Concept and Task Analysis: An Alternative Look at Planning Instruction

Regina M Panasuk

The National Council of Teachers of Mathematics (NCTM 2000) lists the following headings under its process standards:

- Problem solving
- Reasoning and proof
- Communication
- Connections
- Representation

One of the process standards, connections, puts special emphasis on understanding how mathematical ideas interconnect and build on one another to produce a coherent whole. The question is how to help high school students make these connections, particularly students who do not always understand the relationships between the concepts they studied in junior high. This article addresses this question using concept and task analysis and its role in planning mathematics teaching.

Concept and Task Analysis

Concept and task analysis, taken from Gagne's (1970) learning hierarchies and Ausubel's (1968) theory of advance organizers, is a widely accepted idea that is used to inform instruction and facilitate meaningful learning. The purpose of concept and task analysis, or decomposition, is to gain insight into the structure of a given concept or task that students must learn or perform. This is done by breaking down superordinate concepts or tasks into their constituent subconcepts or subtasks (also known as decomposing). This analysis helps classify and categorize students' cognitive processes and actions (Gagne 2001). Many reasoning skills and tasks in which students are engaged are quite complex. They involve various relatively small and distinct subconcepts and subskills that students are expected to learn. For example, operating with positive and negative numbers is a prerequisite for the concept of collecting like terms, which in turn is a prerequisite for solving multistep equations. The flowchart in Figure 1 represents the organization of the subconcepts that constitute a superordinate concept, in this case, the concept of the product of two negative numbers. It also represents the hierarchy of the subconcepts involved in the superordinate concept of the product of two negative numbers. This classification can help the educator grasp the big picture.

Concept and task analysis is the process of progressing from abstract to concrete, and from complex to simple[1]. Students who reason at the concrete or preoperational levels (Piaget 1970) are most successful when they advance from concrete to abstract, as shown in Figure 2a. For these students, if the concept is presented gradually through three different modes, they are able to grasp the relations among the concepts and enhance their thinking skills. The three modes are: concrete (working and operating with real objects, geometrical shapes and simple, hands-on experiments), mental imagery (pictures or diagrams that summarize and represent properties) and abstract (or symbolic) representation (formulas and terminology). Figure 2 illustrates the representations from concrete to abstract (Figure 2a) and from simple to complex (Figure 2b) as two parallel, top-to-bottom processes. At the same time, Figure 2b, when viewed from bottom to top, shows concept and task analysis as a reciprocal process. For example, one way to present a complex concept (such as factoring the binomial $2x + 6$) from concrete to abstract and from simple to complex (a top-to-bottom process) is to decompose or subdivide it into subconcepts (a bottom-to-top process). The binomial $2x + 6$ is factored by applying the distributive property, an abstract concept that is not likely to be understood and successfully applied without knowing the underlying concepts and meaning.

The first subtask (see Figure 2b) might involve multiple subconcepts. For example, it may be necessary to write each term of the binomial as a product to determine the common factor, which in turn requires understanding that multiplication can be represented as repeated addition (for example, $2x = x + x$). The product of two positive quantities can be represented as the area of a rectangle, the adjacent sides of which have the measure equal to the values of the factors of the product. Knowledge of coefficient and of prime and composite numbers may also be required.

Figure 2b also illustrates the inverse operation of factoring [such as presenting the product $2(x + 3)$ as a polynomial], which in turn relates to the concept of order of operations. Students may assume that work inside parentheses should be done first, such as with the example $2(x + 3)$. This would indicate two things. First, those students do not fully understand the nature of variables (x and 3 can be added to obtain a binomial, but until the variable is involved, the binomial cannot be converted to a monomial). Second, those students do not understand the connection between the order of operation and the properties of operations, nor that some problems can be calculated by integrating several concepts (Panasuk, in press). For example, when calculating the value of the numerical expression $\frac{7}{4}\times(0.5+\frac{4}{7})$, $\frac{7}{4}$ can be distributed through through 0.5 (that is, $\frac{1}{2}$) and $\frac{4}{7}$, and then the two products $\frac{7}{4}\times\frac{4}{7}$ and $\frac{7}{4}\times\frac{1}{2}$ can be added to produce $\frac{7}{8}+1=1\frac{7}{8}$.

