delta-k

JOURNAL OF THE MATHEMATICS COUNCIL OF THE ALBERTA TEACHERS' ASSOCIATION

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GUIDELINES FOR MANUSCRIPTS

delta-K is a professional journal for mathematics teachers in Alberta. It is published twice a year to

- promote the professional development of mathematics educators, and
- stimulate thinking, explore new ideas and offer various viewpoints.

Submissions are requested that have a classroom as well as a scholarly focus. They may include

- personal explorations of significant classroom experiences;
- descriptions of innovative classroom and school practices;
- reviews or evaluations of instructional and curricular methods, programs or materials;
- discussions of trends, issues or policies;
- a specific focus on technology in the classroom; and
- a focus on the curriculum, professional and assessment standards of the NCTM.

Suggestions to Writers

1. *delta-K* is a refereed journal. Manuscripts submitted to *delta-K* should be original material. Articles currently under consideration by other journals will not be reviewed.
2. All manuscripts should be typewritten, double-spaced and properly referenced. All pages should be numbered.
3. The author's name and full address should be provided on a separate page. If an article has more than one author, the contact author must be clearly identified. Authors should avoid all other references that may reveal their identities to the reviewers.
4. All manuscripts should be submitted electronically, using Microsoft Word format.
5. Pictures or illustrations should be clearly labelled and placed where you want them to appear in the article. A caption and photo credit should accompany each photograph.
6. References should be formatted using Chicago style.
7. If any student sample work is included, please provide a release letter from the student's parent/guardian allowing publication in the journal.
8. Limit your manuscripts to no more than eight pages double-spaced.
9. A 250–350 word abstract should accompany your manuscript for inclusion on the Mathematics Council's website.
10. Letters to the editor or reviews of curriculum materials are welcome.
11. Send manuscripts and inquiries to the editor: Gladys Sterenberg, 131 Woodbend Way, Okotoks, AB T1S 1L7; e-mail gladys.sterenberg@uleth.ca.

MCATA Mission Statement

Providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics.

delta-K

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Editorial

Gladys Sterenberg

This issue is the first to be peer-reviewed, and it represents the work of numerous people. First, I would like to extend my heartfelt and sincere thanks to those who volunteered to review the articles. I did not anticipate the length of time this process would entail, and their willingness to provide timely and helpful feedback was deeply appreciated. This publication would not have been possible without the strong commitment of these reviewers.

Second, I would like to thank the authors of the articles included here. They responded to reviewers' comments with a willingness to edit and rewrite, demonstrated their desire to improve communication and were genuinely motivated to present provocative ideas to deepen our understanding of mathematics teaching and learning.

Third, I would have ceased in my editorial efforts had it not been for the substantial support I received from members of the MCATA executive. On many occasions, they responded to my pleas and uncertainty with affirmation and practical suggestions. Their help in providing feedback on administrative procedures, editing drafts of review forms and forwarding names of people who might be interested in reviewing articles was essential in putting this review process into action. Truly, it has been an amazing experience to work with so many people who value mathematics education.

The articles featured in this issue represent a range of issues and ideas in mathematics education. Exploring alternative experiences of professional development, Katherine Willson and Roberta McKay share the stories of two elementary mathematics teachers engaged in a constructivist model of learning. Their article highlights the possibilities of long-term involvement in a learning project and offers an alternative approach to traditional ways of thinking about professional development.

Much of what we focus on as mathematics teachers involves pedagogical issues. However, it is also important to grow in our understanding of mathematics and engage in recreational mathematics. To enhance professional development in this area, I have included an article by David Dobbs on statistical probability. Read it with a pencil and paper, engage in the logic of his proofs and rekindle your love for mathematics. When done, the essence of the article can be adapted for use with your students.

The tension between new ways of thinking and traditional approaches in mathematics education is explored in the next two articles. Thomas Varghese and Elizabeth Pace investigate the strengths of traditional instructional strategies and propose an alternative teaching strategy that seeks a balance with traditional approaches. Revisiting traditional Piagetian ways of thinking, Regina Panasuk describes how concept and task analysis can assist in teaching how mathematical ideas interconnect and build on one another to produce a coherent whole. The ideas of these authors provoke much thought in this era of mathematical reform and offer practical suggestions for incorporating process skills.

Assessment and evaluation practices continue to be issues in mathematics classrooms. Reflecting on the validity of tests and test items, Werner Liedtke challenges common assumptions about the usefulness and appropriateness of this information. He urges teachers to take greater care when designing and administering tests. Also included is a commentary by Eden Haythornthwaite on the professional importance of teachers in an era of standardized testing. The article by JoAnn Grand Pooley demonstrates how alternative approaches to assessment can be carried out in the classroom.

Finally, a page of cartoons by Julie Mallet-Paret is included, as well as a page of problems by A Craig Loewen to challenge readers and students alike. Enjoy!

From the President's Pen

Janis Kristjansson

As I write my first message for *delta-K*, I think back on my growing involvement in mathematics education. Math was the most comfortable subject for me when I was a beginning teacher. I had been good at math in school and felt that I taught it well. In addition, math class gave me a period of peace and quiet each day (how times have changed). I gradually realized that, although I was teaching all of the children how to calculate, I was teaching only some of the children how to solve problems. There were always some children who couldn't understand word problems. I was not doing a good job of teaching these children to understand numbers and to know when their strategies and answers made sense.

This disheartening discovery made me re-examine my practice. I sought out a like-minded colleague and together we set out to teach by developing a stronger understanding in our students. This exciting and scary time taught me about the power of working together. We developed our own professional learning community. We joined MCATA and began attending conferences. The more I worked to understand children's mathematical thinking, the more fascinated I became.

I worked as a system math consultant with the Calgary Board of Education for several years before returning to a school-based leadership position. As I continued to learn, I realized how much more there was to know. I began to seek a wider involvement with others who were interested in mathematics teaching and learning.

I became involved in MCATA conference planning and committee work. When I joined the executive, I joined a group of teachers who were passionately committed to the improvement of mathematics teaching and learning in Alberta. With them I have had wonderful opportunities, including helping develop an MCATA brochure on mathematical literacy and, most recently, attending the Canadian Mathematics Society's (CMS) Canadian Mathematics Education Forum, held last May in Toronto.

As president of MCATA, I encourage all of you to become involved in the improvement of mathematics teaching and learning. Whether you team up with another teacher to examine an aspect of your practice, present a session on your findings at a math conference or become involved in local, provincial, national or international associations, you will be welcomed by an enthusiastic group of thinkers and doers.

Canadian Mathematics Education Forum Report

Janis Kristjansson

The Canadian Mathematics Education Forum was held at the University of Toronto on May 6–8, 2005, and was organized by the Canadian Mathematics Society. The forum encouraged a national dialogue among educators at all levels of schooling on issues and concerns in the development and future of mathematics education in Canada. Through MCATA, Alberta Education funded 10 Alberta teachers to attend the forum and discuss the central questions that drive the work of mathematics educators.

Delegates to this forum included representatives from all levels of education—from university math professors to preschool teachers to researchers. Most provincial ministries of education sent representatives, as did a number of larger school jurisdictions. The multiple viewpoints and passionate commitment to excellence in mathematics education in Canada produced rich and fruitful discussions in all of the working groups.

The forum opened with a panel discussion on the central question: Why teach mathematics? The panellists expressed their viewpoints on this fundamental issue with great passion and verve in both French and English. "Mathematics is a living form, a manifestation of energy, spirit and mystery," stated one panellist. "Faire les choses difficiles, c'est amusant" ("It is fun to do difficult things,") said another. Their commitment and excitement continued to inspire everyone in the working group discussions that followed.

The key question was considered from three viewpoints: mathematics and society, mathematics in classrooms and the mathematics education community in Canada. For two days, working groups of 10–20 participants met and discussed in depth a significant issue in mathematics education.

Participants also heard about promising practices and the thoughts of national and international leaders in mathematics education.

At the end of the forum, each group made recommendations for action and/or stated points that they wished to bring to the attention of the larger group. The statements are summarized here:

1. To support mathematics teaching and learning in Aboriginal communities, there needs to be an emphasis on culturally appropriate methodology and content. Parents need to be involved and need to be taught how to support their children's mathematics learning.
2. Curriculum for the early years of education needs to be based on big ideas in mathematics. We need to better understand the mathematics that children bring to school and how children think about mathematics.
3. Mathematical literacy enables participation in a democratic society, is essential for careers in science and technology, and provides an awareness of history and culture.
4. Mathematical experiences need to be meaningful and accessible if students are to succeed. For students struggling in mathematics, early intervention is often the most effective strategy, but this does not mean that we should abandon older learners. It is possible for all students to achieve their potential in mathematics.
5. In creating a curriculum for powerful mathematics, the tension between understanding and the need for technical proficiency can be a catalyst for change. More time needs to be spent in preservice education on what to teach and how to teach it, followed by ongoing inservice professional development. Elementary teachers in particular need to have a deep understanding of basic mathematical ideas. Texts and teaching resources need to support the curriculum and show student work.
6. Technology is essential for exploring, reasoning, modelling and communicating. Teachers need to be enabled to help students learn important mathematical skills through rich tasks in which technology is used effectively as a thinking tool. A national network of leaders in teaching mathematics with technology is needed. There is a great divide between high school and university in the use of technology. It is the considered opinion of

this working group that if you are unfortunate enough to have to learn mathematics in the absence of technology, the only job you will be fitted for is that of a university mathematics professor!

