classrooms.

## Number Strand

***The Curious Incident of the Dog in the Night-Time* by M. Haddon, 2002. Toronto, Ont.: Doubleday. ISBN 0385659792.**

Math Concepts: arithmetic operations can be used to solve problems in logical ways (explicit)

Text Features: novel; main character is a 15-year-old autistic boy; language warning

Teaching Suggestions: students can solve the problems presented in the text; students can discuss the experience of living in a world that is interpreted literally

Program of Studies: Grade 7—SO #14, 15, 16, 21; Grade 8—SO #9, 10; Grade 9—SO #7, 8

***Erin McEwan, Your Days Are Numbered* by A. Ritchie, 1990. New York: Alfred A. Knopf. ISBN 0679803211.**
Math Concepts: numbers can be used to solve problems (explicit)
Text Features: novel; imperial measurements used
Teaching Suggestions: students can construct mathematics questions that arise from situations involving consumer sales; students can use metric measurements to convert decimals into fractions
Program of Studies: Grade 7—SO #4, 6, 6, 7, 13, 14, 15, 17, 18, 21; Grade 8—SO #3, 6, 10, 12, 13; Grade 9—SO #7, 8

***The Essential Arithmetricks* by K. Poskitt, 1999. London: Scholastic. ISBN 0439011573.**
Math Concepts: algorithms can be used to demonstrate proficiency with calculations; understanding numerical patterns can encourage the development of a number sense for decimals (explicit)
Text Features: information text; includes table of contents; cartoon drawings
Teaching Suggestions: chapters can be read and discussed throughout the teaching unit
Program of Studies: Grade 7—SO #3, 4, 5, 6, 13, 14, 15, 17; Grade 8—SO #10; Grade 9—SO #1, 7, 8

***Fabulous Fractions* by L. Long, 2001. New York: John Wiley & Sons. ISBN 0471369810.**
Math Concepts: numbers can be represented as fractions; problems can be solved using arithmetic operations with fractions (explicit)
Text Features: games and activities book; includes contents and index
Teaching Suggestions: problems presented in the text
Program of Studies: Grade 7—SO #4, 5, 6, 7, 21; Grade 8—SO #3, 6, 9, 10; Grade 9—SO #1, 2, 7, 8

***Mathematickles* by B. Franco, 2003. New York: Simon & Schuster. ISBN 0689843577.**
Math Concepts: number operations can be used to express relationships (explicit)
Text Features: poetry; language and number operations are combined into playful equations; colourful illustrations (picture book)
Teaching Suggestions: students can write their own mathematical poetry (for example, crisp air + shadows tall + cat's thick coat = signs of fall)
Program of Studies: Grades 7–9—General Outcomes: develop and demonstrate a number sense; apply arithmetic operations while solving problems

***Much Bigger Than Martin* by S. Kellogg, 1992. New York: Penguin Books. ISBN 0140546669.**
Math Concepts: ratios can be used to solve problems (implicit)
Text Features: narrative picture book
Teaching Suggestions: students can calculate the height of the person throughout the book using ratios to compare the sizes of body parts
Program of Studies: Grade 7—SO #19, 20

***The Number Devil* by H. M. Enzensberger, 1997. New York: Henry Holt & Company. ISBN 0805062998.**
Math Concepts: numbers can be represented in multiple ways; numbers can be used to solve problems (explicit)
Text Features: novel; colourful artwork; humorous
Teaching Suggestions: students can generate and extend the number patterns presented in the text
Program of Studies: Grade 7—SO #1, 3, 4, 5, 6, 13, 14, 20, 21; Grade 8—SO #3, 7, 8, 9, 11; Grade 9—SO #1, 2, 3, 4, 7, 8, 9

***On Beyond a Million* by D. M. Schwartz, 2001. New York: Dragonfly Books.**
Math Concepts: numbers can be expressed as powers with exponents and bases (explicit)
Text Features: picture book; cartoon drawings; sidebars provide additional information
Teaching Suggestions: students can express large numbers in scientific form
Program of Studies: Grade 7—SO #1, 2