Understanding the subconcepts and subskills listed above is the easiest way to successfully complete a problem involving a complex concept. Identifying and categorizing these subconcepts and subskills in a hierarchy of knowledge helps teachers make instructional decisions when planning lessons.

Figure 1
Flowchart of the product of negative numbers subconcepts

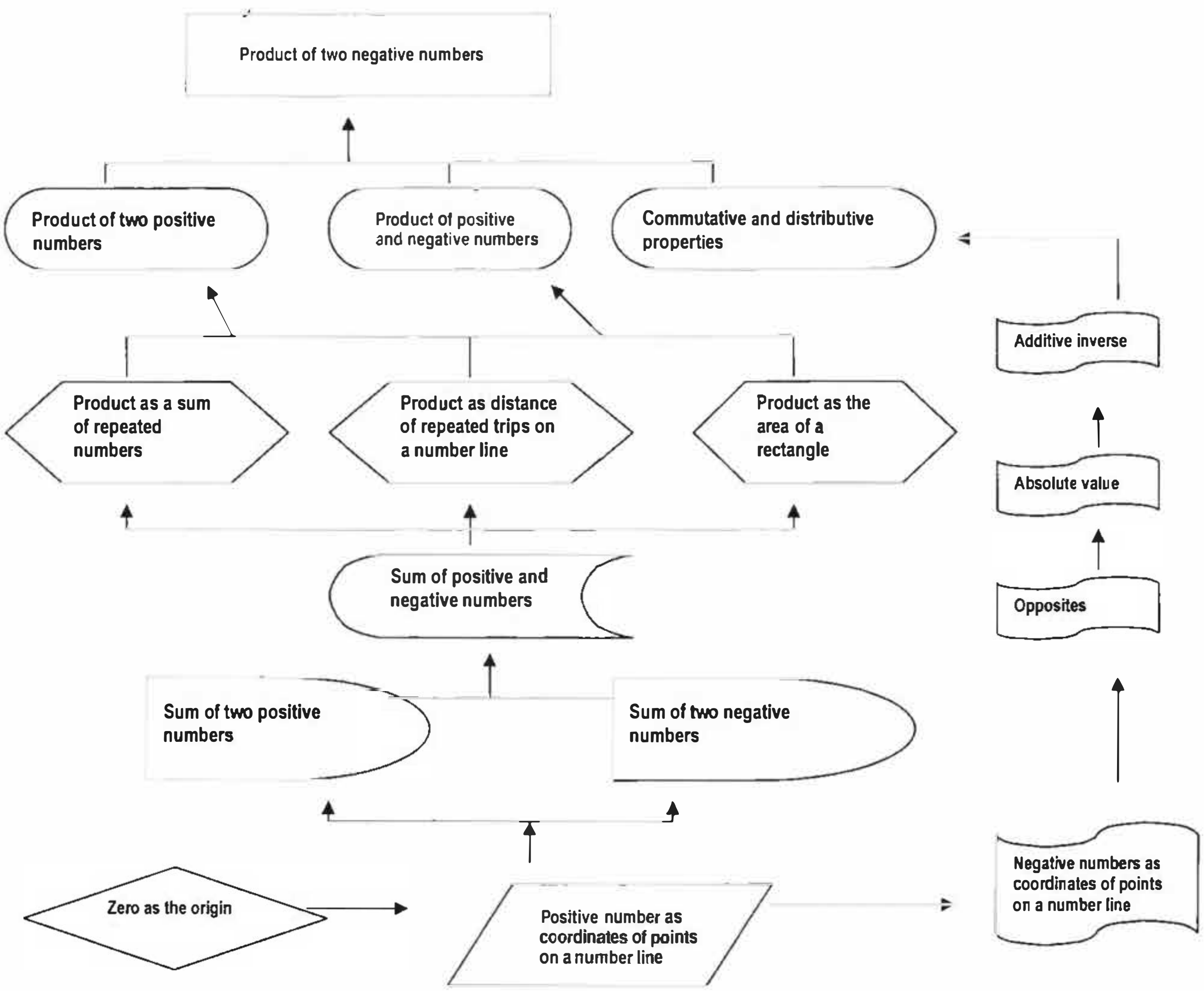
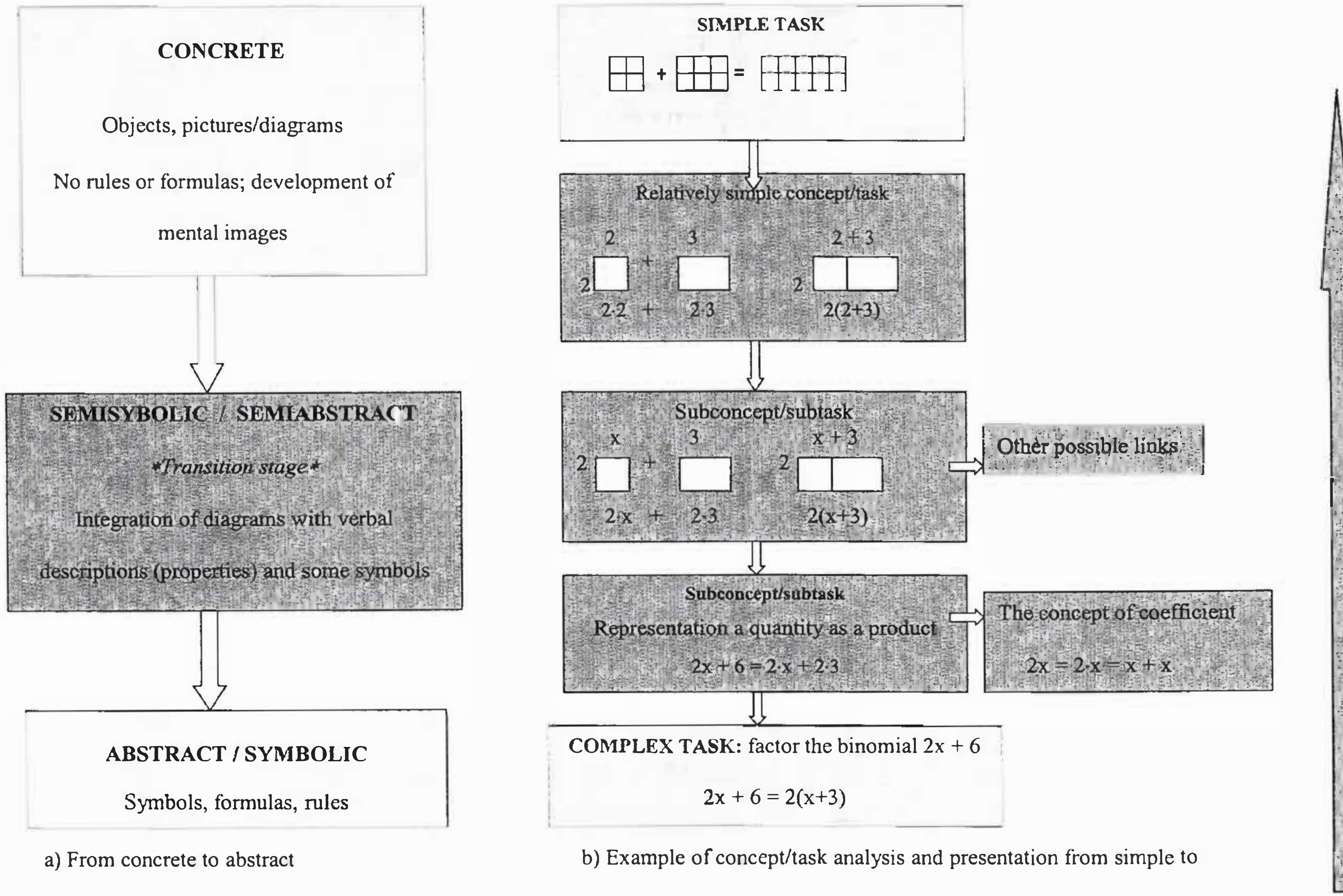