7. A collection of mathematically rich activities will allow teachers to collect data and build a portfolio of mathematics through children's eyes. It will also help build a community of learners that includes both students and teachers.
8. Education faculties, math departments at universities and colleges, math associations, ministries of education and school boards all need to see teachers as researchers and support their work.
9. The following are the criteria for promising programs of support for teachers:
    - Cross-group collaboration from K–postsecondary
    - Preservice and inservice teachers
    - Sustained engagement over time
    - Face-to-face interaction in responding to a need
    - Doing math and experiencing deep math
    - Support connections and good resources
10. A national bilingual association is needed to ensure a quality mathematics education for everyone, to support the teaching and learning of mathematics, to facilitate communication and to make connections among provincial associations.

Another crucial issue was raised by several participants. Mathematics is increasingly seen as an academic gatekeeper, just as Latin was seen in the past. However, advanced mathematics (and algebra in particular) is essentially useless for many paths in higher education. "Algebra for all" is no more useful than "Latin for all." Using mathematics as a gatekeeper bars intelligent people from academic opportunities. This is a concern throughout all levels of the mathematics education community and is one of the emerging issues in Canadian mathematics education.

At the end of the three-day forum, participants returned home invigorated from discussing in-depth issues that matter deeply to them and aware that they are part of a national community. As one participant said, "A number is a door that opens into the farthest reaches of the human imagination."

MCATA members who attended the forum found it fascinating, challenging and informative. They are grateful for the funding provided by Alberta Education that allowed them to attend.

Alberta Education Report: The Right Angle

Deanna Shostak

Curriculum Branch

Alberta Education is providing leadership for the Western and Northern Canadian Protocol (WNCP) in making revisions to the WNCP Common Curriculum Framework (CCF) for K–12 mathematics. A publishers' draft for Kindergarten to Grade 7, based on feedback received during consultations, is on the WNCP website at www.wncp.ca with the final consultation report. Outcomes for Grades 8 and 9 will be finalized in conjunction with ongoing revisions to the Grades 10–12 CCF to ensure a smooth transition from junior high to senior high school.

New Alberta programs of study will be developed based on the revised K–12 CCF. Implementation timelines are published in the *Implementation Schedule for Programs of Study and Related Activities* on the Alberta Education website at www.education.gov.ab.ca.

Alberta Education, in cooperation with WNCP partners, met with educational publishers in July to discuss developing student and teacher resources to support the implementation of programs of study derived from the WNCP CCF for K–9 Mathematics. Alberta Education intends to have resources available for Alberta schools in both English and French prior to provincial implementation.

The Alberta Regional Professional Development Consortia (ARPDC) is continuing the series of workshops on key topics in Grades 7, 8 and 9 mathematics. During the 2005/06 school year, workshops will address the following topics:

- Teaching algebra in junior high
- Teaching measurement concepts
- Strategies for enhancing understanding in the mathematics classroom

In addition to these workshops, the ARPDC will be offering workshops to K–3 and Grades 4–6 teachers in the spring. These workshops will address the following topics:

- Developing number sense
- Teaching pre-algebra concepts

For more information on the availability of these workshops, please contact your local regional consortia.

Learner Assessment Branch

Maintaining Consistent Standards Over Time

Alberta Education plans to make diploma examination results directly comparable from session to session, thereby enhancing fairness to students across administrations. To achieve this goal, a number of questions—called anchor items—remain the same from one examination to the next. Anchor items are used to find out if student populations writing in different administrations are differing in achievement. Anchor items are also used to find out if the unique items (questions that are different on each examination) are different in difficulty from the unique items on the baseline (the first examination to use anchor items). A statistical process, called equating, adjusts for the differences in examination form. The resulting examination scores can be equated regardless of when and to whom the examination was administered.

Baseline examinations were written in Chemistry 30, Physics 30 and Pure Mathematics 30 in January 2005. In January 2006, baseline examinations will be written in Applied Mathematics 30 and Biology 30. For all of these courses, students who write a diploma examination in a later administration will have their marks directly equated to the baseline examination. Examination marks may be adjusted slightly upward or downward depending on the difficulty of the examination written relative to the baseline examination. These equated marks will be reported to students. As a result of equating to the baseline examination, students' marks will accurately reflect their levels of achievement, regardless of the examination administration in which the students wrote.

Due to the security required for fair and appropriate assessment of student achievement over time, both Parts A and B of the January 2006 Applied Mathematics 30, Pure Mathematics 30, Biology 30 and Chemistry 30 diploma examinations will be secured and not released at the time of writing. The June 2006 Applied Mathematics 30 and Biology 30 diploma examinations will also be secured. Only one form of the examination will be written in each administration.

Once again in the autumn of 2006, Alberta Education will provide schools with print copies of released items from past diploma examinations. Alberta Education will also provide assessment highlights to educators, students and parents on the department's website (www.education.gov.ab.ca). Teachers will continue to be able to peruse copies of these examinations during examination administrations.

Learning and Teaching Resources

New Applets and Interface for Applied Mathematics Multimedia Resources

Eleven new discovery applets that address outcomes in applied and pure high school mathematics have been added to the LearnAlberta.ca website. These applets allow students and teachers to manipulate parameters and visualize concepts being addressed. This resource has also adopted a new and improved user interface. In addition to grouping the applets according to strand and substrand, the new interface will allow the user to select applets according to the relevant course and grade: pure math or applied math at the 10, 20 or 30 level.

New Preview Zone launched on LearnAlberta.ca

The Preview Zone is now available on the LearnAlberta.ca website. This zone offers learning resources that are in the final stages of technical development. These resources will be available in the Preview Zone to authorized users on a temporary basis during development and testing. New resources will be added as they become available. Go to the What's New section or the Teachers page on www.learnalberta.ca for a link to the Preview Zone. Your feedback on these resources is valued and encouraged.

Dr Arthur Jorgenson Chair Award Recipient: Rebecca Steel

The Dr Arthur Jorgensen Chair Award is presented by MCATA to encourage students enrolled in education programs in postsecondary institutions throughout Alberta to pursue and commit to mathematics education. The award consists of an invitation to attend two MCATA conferences with all expenses paid (includes a teacher substitute, hotel accommodation and travel), a three-year membership in MCATA, a one-year membership in the National Council of Teachers of Mathematics (NCTM) and a one-year term on the MCATA executive, with all expenses paid to attend executive meetings (meals, travel and accommodation when necessary).

The recipient is selected on the basis of demonstrated academic excellence and a clear commitment to mathematics education. The first recipient of the award was Lisa Hauk-Meeker, who accepted the chair for the 2004/05 school year. Lisa continues to demonstrate her commitment to mathematics education and joins the current MCATA executive as a director at large.

The winner of this year's Dr Arthur Jorgensen Chair Award is Rebecca Steel. Her thoughtful responses to the questions on the application form demonstrate her keen interest in teaching and learning mathematics. By way of introduction, her description of mathematics and her reasons for teaching this subject are presented below:

> I chose to pursue mathematics as a major area of study because I enjoy the beauty of it. I enjoy playing with numbers, observing patterns that form through working with different ideas, theories and concepts that occur in mathematics—those that have been discovered and ideas that are still being investigated. I've always enjoyed mathematics from a young age. As I became more involved with mathematics, the more familiar it became and I soon realized how mathematical concepts connected together. These concepts and how to work with them seemed to come very naturally to me, and still do. Therefore, throughout my grade-school time I was often asked to help several of my friends with the math that we were studying. Through doing this I was better able to understand the concept being presented and I also needed to develop different ways to explain the concepts to my peers. This challenged me quite a bit and I enjoyed taking on this challenge and decided to pursue this further and study mathematics at the university level so I could one day teach it at the secondary level. I also wanted to further my own understanding and go deeper into the concepts that I learned through grade school. To do this, I took several pure, applied and statistic mathematics courses at the university level. I enjoy the problem-solving aspect and the way that one needs to think to solve mathematical problems. Along with this, I enjoy learning and talking about the history of mathematics, the concepts that have been developed, and how they are applied and what they can be used for in the world around us. This enjoyment is something I want to share, especially with students that are going through this stage of learning about the beauty of mathematics and the excitement that there is in it when solving problems and developing conclusions.

We welcome Rebecca to the teaching profession and to the MCATA executive.

Mathematics Professional Development for Elementary Teachers: A Constructivist Model

Katherine Willson and Roberta McKay

Background

The Western Canadian Protocol framework for mathematics focuses on a constructivist approach to teaching and learning mathematics. One particular area of concern for teachers has been the expectation to increase communication processes in the mathematics classroom through student discussions and writing. According to researchers, such as Deborah Schifter (1996), teacher professional development in mathematics needs to change to reflect constructivist principles. Key ideas from constructivist theory include a recognition that students are active learners who come to classrooms with different knowledge, experiences and backgrounds, and that students learn by attaching meaning to what they do; they must be able to construct their own meaning of mathematics. Students need to read, explore, investigate, listen, discuss, explain ideas in their own language and use manipulatives. This helps students create their own links between their informal understandings of mathematics and the formal language and symbolism of mathematics. Similar constructivist principles are needed to inform teacher professional development in mathematics.