## Patterns and Relations

***Anno's Mysterious Multiplying Jar* by M. Anno and M. Anno, 1999. New York: Philomel Books. ISBN 0698117530.**
Math Concepts: patterns can be expressed in terms of variables; variables and equations can be used to express and summarize relationships (explicit)
Text Features: picture book; includes afterword; recursive ending
Teaching Suggestions: as the book is read, students can develop their own system of notation; introduce students to factorial notation
Program of Studies: Grade 6—SO #1, 2, 3, 4; Grade 7—SO #1, 2, 3, 4; Grade 8—SO #1, 2, 3; Grade 9—SO #2, 3

***The Countingbury Tales* by M. de Guzmán, 2000. River Edge, N.J.: World Scientific. ISBN 9810240333.**
Math Concepts: games and beauty often compel mathematicians to develop concepts (explicit)
Text Features: information book; includes table of contents and bibliography; historical; each chapter differs in degree of difficulty
Teaching Suggestions: activities are presented in the text
Program of Studies: Grade 7—SO #1, 3, 4; Grade 8—SO #1; Grade 9—SO #1

***Fascinating Fibonaccis: Mystery and Magic in Numbers* by T. H. Garland, 1990. Palo Alto, Calif.: Dale Seymour. ISBN 0866513434.**
Math Concepts: patterns can be used to describe the world and to solve problems (explicit)
Text Features: information book; includes diagrams, a few proofs and historical notes
Teaching Suggestions: students can express patterns using variables
Program of Studies: Grade 7—SO #1, 2, 3, 4, 5, 6, 7, 8, 9; Grade 8—SO #1, 2, 3, 6; Grade 9—SO #1, 2

***A Gebra Named Al* by W. Isdell, 1993. Minneapolis, Minn.: Free Spirit. ISBN 091579358X.**
Math Concepts: patterns can be expressed using variables (explicit)
Text Features: novel; includes table of contents, a map of mathematics, and a list of characters
Teaching Suggestions: integrate with science unit on the periodic table
Program of Studies: Grade 7—SO #4, 5, 6, 7, 8, 9; Grade 8—SO #1, 2, 4, 6; Grade 9—SO #1, 4, 6

## Shape and Space

***Around the World in Eighty Days* by J. Verne, 1873. United Kindgom: Oxford University Press, 2000.**
Math Concepts: periods of time can be measured (explicit)
Text Features: novel; includes full-page coloured illustrations
Teaching Suggestions: students can construct a timeline of the journey
Program of Studies: Grade 7—SO #3, 4

***Circles: Shapes in Math, Science and Nature* by C. S. Ross, 1998. Toronto, Ont.: Kids Can Press. ISBN 1550740644.**
Math Concepts: everyday phenomena can be described and compared using circles (explicit)
Text Features: includes historical notes; contains contents, circle formulas, answers, a glossary, and an index; metric measurements are given
Teaching Suggestions: Pi is presented incorrectly as 3.14; activities and games are presented in the text
Program of Studies: Grade 7—SO #1, 2

***Four Colours Suffice* by R. Wilson, 2003. Princeton, N.J.: Princeton University Press.**
Math Concepts: design problems can be explored using properties of networks (explicit)
Text Features: historical non-fiction; includes a table of contents, a preface, notes and references, a chronology of events, a glossary and an index; contains photographs and diagrams; the text is dense and is at a high reading level
Teaching Suggestions: students can investigate the four-colour problem and other problems using various maps and diagrams presented in the text
Program of Studies: Grade 8—SO #12

***Holes* by L. Sachar, 2000. New York: Yearling. ISBN 0440414806.**
Math Concepts: everyday phenomena can be described and compared using measurement; the effects of dimension changes in 3-D objects can be described using volume measurements (implicit)
Text Features: novel; national book award winner
Teaching Suggestions: students can calculate the volume of the dirt removed from the holes and the surface area needed for the resulting conical piles
Program of Studies: Grade 8—SO #3, 4, 5, 7, 9; Grade 9—SO #5, 11, 12