Figure 2

Comparison of the relationship between concept/task analysis and presentation from concrete to abstract and simple to complex



a) From concrete to abstract

b) Example of concept/task analysis and presentation from simple to complex

How do teachers identify smaller connections within a subconcept or subtask? What process do teachers go through while analyzing a concept or task and devising a list of subconcepts and subtasks that students are expected to know? How infinitesimal should the links between the subordinates be? To answer these questions and to address concept and task analysis in more detail, consider the concept of solving linear equations with one variable. Assume that the equation $2(3x - 7) - 2(4 + x) = 2$ is a superordinate concept that must be studied during the upcoming curriculum unit. The decomposition of this high-level task into subtasks and operations, and naming the subconcepts underlying each of the subtasks helps teachers comprehend the nature of the concept. This in turn helps order student experience in a logical and coherent manner. The analysis enables teachers to select and sequence the activities required by ordering those activities from simple and less inclusive to complex and more inclusive.

Concept mapping, as shown in Figure 1, can help students record and organize subordinate ideas. Of course, teachers know how to solve the equations they present in class, and concept mapping may be considered unnecessarily utilitarian. The point, however, is not to find the answer but to scrutinize each step, observe whether the equation can be broken into subtasks, recognize the subconcepts involved in each

Figure 3
Using the line segment to represent a variable

A. 5 7

12

B. a 7

$a + 7$

C. 1. a

b

2. a a $= 2a$

3. b b b $= 3b$

4. (2) and (3) together form a new segment

$2a + 3b$

a a b b b

5. Remove one a-segment and two b-segments from (4)

$2a + 3b - a - 2b$

a b

6. The resulting segment $(a + b)$ has the same length as $(2b + 3b - a - 2b)$

$a + b$

a b

Note: The popular student misconception $2a + 3b = 5ab$ can be treated using the variable-segment idea as well. The product of two positive quantities a and b can be illustrated as the area of a rectangle with sides a and b.

a

b

While $2a + 3b$ represents a line (a one-dimensional figure), $5ab$ would represent a rectangle (a two-dimensional figure).

subtask and delineate the subconcepts by developing a hierarchy of prerequisite knowledge that students must be able to combine in a new structure to master higher-level concepts. The number of steps, the range of detail into which to decompose, and the hierarchy of the subconcepts and subtasks can vary, depending on the teacher's best judgment of the students' prior knowledge, needs, ability to execute a chain of simple tasks and ability to explain their actions. Some students display the desired mathematical skills with minimum guidance and need few ideas to grasp the concept. Others require more support and become competent from seeing more increments. For those students, in particular, missing some steps could contribute to the development of misconceptions.

Solving a linear equation involves executing a set of distinct, relatively simple and sequential tasks. The following is an example of how the equation $2(3x - 7) - 2(4 + x) = 2$ could be solved:

$2(3x - 7) - 2(4 + x) = 2$	
$6x - 14 - 8 - 2x = 2$	(application of the distributive property, and operations with positive and negative numbers)
$4x - 22 = 2$	(collecting like terms by applying associative and commutative properties)
$4x = 24$	(addition of 22 to each side of the equation using additive inverse $[a + (-a) = 0]$)
$x = 6$	(dividing both sides of the equation by 4, using multiplicative inverse $[a \times \frac{1}{a} = 1; a \neq 0]$)

These steps, described using minimal language, are important in the process of solving equations and are a well-defined routine that helps establish and maintain certain order and parsimony. However, many students might view this transformation as a mysterious procedure that cannot be understood and must be memorized and imitated. Because of its refined form, the procedure has a specific logic, which in turn makes it difficult to understand when it is presented in a concise way. Presenting a brief and refined procedure can be a barrier to learning for many students. The procedure makes sense when it is part of a system of related concepts. Unfortunately, accurately performing a procedure will not guarantee that a correct solution to the problem will be found.