As professors in the Department of Elementary Education at the University of Alberta, we teach in the areas of language arts and mathematics. With funding from Imperial Oil National Centre for Mathematics, Science and Technology Education (IONCMASTE) and Edmonton Catholic Schools, we initiated a long-term professional development project in mathematics. We wanted teachers, district consultants and ourselves to collaboratively construct meanings about the teaching and learning of elementary school mathematics. We also wanted a professional development project that would demonstrate constructivist teaching and learning principles. A key focus was the role of discussion and writing in the construction of meaning. Not only did we recognize the importance of this in all subject areas, but we believed that a focus on discussion and writing in a mathematics professional development project was particularly important because of the emphasis in the Western Canadian Protocol for mathematics on increasing communication to enhance the learning of mathematics.

The Mathematics Professional Development Project: Year One

We began the project by meeting with the language arts and mathematics consultants from Edmonton Catholic Schools to discuss constructivist professional development. We agreed that the sessions for teachers should be held during the school day to demonstrate the value of teachers' time and the activities in which we would be involved. We planned to meet one half-day per month during the school year. Ten teachers were chosen in consultation with the school district consultants and invited to participate. The mathematics and language arts consultants also participated in the professional development sessions.

The format of our sessions required much thought and a willingness to take risks on our part and on the part of the teachers. We needed to think about what a social constructivist model of professional development in mathematics might look like. This mirrored the teacher's process of considering what the constructivist teaching of mathematics might look like. We knew that we would not be giving inservices or conducting workshops. Although these traditional professional development structures have a purpose, they would not allow us and the teacher to construct meanings about the teaching and learning of mathematics. For this reason, teacher dialogue and journal writing about experiences teaching mathematics would be the main focus of the professional development sessions. In order to reinforce our own role as learners and group members rather than outside experts, we decided to participate in the dialogue and journal writing as well.

The Professional Development Sessions

Participants were given a journal at the first session, and each session had the following format. Participants reflected and wrote in their journals about the mathematical issues (successes and concerns) that had been a part of their teaching experiences over the past month. A sharing session followed, which led to rich discussion—something that usually does not happen during the typical one-shot inservice model of professional development. The teachers told us that because they met on a regular basis over an extended period of time, a feeling of trust was created among the group members that enabled them to share mathematics stories that normally they would not have shared. During each session, teachers were also given an article from a professional journal to read and discuss. This was originally planned as a back-up activity in case there was a lack of discussion, but the current articles written for a professional audience were excellent for provoking lively discussion. In fact, most of the teachers continued to mention these articles throughout the year. At the beginning of the year, teachers were also given a professional book called *What's Happening in Math Class*? by Deborah Schifter, which addresses current teaching and learning issues in elementary mathematics classrooms. The professional development sessions culminated with teachers writing vignettes about the major issues that had surfaced for them through the discussions, readings and journal writing. Many of the teachers were not confident about their writing and wondered if they had anything to say about teaching mathematics that would be of interest to anyone else. They wrote the vignettes, then read each other's and provided feedback. Through the questions and comments they provided to each other, they realized that they were writing about topics of interest to other teachers. We also provided oral and written feedback throughout several drafts. The teachers expressed many interesting insights about the teaching and learning of mathematics. Writing vignettes was a powerful tool for stimulating mathematical dialogue among teachers and promoting reflection on their own practice.

The Mathematics Professional Development Project: Year Two

As a result of the positive feedback from the teachers and consultants, the project was funded for a second year. The format remained the same, although the process for selecting teachers was slightly different. Five teachers from year one expressed an interest in continuing the project, and they were joined by five new teachers. The professional book provided to the teachers in year two was entitled *Math Is Language Too*, by Phyllis Whitin and David Whitin. The key components for each of the sessions remained the same: professional readings, journal writing, and discussions and reflection about the teaching and learning of mathematics in the classrooms. The year once again culminated with the teachers writing vignettes.

The Vignettes

The following are two vignettes written by participants that illustrate the power of this long-term reflective professional development model. In the first vignette, "Drill-and-Practice to Constructivism: My Math Journey," an experienced teacher reflects on her years of teaching mathematics primarily through drill-and-practice, and her frustrations with the number of students who did not succeed in mathematics. She describes being intimidated by shifting to a constructivist orientation for the new mathematics curriculum. However, students' responses to an activity that had several constructivist elements convinced her to switch to a constructivist approach in teaching and learning mathematics.

Drill-and-Practice to Constructivism: My Math Journey

How did teaching math become so complex? Twenty years ago or so, teaching basic facts required exactly one strategy, one teaching method and one method of evaluation. Drill-and-practice was the norm. Now the teaching of mathematics requires a more inclusive approach—an approach that asks the students to be active participants in constructing their own knowledge. Within this approach, the teacher is no less important in her role as educator. However, the nature of her role has changed. She is now asked to facilitate the learning and she is given many more open-ended ways to achieve this end. This is not an easy road; it is very hard to be a learner. This story is about my journey and the best learning I have done in a very long time.

I have been a teacher for almost 20 years. I began teaching in a special education classroom in an inner-city school. I taught 10 to 12 students in a segregated room who, at that time, were classified as educable mentally handicapped (EMH) students. Teaching math was the biggest challenge. Not only were many of the grade-specific math concepts difficult for most of the students, but the students had difficulties retaining what was taught. We had at least three math groups each day in an attempt to individualize instruction as much as possible. Practice through drill was the means to learning.

After my first seven years, I moved to a school on the south side that also ran the district EMH program, later renamed Educational Experiences 3 (EE3). The curriculum in special education had undergone changes. A living and vocational skills component was added. In math, each student now used an appropriate grade-level textbook and there was a focus on preparing students for real life. A huge addition to the program was involving children in hands-on activities to enhance the drill-and-practice. However, my guess is that many teachers' lessons continued to be teacher-directed.

After three years at that school, I left the special education classroom. Over the next four years, I taught Grades 3, 4 and 4/5. I then moved to my new school, where I have taught a combined Grade 4/5 class for the past two years.

Like most teachers, I taught the mathematics concepts, did the drill-and-practice and tested the students for retention of the material. I used current curriculum, followed the program of studies, taught the mathematics guide page-by-page and corrected the daily assignments promptly. Like many teachers, I felt that students' problems with mathematics had more to do with faults in learning than in teaching. Surely it had nothing to do with my teaching strategies or the curriculum.

After much thought and discussion with colleagues, I decided that the way to help my students who were having difficulties was more drill-and-practice. That had to be the answer. They didn't understand math because they did not have enough practice. So I gave them lots of drill—directed drill, timed drill, drill in groups, drill in workbooks, drill from the board and, most recently, computer-generated drill. After the chapter-end tests, several students still did not do as well as I had expected. What should I do next? What was causing a lack of success for some of the students?

It never occurred to me during these years of my career that the students were not *learning* math; they were simply *doing* math. Math was a school subject taught to them. Concepts were explained, drawn on the board, repeated as necessary and, of course, practised frequently. I planned carefully how to introduce the concepts. I used a clock, a thermometer, play money, metre sticks and rulers, fraction pieces, and bundles of 10s and 100s. I do not know if what I said or what I did ever became meaningful to the students. I do not recall using manipulatives often and, certainly, I do not recall students writing, discussing or asking questions other than "What page number do we do?" That was the way of sound teaching.

As the department of learning and the publishers became more progressive, the content of math programs and math textbooks became more relevant and more interesting to the students. Large, colourful pictures of children doing activities were placed in the math textbook to grab the students' attention and make the concepts more meaningful and relevant. However, students were still not expected to be active participants in their own learning, and some students still did not succeed in math. The pedagogy was starting to change but we were not there yet. Math was still something teachers taught to students.

Three years ago, I was handed a new mathematics curriculum along with new mathematics materials, both with a constructivist focus. It was the program of the '90s and it was scary. The students still had some practice exercises, but oh, how the lessons had changed! Even the look of the teacher's guidebook had changed! The student book was almost unrecognizable. Where were the rows and rows of practice exercises? The students were now expected to do something to generate their own learning. They were asked to explain, question, think about and talk about mathematics during math class. More importantly, the new program offered students ways to relate math learning to their own lives. The teacher would be more of a facilitator of learning rather than a deliverer. Both the students and the teacher had to learn to think in new ways. No longer were there pages of isolated drill-and-practice exercises. The practice was included in a problem-solving context so that the students could apply the learning to their everyday lives.

I recall one lesson that I taught in the spring. I was using the Quest 2000 math program with my Grade 4 class, and I had the students make kites to apply their understanding of perimeter and area. The math unit had begun with several classes that developed ideas of area and perimeter, and then explored the relationship between them. The students were drawing and cutting while questioning, comparing and problem solving. Through these activities, they developed an understanding of how area and perimeter are related.