***The Librarian Who Measured the Earth* by K. Lasky, 1994. Boston, Mass.: Little, Brown. ISBN 0316515264.**
Math Concepts: similar triangles may be used to solve problems; angle measurements are linked to the properties of parallel lines (explicit)
Text Features: biography of Eratosthenes; picture book; includes the author's note, an afterword and a bibliography
Teaching Suggestions: students can replicate Eratosthenes' system of measurement using e-mail partners from another city
Program of Studies: Grade 7—SO #1, 2, 5, 6, 7, 9; Grade 8—SO #3; Grade 9—SO #1, 3, 4, 8

***The Library of Alexandria* by K. Trumble and R. M. Marshall, 2003. New York, Clarion Books.**
Math Concepts: mathematics develops within a cultural context (implicit)
Text Features: information book; includes a table of contents, maps, family trees, names and terms, a bibliography, suggested reading lists, and an index; full-page colourful and detailed illustrations; includes short biographical notes on Euclid and Archimedes
Teaching Suggestions: students can determine the volume of a sphere that fits exactly into a cylinder
Program of Studies: Grade 8—SO #4, 7; Grade 8—SO #9; Grade 9—SO #5

***Polyhedron Origami for Beginners* by M. Kawamura, 2001. Tokyo: Nihon Vogue. ISBN 4889960856.**
Math Concepts: 3-D objects can be described and analyzed according to their characteristics and their relationship to 2-D shapes (explicit)
Text Features: activity book; contains brightly-coloured photographs and diagrams; includes step-by-step instructions

Teaching Suggestions: students can construct, identify, and classify polyhedrons
Program of Studies: Grade 8—SO #8, 9

***Sir Cumference and the Dragon of Pi* by C. Neuschwander, 1999. Watertown, Mass.: Charlesbridge. ISBN 1570911649.**
Math Concepts: properties of circles can be used to solve problems; everyday phenomena can be described and compared using measurement (explicit)
Text Features: narrative adventure; the play on words for characters' names reinforces vocabulary
Teaching Suggestions: imperial measurements are used; the mathematically incorrect use of three and one-seventh to describe Pi is corrected on the last page of the book
Program of Studies: Grade 7—SO #1, 2

***This Book Is About Time* by M. Burns, 1978. Boston, Mass.: Little, Brown and Company. ISBN 0316117501.**
Math Concepts: periods of time can be measured (explicit)
Text Features: information book; includes a table of contents, an introduction, and a conclusion; line drawings
Teaching Suggestions: activities are presented in the text
Program of Studies: Grade 7—SO #3, 4

## Statistics and Probability

***Why Do Buses Come in Threes?* by R. Eastaway and J. Wyndham, 2000.**
Math Concepts: everyday phenomena can be described using probability (explicit)
Text Features: information book; includes a table of contents, a foreword, an introduction, references, and an index; contains dense text
Teaching Suggestions: students can investigate the questions posed in each chapter
Program of Studies: Grade 7—SO #9, 10, 11; Grade 8—SO #8, 9, 10; Grade 9—SO #8, 9, 10

## Puzzles and Problems

***50 Mathematical Puzzles and Problems* by G. Cohen, ed., 2001. Emeryville, Calif.: Key Curriculum Press. ISBN 1559534982.**
Math Concepts: logic, symmetry and numbers can be used to solve problems (explicit)
Text Features: collection of puzzles from the International Championship of Mathematics and Logic; includes a preface, a table of contents and solutions
Teaching Suggestions: puzzles are presented in the text
Program of Studies: focuses on number and shape and space strands

***How Math Works* by C. Vorderman, 1999. New York: Reader's Digest.**
Math Concepts: everyday phenomena can be described using mathematics (explicit)
Text Features: activity and information book; historical notes are included; colourful pictures and diagrams; includes a table of contents, a glossary, answers to puzzles and an index
Teaching Suggestions: activities are presented in the text
Program of Studies: all four strands are addressed

***The Man Who Counted* by M. Tahan, 1993. New York: W. W. Norton.**
Math Concepts: throughout history, people have engaged in solving mathematical problems; there are connections between philosophy, religion and mathematics (explicit)
Text Features: narrative; set in the 13th century on the road to Baghdad; answers are provided within the text; historical references to traditional and classic problems are made
Teaching Suggestions: students can investigate the problems as they are introduced and prior to reading the answer
Program of Studies: focuses on the number strand