A close look at each of the four steps will help reveal the underlying subconcepts that make up the procedure.

Distributive Property

The distributive property is a complex concept, particularly when used with expressions with variables. Errors are often made when the distributive property is applied, perhaps because students have not yet developed a solid visual image of the property (see Figure 2b). It is common for students to forget to distribute a negative sign and/or view the expression $(x - y)$ exclusively as subtraction. Some have difficulty applying the difference property, $(x - y) = x + (-y)$, which encourages thinking of subtraction as addition. To help students understand that they are actually distributing –2, for example, they can be encouraged to perform the expansion using smaller increments, such as $2(3x) + 2(-7) + [-2(4)] + (-2)x = 2$. The transition from this expression to $6x + (-14) + (-8) + (-2)x = 2$ may seem trivial at first glance, but it involves several subconcepts that must not be overlooked. These subconcepts are the order of operation that was discussed earlier, the multiplication of positive and negative numbers, the concept of coefficient, and the rule that multiplication does not distribute over multiplication or division but only over addition and subtraction. Students must know that $(2)(3x) = (2 \times 3)(2 \times x)$ and be able to draw a chart similar to that of Figure 2b. Grades 6, 7 and 8 mathematics teachers who were involved in a study of lesson planning and concept and task analysis indicated that they "entered the study with a wrong assumption." They had been convinced that proceeding with "baby steps" was "too elementary" for the students (Panasuk, Stone and Todd 2002, 821).

Collecting Like Terms

The concept of collecting like terms can be dissected into relatively simple subconcepts, including coefficient; variable; addition and subtraction of positive and negative numbers; and commutative, associative and distributive properties. Each of these subconcepts can be interpreted through different representations. For example, a variable can be represented as a line segment—the method invented and used by ancient mathematicians (and a shift from abstract to concrete). This can help students visualize and operate with the terms of the expressions (see Figure 3). Representing the terms $2a$ and $3b$ in the expression $2a + 3b$ as two segments of length a and three segments of length b may help some students manipulate terms of polynomial expressions. The segments are easily drawn or diagrammed, which makes it easy to see the resulting segment and name its length. A different colour can be used to represent a negative number. To solve the equation $2(3x - 7)$

$-2(4 + x) = 2$, for example, $6x$ and $-2x$ will have to be added. Set up a diagram with six segments of the same length and colour (representing $6x$) and two segments of the same length but different colours (representing $-2x$). Pairs of segments of the same length but different colour behave as additive inverse and cancel each other, leaving a sum of $4x$.

When students lack basic knowledge, teachers sometimes have to revert to the rudimentary level. For example, the expression $6a - 2a$ could be rephrased as "6 apples – 2 apples = 4 apples" and then converted back to $6a - 2a = 4a$. (Dollars could be used to illustrate negative difference: \$2 – \$4 = –\$2.)

Further Decomposition

Further decomposition of the one-step linear equation can also incorporate visual representations of the unknown,[2] be replaced by a word that represents the object or be represented as a line segment. When solving one-step linear equations becomes an obstacle, referencing diagrams like those in Figure 4 could be helpful.

Incorporating Concept and Task Analysis into Planning and Teaching

How can a teacher integrate the knowledge gained from concept and task analysis into planning and teaching? Concept and task analysis is one of the most important ingredients in lesson planning (Panasuk 2002; Panasuk, Stone and Todd 2002). It helps teachers make instructional decisions and helps students recognize and build connections among subconcepts. Teachers who have worked with concept and task analysis possess a better understanding of the prerequisite knowledge and skills that are needed. Teachers are better prepared to identify students' prior knowledge and facilitate learning by choosing and sequencing the activities, problems and/or exercises that integrate the prerequisite knowledge.