The culminating lesson---constructing a kite—was wonderful. The students were asked to design the largest kite they could with a one-by-one-metre sheet of white paper. The only condition was that the area had to be larger than the perimeter. What fun that was! All of us were on the floor—measuring, cutting and, perhaps most important, discussing. Mistakes were made and some kites had to be remeasured and redone, but that became a critical part of the learning and our discussion. The students decorated the kites however they chose and we attached string tails and string guides. The best part was trying to fly the kites. They never did fly, but the students learned more about perimeter and area than they would have from 30 drill-and-practice worksheets. Could this really be the way to learn math? Was math really meant to be fun?

This new approach caught me off guard. Until then, I had to plan well for math out of fear that something would happen that I could not control. The students were now more involved in the lessons and added their own ideas, understanding and direction to our math classes. Math class involved more discussion and became less teacher directed. I had no idea at that time how valuable and educationally sound this type of teaching could be. I also had no idea that making a kite could be a form of assessment—a test that allowed everyone to succeed. Now my students loved math, and that became a more meaningful measuring stick of my program than could any number of test scores.

As the year finished and I became more confident with my math program, I realized that, for me, using a constructivist approach was how math should be taught and learned. My Grade 4 class still had an average performance on their formal assessment, but they had discovered that math belonged in their lives and they were confident in their approaches to problem solving. With this type of math instruction, their learning and skills would grow. I discovered that learners must be active in their learning if it is to be meaningful and retained. Students must be allowed to construct their knowledge and then apply it to real-life situations. If those situations happen to be fun, it's a bonus!

I now try hard to plan constructivist strategies into my math lessons, even though I have a combined class in which the expectation to complete two math programs is stressful. Whether they bring in objects to illustrate a concept, read math-related literature or use their bodies to create place-value situations, students are encouraged to be active in their learning. We now discuss and share strategies, construct models, write in math journals, write our own problems to apply our knowledge and do drill-and-practice exercises. Drill-and-practice still has a place in math—just not first place anymore. This has been my math journey and the best learning I have done in a long time.

The second vignette, "Walking the Talk: Lessons Learned About Effective Professional Development," illustrates the experience of a teacher in her new role as a district mathematics consultant. She explains that the most challenging part of her role is helping teachers understand and implement constructivist principles of teaching and learning. She recognizes that, in order to implement constructivist teaching, professional development must model constructivist principles. She highlights four key insights that have influenced her planning of professional development for teachers.

Walking the Talk: Lessons Learned About Effective Professional Development

Many educators are asking why the constructivist movement hasn't had more of an influence on classroom instruction in mathematics. Studies document the gains children can make when we apply constructivist principles in our teaching practices. And yet many teachers still tend to teach mathematics in the way they themselves were taught—memorization and drill—even though they readily admit that they never learned much from this type of instruction.

Helping teachers understand and implement constructivist principles in mathematics instruction is the most challenging part of my position as a mathematics consultant. The back-to-the-basics movement has complicated my task even further. But I have learned a lot over the past year about the type of professional development that promotes change in teachers' attitudes and practices in mathematics instruction.

Walking the talk, or practising what you preach, is more difficult than we think. Some teachers request activities, blackline masters and preplanned lessons to assist them with their mathematics instruction. I understand this in my role as a consultant: lack of time for professional development sometimes tempts me to resort to the one-shot, how-to inservices that may do little to change teaching practices and improve student learning. Simply telling someone how to apply constructivist principles is tantamount to telling a child how to compare fractions—neither lead to true understanding. The structure of professional development must also reflect constructivist principles.

I have come to this conclusion through my experiences with the mathematics professional development group—a collaborative project between our district and the University of Alberta. When I joined the group in its second year, I thought that the teachers had been chosen because of their strength in mathematics instruction. I soon learned that the group had been on a journey of discovery that had profoundly changed how they thought about the teaching and learning of mathematics. I also observed first-hand the constructivist principles being demonstrated throughout the professional development sessions. As the year progressed, I began applying this model of professional development to my work with teachers in schools. Although there is still a lot to be learned, I have highlighted below four of my key insights concerning teacher professional development.

1. Concentrate on the big ideas of mathematics instruction and discover them through real experiences

Too often, teachers only consider the specific outcomes of a program, such as multiplying decimals, recognizing number patterns or using the vocabulary of probability. We are professionally obligated to cover these outcomes over the course of a year and may lose sight of the constructivist principles on which the program is based. Teachers must understand these principles to know how to successfully teach these outcomes.

Discussions in our professional development group focused on principles of constructivist learning. The teachers shared stories of their classroom experiences, which were used to illustrate big ideas about learning rather than show how to teach specific skills.

One teacher shared student posters from a colleague's classroom. The colleague had told the students that aliens had invaded the earth and one of the first changes they wanted to make on the planet was eliminating multiplication. The students were asked to create posters for a campaign to save multiplication from worldwide obliteration. The posters included pictures and ideas, such as multiplication being a more efficient way to add a series of the same number and finding the area of a room. Another poster argued that figuring out the cost of 12 chocolate bars would be more difficult without multiplication. The teacher shared the activity with the professional development group to illustrate how communication in mathematics is not just a way of assessing understanding but also a way of helping students refine their thinking and make sense of mathematics.

Teachers also distinguished between learning *about* problem solving and learning *through* problem solving. Introducing new mathematical concepts in problem solving contexts was modelled and discussed in the professional development group, and applied in all of our sessions. We discussed solutions to real problems that we face in our schools, such as homework issues and the role of the Kumon mathematics program. After being involved in this approach, teachers realized how powerful a model it was and why their students were enthusiastic when solving real-life problems in cooperative groups.

There were many questions about teaching basic facts, such as the multiplication tables. The discussion on teaching basic facts went beyond drill to explore the developing of metacognitive strategies, which help children become proficient in mental mathematics and to see the connection between numbers.

By focusing on these constructivist principles rather than on a mandated document, teachers become resource-free. They use classroom resources as tools rather than instruction books, and they implement programs as intended.

2. Allow for time for reflection

Each session of our professional development group began with time—precious time—for writing in our journals. We wrote about what was happening in our mathematics classes, our successes and our concerns. We could then share what we had written. I often asked myself if everything I had been so busy doing was actually accomplishing something.

Imagine if our children did this as well. Before asking them what they learned that day at school, we could ask them to write down their thoughts and questions about what they had done that day. Instead of just giving the usual reply, "Oh, nothing," children might be surprised at what they had learned.

I am beginning to understand that we don't really learn anything until we reflect on it.

3. Encourage teachers to discuss with, collaborate with and learn from each other

As the year progressed, the professional development group became more comfortable talking about difficult issues and successes. We rarely have the opportunity to collaborate with other teachers in our profession. We are isolated in our classrooms for most of the day and we often spend staff meetings discussing business matters. Even professional development seminars that may capture our attention do not often include time for collaboration.

I learned a lot from my colleagues. We left each session with a new idea or a new question to ponder. Discussions about the tough issues we face gave me insight into what is on teachers' minds. We were given time to discuss and refine our thoughts, organize our understanding of mathematics and share our wisdom about practice.

When I first started working as a consultant with school staffs, I found that many of them were reluctant to talk. Perhaps the teachers expected to listen to an expert tell them how to teach mathematics, even when many of them had their own expertise. Now, I often begin my workshops by stating an idea or concern raised in our professional development group. After a short pause, teachers start talking—to each other. It is as if they need permission to express their own thoughts about effective mathematics instruction. Teachers who spend all day next door to each other find out that their colleagues have wonderful ideas to share. The Grade 2 teacher finds out that her concerns are not so different from the concerns of the Grade 6 teacher. As they begin to search for solutions together, teachers are eager to discuss math and collaborate to make changes for their students. A knowledge-building community is created within the school.

4. If you value something, give it time

Our professional development group met for a half day, once a month, over the course of a school year. Teachers were released during class time to meet. We truly valued our time together.

This type of ongoing professional development is very effective. We felt more accountable for our journey during the year. Teachers were encouraged to become researchers in their own classrooms, constructing their own understanding of why certain practices were more effective than others. Nothing transforms a teacher more than putting theory into practice and then evaluating the results.

We also took the time to investigate some of the best theory. During each session we read a chapter from our professional reading book or an article from recent National Council of Teachers' of Mathematics (NCTM) journal or another professional publication. Professional reading was valued and time was provided during each session to read an article. It is amazing how one well-chosen article can change how you think about teaching and learning. Reading articles during the session alleviated the pressure to prepare for each session.

I have used many of the great articles that were shared within our professional development group in my work with the schools in our district. Teachers appreciate the time to just sit and read during an inservice. I have also learned to be more patient. Real change takes careful planning and time.

Conclusion

As I reflect on the past year and start making plans for my work with schools in the fall, I am encouraged by what I have learned yet also anxious about some of the challenges I face.

With all the pressures and new initiatives that teachers face, how can I help them find the time for ongoing professional development with needed follow-up and self-reflection? How do I avoid becoming the "guest speaker" at school professional development sessions and instead shift the responsibility for improving mathematics instruction to the classroom teacher? Do teachers who may never have had the opportunity to construct a deeper understanding of mathematics have the confidence and knowledge to guide children in their learning? What can I do to assist teachers in gaining this knowledge? Perhaps some teachers will learn with their students. Finally, I wonder what other professional development experiences will help teachers, parents and students make the paradigm shift that this constructivist movement in mathematics requires?