***Marvels of Math* by K. Haven, 1998. Englewood, Colo.: Teacher Ideas Press. ISBN 1563085852.**
Math Concepts: mathematics develops in a social context and is a dynamic cultural activity (explicit)
Text Features: biographies; a collection of 16 historical stories; includes a table of contents, an introduction and an index; brief summaries, terms to know; follow-up questions and activities are included for each story
Teaching Suggestions: activities presented in the text tend not to support constructivist approaches and need to be adapted
Program of Studies: all four strands are addressed

***Math Trek: Adventures in the Mathzone* by I. Peterson and N. Henderson, 2000. New York: John Wiley & Sons. ISBN 0471315702.**
Math Concepts: numbers, arithmetic, geometry and algebra can be used to solve problems and investigate patterns (explicit)
Text Features: narrative; weak plot; includes a preface, answers, a glossary, further readings and an index; contains photographs, diagrams, drawings and tables

Teaching Suggestions: problems are presented in the text
Program of Studies: focuses on number, shapes and space, and patterns and relations strands

***Women and Numbers* by T. Perl, 1993. San Carlos, Calif.: Wide World Publishing/Tetra. ISBN 093317487X.**
Math Concepts: women are actively engaged in creating new mathematics; numbers can be used to solve problems (explicit)
Text Features: biographies; includes a table of contents, timelines and solutions to activities; the historical backgrounds of conceptual developments are provided
Teaching Suggestions: activities are presented in the text
Program of Studies: all four strands are addressed

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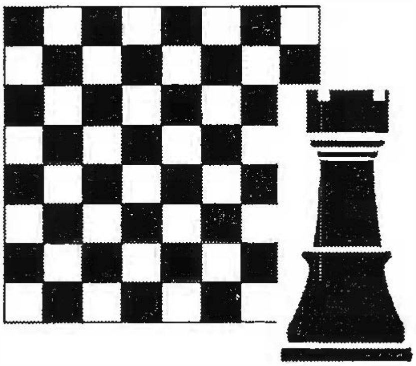
# A Page of Problems

A. Craig Loewen, The University of Lethbridge

## HIGH SCHOOL

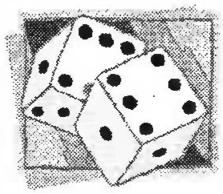
How many rectangles are there on a chessboard?



*Source:* Mason, J. 1985. *Thinking Mathematically.* Don Mills, Ont.: Addison-Wesley.

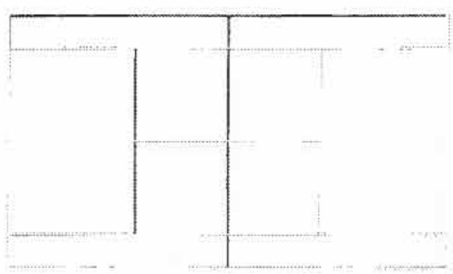
## JUNIOR HIGH

Jane is playing a dice game with Frank. One die is a regular 6-sided die, the other die also has six sides, but has 3 ones and 3 sixes. The game is played by rolling your die five times and summing the values rolled. The player with the highest sum after five rolls wins. Which die should Jane choose if she wants to win?



## ELEMENTARY

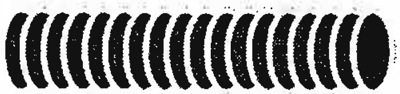
How many rectangles of all sizes are there on a tennis court?



*Source:* Loewen, A. C. 1993. *Mathematical Problem Solving in the Primary Grades.* Barrie, Ont.: Exclusive Educational Products.

## HIGH SCHOOL

1000 coins are placed in a row with the "tails" side up. The first person turns every coin over. The second person turns every second coin over. The third person turns every third coin over, etc. After 1000 people, which coins will show "heads"?



*Source:* Kantecki, C., and L. E. Yunker. 1982. "Problem Solving for the High School Mathematics Student." *Math Monograph no. 7.*

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