Although task analysis is important for each element of planning—including objectives, homework, developmental activities and mental mathematics

Figure 4
Representations of the concept of solving linear equations

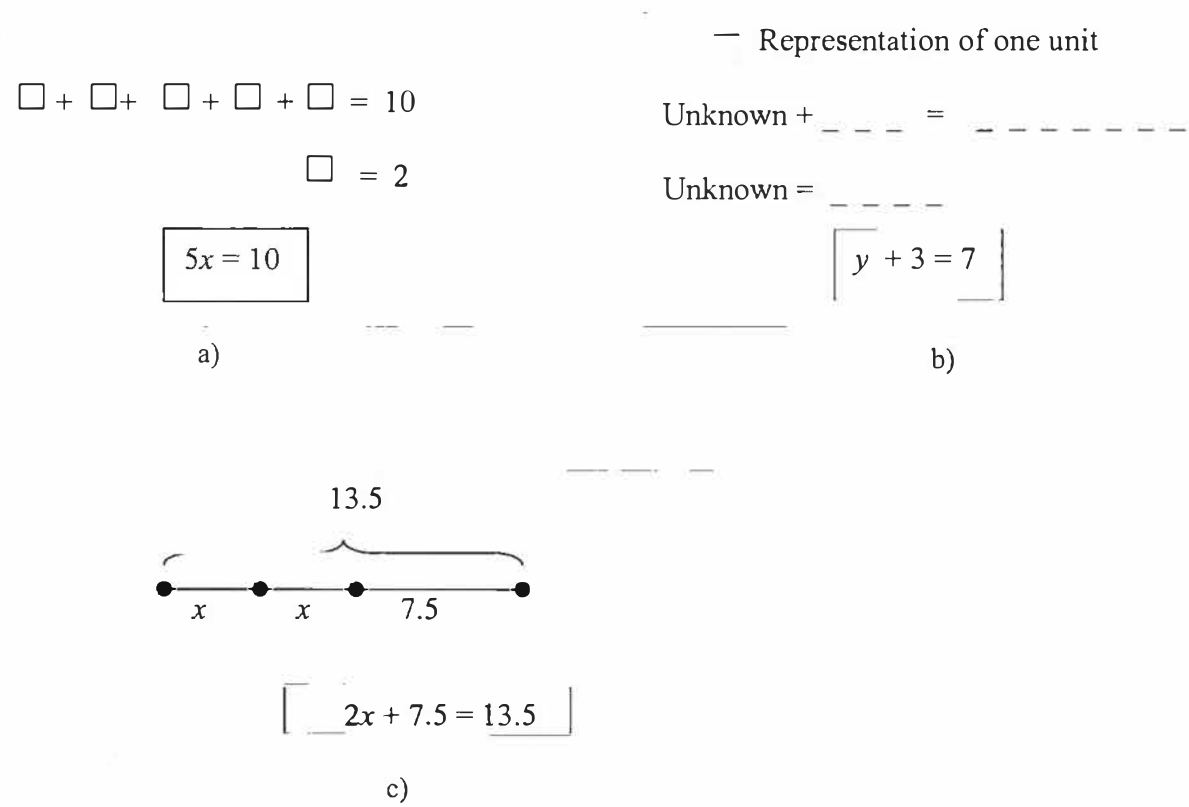


Figure 5

Possible mental mathematics problems for the unit on like terms and operations

Level 1

1. Combine together like (similar) shapes

   ▲ + ▲ + ▲ + ▲ + ■ + ■ + ■

   ♥ + ♥ + ♥ + ○ + ○ + △

2. Combine similar items:

   2 donuts + 3 sodas + 2 donuts + 3 sodas

   5 baseballs + 3 clubs + a baseball

3. True or false:

   2 girls + 3 girls = 5 girls

   $2g + 3g = 5g^2$

   $2g + 3g = g \times (2 + 3)$

Level 2

4. Name the coefficient of each term:

   $5d, -3x, y, -mn$

5. Name the coefficient of each term:

   $5d - 3x + y - mn$

6. True or false:

   $5d - 6y - 3d = 2d - 6y$

7. Find the perimeter of the following figures:

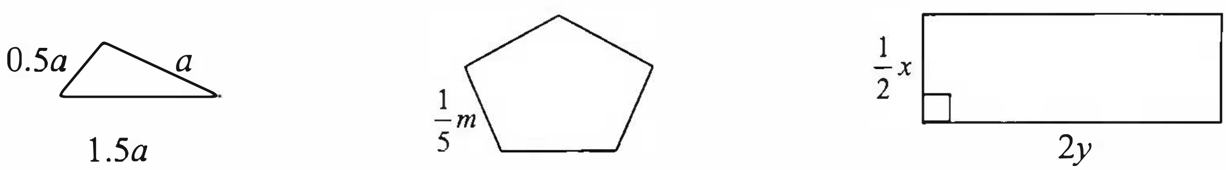


Figure 6

Example of mental mathematics problems to precipitate solving multistep linear equations

See Figure 2b

Operations with positive and negative numbers; the commutative and associative properties

See Figure 4

Possible links

Distributive property:

True or false:

$2(x + 3) = 2x + 6$

$3(2x - 1)= 6x - 1$

$-2(4 + x) = -8 + 2x$

a)

Collect like terms:

$3s + 4s - 2s + 12$

$6x - 14 - 8 - 4x$

$7p + p + 3r - 2p - r - 5p$

b)

Solve one-step equations:

$5x = 10$

$y + 3 = 7$

$4t = 13$

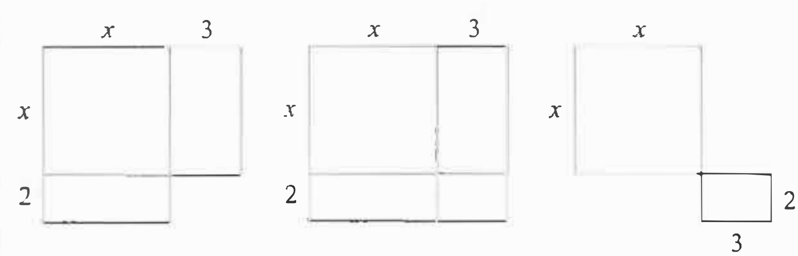
c)

Big concept: The students must be able to solve equations such as

$$2(3x - 7) - 2(4 + x) = 2$$