I am left with many questions that need to be answered. Yet, learning how to "walk the talk" has been a fulfilling journey. I still have much to learn about the constructivist model in mathematics professional development. My experiences this year have given me a direction to follow that can make a difference for teachers and their students. I look forward to the challenges and will be forever grateful for this learning experience.

Concluding Comments

Feedback from teachers about the professional development project was overwhelmingly positive. They stated that reflecting on their practice over time affected their thinking about the teaching and learning of mathematics. They also found the writing of vignettes to be a powerful tool to stimulate mathematical dialogue and promote reflection on their practice.

Vignettes from both years of the project have been compiled into a monograph entitled *Teacher Vignettes: Elementary Teachers Reflect on the Teaching and Learning of Mathematics* (Willson and McKay 2003). Vignettes from the monograph have been shared with graduate and undergraduate students as well as inservice teachers and professional development providers. They are an excellent springboard for discussing the teaching and learning of mathematics. Teachers have told us that they readily identify with the issues that their colleagues have grappled with regarding the implementation of a constructivist philosophy in the teaching and learning of mathematics.

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The Probability of a Statistical Oddity

David E Dobbs

Introduction: A Paradoxical Loss

The accompanying table summarizes the batting performances of four Major League Baseball players during a recent season.

After the first half of the season, the race for the batting average title was not close. The preseason favourite, Player A (with a batting average of 0.295 after 81 games), trailed a journeyman, Player B, who had a batting average of 0.320 after 81 games. Also contending was Player C, whose batting average after 81 games was 0.294. After midseason, Player A took some time off to nurse an old injury. Following some rest and recuperation, Player A returned for the last 23 games of the season, amassing a second-half batting average of 0.333 and an overall batting average of 0.3025 (rounded off as 0.303) for the entire season. When the results came in, Player A had overcome the lead that Player B had enjoyed at midseason. Player A reasoned, "Yes, Player B beat me during the first half of the season, but I beat him more decisively during the second half of the season." In addition, Player A was not surprised that he had beaten Player C. Player A reasoned, "After all, I beat Player C in each half of the season, so of course I beat him overall for the entire season." Player A prepared to be awarded yet another batting title. But it was not to be.

The actual winner of the batting title was Player D. This young professional had been brought up too quickly from the minor leagues at the beginning of the season. Because Player D had a batting average of only 0.240 after 23 games, he was sent back to the minor leagues until midseason so that he could work on hitting a curve ball. Player D returned to the major leagues after the 81 game of the season. Hitting lead-off, he compiled 325 at-bats and a batting average of 0.320 for the second half of the season. This effort produced a batting average of 0.305 for Player D for the entire season.

Player A was dismayed and confused. How, he wondered, could he have lost the batting title to the unheralded Player D? Player A reasoned, "I beat Player D in each half of the season, so how and when did he end up beating me overall for the entire season? What are the chances that this kind of paradoxical loss could happen?"

In this article, I answer the questions raised by Player A. By doing so, I also provide enrichment material for mathematics classes at various levels. Table 1 can be used to help beginning students practise

Table 1

Player A	At-Bats	Hits	Batting Average
First half of the season	325	96	0.295(38...)
Second half of the season	75	25	0.333(3...)
Totals for the entire season	400	121	0.3025 (recorded as 0.303)
Player B	**At-Bats**	**Hits**	**Batting Average**
First half of the season	200	64	0.320
Second half of the season	200	52	0.260
Totals for the entire season	400	116	0.290
Player C	**At-Bats**	**Hits**	**Batting Average**
First half of the season	160	47	0.294
Second half of the season	240	67	0.279(16...)
Totals for the entire season	400	114	0.285
Player D	**At-Bats**	**Hits**	**Batting Average**
First half of the season	75	18	0.240
Second half of the season	325	104	0.320
Totals for the entire season	400	122	0.305

calculating averages (technically called *means* in statistics). This activity also points out how a player's overall batting average is weighted toward the batting average that he obtained during the half-season in which he had the greater number of at-bats. This activity also gives students practical experience with rounding off to three decimal places. It is important to realize that someone's batting average for the entire season is not simply the arithmetic mean of his batting averages for the two half-seasons. In effect, the table entry "Totals for the entire season" treats the season as a whole because the batting average for the entire season for each player (A, B, C or D) is calculated as [total number of hits]/400.

We require more than elementary algebra to answer the questions raised by Player A. He asked about the chances of a paradoxical loss (that is, the chance of losing the batting title to a player whom one had beaten in each half of the season). First, I made some reasonable assumptions, which are specified below. Then I found the chance (or probability) by using analytic geometry. I compared the area of a certain rectangle with the area lying between a horizontal line and a branch of a certain rectangular hyperbola. To identify the parameters involved, I used the SOLVER feature on a Texas Instruments (TI) graphing calculator. This part of the analysis would fit well in a precalculus class.

Finding the area between the line and the hyperbola depends on two mathematical matters. The first determines that the graph of the hyperbola is rising—that is, described by an increasing function. I give three proofs of this fact in Theorem I, and each proof is designed for an audience at a different level. Proof 1 can be given to an algebra class; it reinforces the long division (or synthetic division) of polynomials and the core meaning of fractions. Proof 2 involves inequalities, and thus could be given to a precalculus class. Finally, Proof 3 involves the sign of a derivative, and thus may be the method of choice for a calculus class.

The second mathematical matter involves calculating the area between the line and the hyperbola. This is the only part of the analysis that requires calculus. After this area was described by a definite integral, I calculated that integral by using the numerical integrator (fnInt) on a TI graphing calculator. In this way, the entire experience reinforces two important technological functions—SOLVER and fnInt—on a TI calculator.

I carried out the above analysis three times to answer several precise questions that are suggested by Player A's queries. The first result is that there is a probability of only 0.00635 (that is, 0.635 per cent) that Player A could, paradoxically, be beaten by a player (like Player D) with a first half-season batting average of only 0.240 and a second half-season batting average of less than 0.333. The second result is that there is a probability of about 0.0368 (that is, 3.68 per cent) that Player A could, paradoxically, be beaten by a player (like Player C) with a first half-season batting average of 0.294 and a second half-season batting average of less than 0.333. Finally, I found that there was a probability of 0.3679 (that is, 36.79 per cent) that Player A could be beaten (not necessarily paradoxically) by a player with a first half-season batting average of 0.294.

To make the analysis more realistic, I only considered players who had between 50 and 350 at-bats in each half-season and (like Players A, B, C and D) at least 400 at-bats for the entire season. I also assumed that no player would have a batting average exceeding 0.500 during any half-season. Even more realistic analysis is possible, but at the cost of considering the calculus of functions of several variables and using computer technology to evaluate various multiple integrals. This is explained, along with some philosophical musing, in the closing comments. As a final consolation for Player A, I will also provide a theorem explaining that paradoxical losses are not possible for players with the same number of at-bats in each half-season.

The Probability of Losing Paradoxically as in the Above Example

Next, I determined the probable paradox that Player A will lose to a random player, such as Player G who (like Player D) has a batting average of 0.240 for the first half of the season and less than 0.333 for the second half of the season. The next table summarizes the batting performance of such a Player G.

Table 2

Player G	**At-Bats**	**Hits**	**Batting Average**
First half of the season	x	$0.240\,x$	0.240
Second half of the season	$400 - x$	$(400 - x)$	$r(<.333)$
Totals for the entire season	400	$.240x + (400 - x)r$	$\frac{.240x + (400 - x)r}{400}$

Note that I indicated the randomness of Player G by letting x denote the number of at-bats for Player G during the first half of the season. Now, Player A will only lose to Player G if the batting average of Player G for the entire season exceeds that of Player A—that is, only if

$$\frac{.240x+(400-x)r}{400}>.3025.$$

An equivalent inequality is

$$r>\frac{121-.240x}{400-x}.$$

The modern graphical approach to solving inequalities requires that we understand the graph of the equation $r=\frac{121-.240x}{400-x}$ in the xr-plane. As will be shown in Proof 1 of Theorem 1, this equation in the xr-plane can also be expressed as $r=.240+\frac{25}{400-x}$ by long division, from which is obtained the equivalent equation $(x-400)(r-0.2400)=-25$. By translating the x- and r-axes 400 units to the right and 0.240 units upwards, a new x^*r^*-coordinate system is obtained with an origin at the point formerly known as (400, 0.240). The equation $(x-400)(r-0.2400)=-25$ can now be expressed as the equivalent equation $x^*r^*=-25$. Because the product of the variables is a nonzero constant, the graph is a rectangular hyperbola.
The above graph (of a hyperbola) is rising for the relevant values of x (namely, for $50\le x\le 350$), and this is established in Theorem 1, Proof 1. As is explained in the introduction, the three proofs of Theorem 1 are designed for classes with varying backgrounds. (A fourth proof, which is more geometric, follows by noticing that for $c<0$, the left-hand branch of a rectangular hyperbola $x^*r^*=c$ is rising.)

Theorem I

The function $r=f(x)=\frac{121-.240x}{400-x}=.240+\frac{25}{400-x}$ is increasing for $0<x<400$. More generally, if a, b and c are positive constants, then the function given by $y=a+\frac{b}{c-x}$ is increasing for $0<x<c$.

Proof 1

Using long division (or synthetic division), observe that when $0.240x-121$ is divided by $x-400$, the quotient is 0.240 and the remainder is -25. Thus,

$$\frac{121-.240x}{400-x}=\frac{.240x-121}{x-400}=.240+\frac{25}{400-x}.$$

This expression takes the asserted form, with $a=0.240$, $b=25$ and $c=400$. To prove that this expression is increasing, increase x (strictly between 0 and c, and note the effect). As x increases, $400-x$ decreases (but remains positive)—and so the ratio $\frac{25}{400-x}$ of positive quantities increases, a situation that is unaltered by the addition of the constant 0.240. Thus, this first proof is based on the behaviour of a fraction that has a constant positive numerator and a varying positive denominator. (Students may have developed this understanding as a result of studying instances of inverse proportionality, such as Boyle's Law in chemistry.)

Proof 2

Using the rules for operating on inequalities, find that if $0<x_1<x_2<c$, then $a+\frac{b}{c-x_1}<a+\frac{b}{c-x_2}$. (One could work directly with the original form of $f(x)$ as well. I leave the details of that similar proof to the reader.) Observe that $0<c-x_2<c-x_1$. Multiply this inequality with the positive number $\frac{b}{(c-x_2)(c-x_1)}$, to obtain $\frac{b}{c-x_1}<\frac{b}{c-x_2}$. The inequality is unaltered if the constant a is added to both sides, thus completing the second proof.

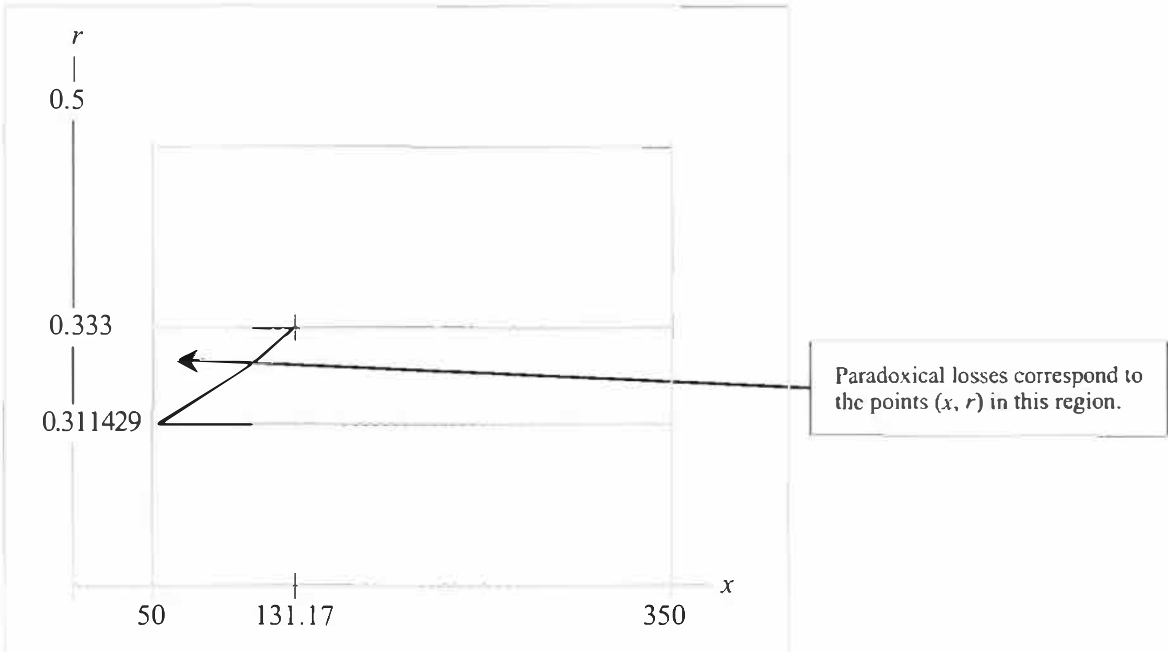
Proof 3

For this calculus-based proof, find that the function given by $y=a+\frac{b}{c-x}$ has a positive derivative. Using the formulas of different calculus, that derivative is found to be $\frac{b}{(c-x)^2}>0$.

The next figure graphs the feasible points (x,r) that have been discussed, as well as the subset of points that describe a paradoxical loss. This subset consists of the points (x,r) that lie above the rising graph of the hyperbola and below the horizontal line $r=0.333$. Using the evalF function on a TI graphing calculator, the hyperbola is found to intersect the vertical line $x=50$ at $r=.240+\frac{25}{400-50}\approx.311429$. Moreover, by using the SOLVER function on a TI calculator, the hyperbola is found to lie above the horizontal line $r=0.333$ for $x>131.1728$. Thus, paradoxical losses correspond to the points (x,r) such that $50\le x\le 131$ and $.311429\le r<.333$. Theorem 1 justifies the appearance of Figure 1.

Next, determine the probable paradox of Player A losing to someone with a batting average of only 0.240 for the first half of the season (such as Player D). To determine the probability of such a paradoxical loss, the ratio of two areas must be found. (This approach to calculating probability through so-called "geometrical

Figure 1



methods" likely originated in the classic textbook *Higher Algebra* (Hall and Knight 1960, 401–02, especially the example on page 402). This approach is justified if all the feasible points are deemed to be equally likely. In this case, one takes a uniform probability density function, and the usual calculus-based method for computing probability relative to a continuous distribution reduces to taking a ratio of areas. Specifically, the probability in question is the ratio obtained by dividing the area that is above the hyperbola and below the line $r = 0.333$ by the area of the rectangle that encloses all the feasible points. The area of that rectangle is (base)(height) = (350 – 50)(0.5 – 0) = 150. The other area is found by this article's only essential use of calculus—the definite integral

$$\int_{50}^{131} .333 - \frac{121 - .240x}{400 - x}\,dx \approx 0.95245562796.$$

Accordingly, the probability of Player A losing to someone with a first half-season batting average of 0.240 is approximately

$$\frac{0.95245562796}{150} \approx 0.006349704186 \approx 0.635\%.$$

The low value of this probability, which is less than 1 per cent, somewhat justifies the skepticism (if not the disappointment) that Player A felt when informed of his defeat by Player D. The next two sections will show that one should be less sceptical that foes who are more formidable than Player D could also defeat Player A.

We have already seen that Player D can defeat Player A with a paradoxical loss. The above work helps construct and identify another example of a victor: Player D*. The following table describing the batting performance of Player D* can be obtained by taking $50 \le x \le 131$ in the above data for Player G.

Although the mathematical analysis led to a maximum value of x that was slightly greater than 131, it is desirable to have an example of the above phenomenon, along with a table of batting average performances that lists a whole number of at-bats and a whole number of hits for each half of the season. The preceding table displays the example (Player D*) with the largest integral value (namely, 104) for x.

The Probability of Losing Paradoxically to a More Worthy Opponent

Next, calculate the probably paradox of Player A losing to someone like Player C, who had a batting average of 0.294 during the first half of the season. The same reasoning from the preceding section can be used with the following changes. First, replace 0.240 with 0.294. The result is the function $r = f(x) = \frac{121-.294x}{400-x} = .294 + \frac{3.4}{400-x}$ instead of $r = f(x) = .240 + \frac{25}{400-x}$. Replace the number 0.311429 from Figure 1 with 0.3037..., and replace 131.18 with 312.8205 (rounded down to 312). As before, the probability is the ratio of two areas, the denominator is still 150 and the numerator is given by the definite integral

$$\int_{50}^{312} .333 - \frac{121-.294x}{400-x}\,dx \approx 5.52397244398.$$

Hence, the probability of Player A losing to someone with a first half-season batting average of 0.294 is approximately

$$\frac{5.52397244398}{150} \approx 0.03682648296 \approx 3.68\%.$$

An example of someone (Player D**) who can inflict a paradoxical defeat on Player A after recording a batting average of 0.294 during the first half of the season is described in the next table.

Although the mathematical analysis in this section led to a value of x that was slightly greater than 312, it is desirable to have an example of the above phenomenon with a table of batting averages that lists a whole number of at-bats and a whole number of hits for each half-season. The preceding table displays such an example, with x taking the largest possible integral value closest to 200 (that is, 197).