(Panasuk, Stone and Todd 2002; Panasuk and Todd, in press)—the following will only address its application to mental mathematics. Mental mathematics is an effective teaching tool that can test and advance both basic skills and higher-order thinking (Reys, Reys, Nohda and Emori 1995; Panasuk 2002). Unfortunately, mental mathematics is often associated with elementary-level grade mental computations that are not necessarily connected to the lesson. Mental mathematics can be designed for any topic, concept or grade, and can effectively meet different instructional purposes. The usual recommendation is that mental mathematics activities not exceed 7–10 minutes and be fast-paced and simple enough for all students to be able to respond to them. Figure 5 lists mental mathematics problems that can be used to teach about collecting like terms. Students are encouraged to complete a set of relatively simple tasks that are organized from simple to more complex. Students who have achieved a higher level of abstract reasoning will have outgrown examples 1, 2, and 3 in Figure 5. However, students who have not fully grasped the concept of like terms would still benefit from this rudimentary representation.

Table 1
Connecting mental mathematics and homework problems

Examples of homework problems (not for the same lesson)	Next class mental mathematics problem
Solve the equation: $3x + 7 = -2$	Check whether $x = 3$ is the solution of the equation $3x + 7 = -2$
Collect like terms: $-3x - y + 2x + 7y - x$	True or false: $-3x - y + 2x + 7y - x = (2x - 4x) + (7y - y)$
Solve the inequality: $-12y < 3$	Given the inequality $-12y < 3$, is $y < -4$ true or false?
Represent the product as a trinomial: $(x + 2)(x + 3)$	Which figure correctly represents the product $(x + 2)(x + 3)$? 