The Probability of Losing to a More Worthy Opponent Whom One Had Defeated During the First Half-Season

Next, calculate the probability that Player A will lose (not necessarily paradoxically) to someone who had a batting average of 0.294 during the first half-season. The same reasoning from the preceding section can be used with the following changes. Because nonparadoxical losses are allowed, the maximum permissible batting average for the second half of the season is 0.500. This changes the integrand. In addition, the upper limit of integration changes from 312 to 350 because the (rising) graph of $r = f(x) = \frac{121-.294x}{400-x} = .294 + \frac{3.4}{400-x}$ remains below the horizontal line $r = 0.500$, the point being that $.294 + \frac{3.4}{400-350} \approx .362 < .500$. As before, the probability is the ratio of two areas, the denominator is still 150 and the numerator is given by the definite integral

$$\int_{50}^{350} .500 - \frac{121-.294x}{400-x}\,dx \approx 55.1839054932.$$

Hence, the probability of Player A losing (not necessarily paradoxically) to someone with a first half-season batting average of 0.294 is approximately

$$\frac{55.1839054932}{150} \approx 0.367892703288 \approx 36.79\%.$$

Closing Comments

A more realistic analysis of the above probabilities may require the consideration of more than just areas. In the above analysis, it was assumed that all points (x,r) in the feasible region were equally probable. This means that constant (or uniform) probability-density functions were implicitly used. An analysis

Table 3

Player D*	At-Bats	Hits	Batting Average
First half of the season	104	25	0.240
Second half of the season	296	97	0.328
Totals for the entire season	400	122	0.305

Table 4

Player D**	At-Bats	Hits	Batting Average
First half of the season	197	58	0.294(416...)
Second half of the season	203	64	0.315(27...)
Totals for the entire season	400	122	0.305

of batting-average data from major league baseball may show that this assumption is inappropriate. In that case, certain points would have to be weighted more heavily than others by using nonconstant (that is, nonuniform) probability-density functions. Such nonuniform density functions would arise as factors in integrands figuring in the numerators (and implicitly in the denominators) of refinements of the above probability calculations.

A more realistic analysis would allow for competition between players who have a different number of at-bats for the entire season. More independent variables would need to be introduced to address such considerations. The resulting graphs would not be planar and the resulting calculations would involve multiple (or iterated) integrals. Computer technology would be needed for the analyses and the calculating of the more realistic probabilities. From a qualitative point of view, these more realistic modelling activities would likely lead to the same conclusions that were drawn from the more accessible work described above.

One role of mathematics and philosophy is to attempt to resolve a paradox by clarifying the (often unstated) assumptions that underlie the paradox's original formulation. Consider, for instance, the paradox of Zeno that claims that flight is impossible. Zeno reasoned that at any given instant an arrow cannot be in motion. Cameras seem to support this view by presenting images of objects that are momentarily frozen in one place. Zeno reasoned that, because time is nothing more than a succession (or set, as we might say now) of instants, there is no time during which motion is possible. The flaw in Zeno's argument is that moving objects can, in fact, have nonzero instantaneous velocities. This great insight of differential calculus was the key concept that defeated Zeno's paradox. What insight, if any, can defeat the paradox expressed by Player A?

Player A, like every student of mathematics, needs to understand that batting averages are global entities that summarize performances over extended periods of time. Player A was surprised at losing to Player D because he had not accounted for competition from a player who had a different number of at-bats than he did during each half-season. I hope that the following result will be of some consolation to Player A (and the reader). The result justifies the intuition of anyone who shared Player A's confusion; Theorem II shows that paradoxical losses are impossible on a level playing field—that is, when both players have the same number of at-bats in each half of the season.

Theorem 2

Two players cannot be involved in a paradoxical loss if they have the same number of at-bats in each half of the season.

Proof

Suppose that Players E and F each have b at-bats during the first half of the season and B at-bats during the second half of the season. Suppose that Player E has h_1 hits during the first half of the season and h_2 hits during the first half of the season (and Player F, respectively, has H_1 and H_2). Finally, suppose that Player E has a lower batting average than Player F during each half-season—that is,

$$\frac{h_1}{b} < \frac{H_1}{b} \text{ and } \frac{h_2}{B} < \frac{H_2}{B}.$$

Player E therefore has a lower batting average than Player F for the entire season—that is, that

$$\frac{h_1 + h_2}{b+B} < \frac{H_1 + H_2}{b+B}.$$

Now, since $b > 0$ and $B > 0$, it follows that $h_1 < H_1$ and $h_2 < H_2$. Therefore, $h_1 + h_2 < H_1 + H_2$. Because $b + B > 0$, the required inequality follows and the proof is complete.

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A Linear Combination of Traditional Instruction and TPSWriC for Math-Anxious Students

Thomas Varghese and Elizabeth Pace

Introduction

Everyone in a classroom is unique. Some students like mathematics; others do not. Mathematics comes easily for some; others struggle with it every step. For others, mathematics is a stress inducer, and others even see it as a punishment. Mathematics anxiety has been discussed periodically in the professional literature for several decades. In 2001, the No Child Left Behind legislation was passed in the United States to ensure that every child meets high standards of mathematical proficiency, and this has required educators to give increased attention to the study of mathematics anxiety. Fotoples (2000) noted that, while the United States and the rest of the world have caught up in the growth of communications and electronics, expertise in mathematics has suffered. Many employers find employees' math expertise inadequate and their skills in basic arithmetic insufficient. Along with proficiency in number sense and computational skills, other domains of mathematics are also essential for educational advancement and career opportunity. Many jobs require substantial mathematics, such as forecasting, budgeting, modelling and statistical analysis. A lack of mathematical understanding can affect career-selection options and advancement. It can also make it difficult to respond positively to changes in the workplace environment. The mathematics competency of every student is a matter of worldwide concern. We have noticed through our classroom experience that math-anxious students have the greatest difficulty in learning mathematics skills. Hence, as teachers, we should endeavour to more effectively teach math-anxious students.

Mathematics Anxiety

Wood (1988) defined mathematics anxiety as the general lack of comfort when required to perform mathematically. Ma (1999) noted that mathematics anxiety can take multidimensional forms, such as dislike (an attitudinal element), worry (a cognitive element) and fear (an emotional element). He also reported the negative consequences of being anxious about mathematics. These consequences include the inability to do mathematics, a decline in mathematics achievement, the avoidance of mathematics courses, and associated feelings of guilt and shame. Limiting one's participation in mathematics courses potentially limits future course selection and career paths.

Research studies (Fiore 1999; Handler 1990; Norwood 1994) have indicated that teachers play a major role in contributing to mathematics anxiety in their students. Tobias (1978) and Stodolsky (1985) noted that the beginning of math anxiety can be traced to negative classroom experiences. Greenwood (1984) suggested that mathematics anxiety results more from the way the subject matter is presented than from the subject matter itself. Teachers can create anxiety by placing too much emphasis on the memorizing formulae, learning mathematics through drill-and-practice and applying rote memorized rules (Greenwood 1984). Rote memorized rules and manipulation of symbols can be stumbling blocks to a child's learning (Skemp 1986). Mathematics anxiety can therefore be considered a function of teaching methodology.

Instructional Strategies

Given the effect of negative classroom experiences on the development of math anxiety, it is critical to examine classroom practices and modify instructional methods to ensure quality mathematics teaching. It is important that students experience success in the classroom. If students experience success in learning mathematics, they will be inclined to like the subject and the teacher. Success and motivation depend on self-efficacy, and self-efficacy is based on positive past learning experiences. There are a variety of nontraditional (or alternative) instructional strategies that promote success and thereby mitigate mathematics anxiety (Newstead 1998; Norwood 1994; Vacc 1993; von Glasersfeld 1991).

Nontraditional instructional approaches that involve more personal and process-oriented teaching and that emphasize understanding rather than drill-and-practice are believed to reduce mathematics anxiety (Newstead 1998). It has also been suggested that encouraging students to work with peers in small, cooperative groups may have important, affective consequences, including a reduction of anxiety (Vacc 1993; von Glasersfeld 1991). Problem solving and discussing strategies for problem solving may also reduce mathematics anxiety (Skiba 1990; Vacc 1993).

In contrast, emphasis on discussion, explanation and justification of strategies can cause anxiety in children. These children often prefer a classroom in which they are given strong direction from a teacher (Newstead 1998). Perception of strong direction from a teacher lessens previously formed anxiety and discomfort (Clute 1984; Newstead 1998; Norwood 1994).

Berliner and Rosenshine (1987) reported that it is most effective to teach in a systematic manner, providing instructional support for the students at each stage of learning. Guided practice followed by independent practice fosters fluency. This strategy involves reviewing material and then presenting new material with diverse and concrete examples, which the students solve with the teacher's guidance.

According to Newstead (1998), it is difficult to conclude that traditional teaching approaches always lead to math anxiety while alternative classroom approaches (which emphasize discussion, understanding and problem solving) do not. Some pupils thrive in the security and structure of a formal classroom while others are interested in the responsibility and creativity associated with problem solving and discussion. Both approaches can cause mathematics anxiety. Hence, a combination of both traditional and nontraditional approaches may help different types of learners.

The following section suggests an instructional strategy that is a combination of both traditional and nontraditional approaches.

A Linear Combination of Traditional Mathematics Instruction and TPSWriC

Bandura (1982) believed that people's perception of their own effectiveness plays a major role in their behaviour. We propose an instructional strategy that will cultivate in students a positive perception of ability in a mathematics classroom. This strategy begins with a structured lesson in which procedures and rules are clearly explained through guided practice, followed by independent practice in which students verbalize their thought process and write down what they understand. This strategy is a combination of traditional instruction (direct teaching) and an alternative method (TPSWriC—pronounced *T-P-S-Rick*—the *W* is silent).