For a lesson in solving linear equations, such as the earlier example of $2(3x - 7) - 2(4 + x) = 2$, a set of mental mathematics activities should be created to reflect the key subconcepts that are linked together into a larger concept later on in the lesson. These mental mathematics problems can include those shown in Figures 6b, c and d. Again, if further reduction of the complexity of the problems and level of presentation is needed, the examples given in Figure 2b and Figure 4 can be used. Mental mathematics activities are best used when they are prepared in advance and displayed to the whole class, such as on an overhead transparency. Different aids, such as individual white boards, are helpful in engaging all of the students. This instructional efficiency creates more time for students to think, communicate and explain.

To make a transition from relatively simple tasks that can be done mentally (see Figures 6b, c and d) to another context (such as solving the equation), students could be encouraged to combine their prior knowledge and skills into a new strategy (see Figure 6a). This instructional decision can be more effective than simply giving the steps of the procedure and working through some examples, which is rarely effective and contributes to a lack of understanding of subconcepts, which is compensated for through imitation and/or rote memorization (NCTM 2000).

When appropriate, mental mathematics can be a way to review homework to ensure that the association between lesson components has been made. Table 1 displays examples of homework problems and their mental mathematics counterparts. When combining their homework review with mental mathematics, students are encouraged to communicate, analyze and evaluate their thinking, and to compare their strategy with other approaches. Students are also encouraged to reveal their misconceptions, if any.

After completing concept and task analysis, teachers can provide students with the most useful experiences for meaningful learning. Concept and task analysis helps teachers plan gradual progression from one level of representation to another, ultimately helping students develop an understanding of the concepts and learn the skills to become confident when solving other problems.

Notes

1. Simple does not always mean concrete. The concept $(a + b)^2 = a^2 + 2ab + b^2$ can seem simple, but it is actually an abstract (symbolic) representation of a square of binomial.

2. The notions of *unknown* and *variable* are often used interchangeably, but there is a distinction between the two. A variable, represented by a letter, is a cumulative representation of a set of numbers, such as $a + b = b + a$, where a and b are any real number. An unknown, also represented by a letter, is a number that must be identified, such as when solving the equation $x + 2 = 3$. In this equation, x stands for a unique number that would make the equation a true numerical statement.

References

Ausubel, D. 1968. *Educational Psychology: A Cognitive View.* New York, NY: Holt, Rinehard and Winston.

Gagne, R. 1970. *The Conditions of Learning* (2nd ed). New York, NY: Holt, Rinehart and Winston.

———. 2001. "Preparing the Learner for New Learning." *Theory into Practice* 19, 6–9.

National Council of Teachers of Mathematics (NCTM). 2000. *Principles and Standards for School Mathematics*: Reston, Va: NCTM.

Panasuk, R. 2002. "Connected Mental Mathematics: Systematic Planning." *Illinois Mathematics Teacher Journal* 53, 3–9.

———, in press. "The Order of Operations: Connection to the Real Life and Algebra. *Illinois Mathematics Teacher Journal.*

Panasuk, R, W Stone and J Todd. 2002. "Lesson Planning Strategy for Effective Mathematics Teaching." *Education* 122, 808–26.

Panasuk, R, and J Todd, in press. "Effectiveness of Lesson Planning: Factor Analysis." *Journal of Instructional Psychology.*

Piaget, J. 1970. *Science of Education and the Psychology of the Child.* New York, NY: Orion Press.

Reys, R, B Reys, N Nohda and H Emori. 1995. "Mental Computation Performance and Strategy Use of Japanese Students in Grades 2, 4, 6 and 8." *Journal for Research in Mathematics Education* 26, 304–26.

Regina M Panasuk is a professor of mathematics education in the graduate school of education, University of Massachusetts Lowell (UMass Lowell). At UMass Lowell, she has taught mathematics in junior high and high school. At the university she teaches content and pedagogy courses for master's and doctoral students, and supervises teaching practicum and doctoral dissertations. Her research interests include cognition, mathematics-teacher preparation and teacher professional development.