Norwood (1994) found that students with mathematics anxiety are more comfortable in highly structured classrooms than less-structured classrooms. Math-anxious students do not trust their instincts or intuitions, and therefore do not prefer to work with a discovery approach to learning mathematics. The proposed strategy therefore begins with a structured, direct-instruction component.

Direct Instruction

The role of direct instruction (or lecture method) in a mathematics classroom has been clearly documented. Many authors (Gunter, Estes and Schwab 1995; Krantz 1998; Wu 1998) have pointed out that lecturing is an effective way of teaching mathematics. When teachers give specific instructions and take students through a task step-by-step, students can master the skill. Krantz noted that the lecture is a powerful teaching device that has stood the test of time. It has been used to "good effect for more than 3,000 years" (Krantz 1998, 12). Although this method has received much criticism from contemporary educators, the survival of the lecture is itself evidence that it has unique strengths as a teaching method (Freiberg and Driscoll 1996; Henson 1988). Borich (2004) indicated that a lecture should neither be a lengthy monologue nor an open, free-wheeling discussion. It should be a quickly paced, highly organized set of interchanges controlled by the teacher and focused on acquiring a limited set of predetermined facts, rules or action sequences. By adding wit and humour in a presentation, a teacher can arouse curiosity and amusement in the listener.

Teaching the first part of a lesson systematically (through direct instruction) and the second part through guided practice is a strategy based on the research by Berliner and Rosenshine (1987). Students who prefer a systematic, structured classroom to a nonstructured classroom will benefit from this instructional methodology (Vinner 1994).

Having a discussion at the beginning of a new chapter or concept alienates certain students. Norwood (1994) also remarked that math-anxious students feel uncomfortable while being taught with nontraditional methods. Once they have had success in a structured classroom, math-anxious students will be more open to nonstructured classrooms. Vinner

(1994) suggests that procedures and rules (which are cognitively simpler, clearer and easier to handle) give math-anxious students an emotional security. Math-anxious students should initially be given something to help them solve a problem, rather than having them create their own solution procedure (Vinner 1994). However, once they get the problem-solving skills, math-anxious students are more willing to begin discussing problems at a deeper level.

Hence, we propose beginning with a direct instruction component. The underlying assumption is that with direct instruction, "all students can learn to think mathematically when carefully taught" (Marchand-Martella, Slocum and Martella 2004, 209).

Marchand-Martella, Slocum and Martella (2004) noted that many educators equate child-centred instructional strategies (such as discovery learning) with conceptual understanding, and teacher-directed methods with rote learning and mindless computation. However, "this view promotes a false dichotomy between basic skills and conceptual understanding" (Marchand-Martella, Slocum and Martella 2004, 210).

We should not "conveniently forget" that not all mathematical concepts lend themselves to discovery (Carnine 1990, as cited in Marchand-Martella, Slocum and Martella 2004). Carnine indicated that many students, particularly low-performing students, learn more quickly when given a clear, concise explanation of what to do and how to do it. When explicit strategies are not provided, students often make up their own mathematical rules, which are often creative but incorrect. Moreover, discovery learning does not provide a gradual transition from structured to independent work (Marchand-Martella, Slocum and Martella 2004). In mathematics, skills and understanding are completely intertwined. Precision and fluency in the execution of skills are the requisites for conceptual understanding (Wu 1999). "From the intuitive to the abstract, and primitive skills to sophisticated ones; such is the normal progression in mathematics" (Wu 1999, 16). Once students understand a topic or concept, and attain fluency in the required skills through the guidance of their teacher, they can confidently move on to discussion and questioning. They will also be more inclined at this point to explore the concepts using alternative strategies. Hence, the second part of this strategy is nonstructured.

TPSWriC

During the second part of the lesson, students move from structured practice to more open (or independent) practice. This phase emphasizes thought processes and their verbalization, and a written response. This is based on a five-step process called TPSWriC (T—think, P—pair, S—share, Wri—write and C—compare). *Pair and share* is an old technique to which we suggest adding the components *think*, *write* and *compare*.

Math-anxious learners should be allowed to continually move ahead and make progress. Through appropriate instructional strategies, teachers can help math-anxious students continue to think, write and try, even when the students feel they can't proceed.

T (Think)

Students think about every question and develop an idea or strategy to solve the problem. This enables them to briefly work on their own without the pressure to arrive at the right answer. Students reflect on the new concept or topic introduced in the class and explore alternative strategies based on the knowledge they acquired in the first part of the lesson.

P (Pair) and S (Share)

Students form groups and share their ideas for solving the problem. Cooperative learning is an excellent tool for helping those who suffer from mathematics anxiety (Vacc 1993). Two to four students can share responsibility for solving a problem instead of just one, and it becomes okay to make mistakes and ask questions. There is less pressure to find one method or one right answer, and different approaches are more likely to be proposed and explored. This helps develop a richer understanding of the topic. Group interaction or cooperative learning promotes female and minority students' self-esteem, motivation and achievement (Croom 1995). Also, when students participate in cooperative learning, their attitudes toward their classmates improve, particularly toward those from different ethnic backgrounds (Slavin 1986). Thus, students will learn to respect the point of view and accept the differences of other students.

Wri (Write)

Students may have arrived at an answer or a way of solving a problem by their own thought processes, group work or both. Writing helps students make sense of mathematics and helps them practise inferring, communicating, symbolizing, organizing, interpreting, linking, explaining, planning and reflecting (Countryman 1992). Verbalizing during the discussion and then writing down the methodology (or thought process) activates the mind and reinforces the problem-solving strategy.

C (Compare)

The Russian psychologist Vygotsky pointed out that students accomplish different things when they work on their own rather than with the guidance of a teacher. The work students do under the guidance of a teacher enables them to succeed on their own later on (Vygotsky 1978, as cited in Epp 1998). Once students develop their written response through their own thought processes and/or through discussion, they need to verify that they are on the right track. Students can compare the methodology and/or answer given by a teacher with their own methodology and/or answer. This step paves the way for classroom discussion. Discussion allows students to justify their own answer and gain insight into the thought processes of others. As well, Vygotsky (1978, as cited in Croom 1995) suggested that group interaction helps develop mental operations or processes in children because they tend to internalize what is discussed.

Summary

Wu (1999) noted that some mathematics educators want to stop teaching basic skills and instead teach mathematical understanding. However, precision and fluency in basic skills form the basis for conceptual understanding (Wu 1999). This instructional strategy is intended to help math-anxious learners by providing them with a firm foundation in the required concepts and then encouraging them to attain higher-level thinking skills. We are aware that even mentioning procedures and rules can cause open hostility in some mathematics education circles. Some educators believe that having math-anxious students invent their own algorithms promotes conceptual understanding. However, correctness and generality become two major concerns when students are allowed to make up their own algorithms (Wu 1999). As Wu noted, it is also a Herculean task to examine 30 different algorithms in a class of 30 students and then work with each student on their own understanding. Open-ended problems can be used to build confidence in mathematical problem solving, but only after students have learned the required skills to tackle the problem. The advantage of the proposed strategy is that it helps students achieve a firm foundation in a concept, topic or skill and then encourages them to work both independently and cooperatively at various stages.

Direct instruction helps teachers present new content by breaking it into smaller segments. During this part of the lesson, teachers can ask direct questions to get an idea of students' understanding of new concepts or skills. TPSWriC transforms the class into an environment with a nonstructured pace. This strategy encourages students to get involved in their own thought processes, verbalize these processes and reflect on their written responses through a comparison with the teacher's suggested answer. This overt verbalization (especially through group discussions) not only helps students attend to their own strategies but also boosts their self-confidence. Students who cannot find an answer on their own can listen to others in the group. Through this strategy, teachers are responsible not only for the intellectual development of students but also for emotional support. Rather than complaining that the students lack ability, teachers can teach effectively by building confidence in their students. This has a positive effect on both the social and academic skills of students.

Classrooms that work in this way lessen the anxiety and discomfort for some students (Vinner 1994). The security and structure of a lesson are initially maintained. Once a student knows the content, his anxiety is lessened. At the same time, pupils who want to understand better by asking "why" and "how" are given opportunity to do so through discussion and through developing their own thought processes. Classroom discussion helps students develop an understanding of themselves and others. Social skills—such as showing encouragement, giving direction and asking for help—are facilitated by both the teacher and other students. This not only reduces subject-related anxiety but also shows students that their peers possess both strengths and weaknesses, which boosts the morale of math-anxious students.

Many of the authors cited in this article have successfully used this strategy as a dominant form of instruction in their classrooms. The advantage of this strategy is that the strengths of traditional instruction are supplemented by the strengths of nontraditional instruction.

Through this strategy, students are more likely to achieve the goals of the National Council of Teachers of Mathematics (NCTM 1989): to value mathematics and develop confidence, to reason and communicate mathematically and to solve problems. Thus, no child is left behind.

> Let us teach our children mathematics the honest way by teaching both skills and understanding.
>
> —Wu 1999, 52

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Concept and Task Analysis: An Alternative Look at Planning Instruction

Regina M Panasuk

The National Council of Teachers of Mathematics (NCTM 2000) lists the following headings under its process standards:

- Problem solving
- Reasoning and proof
- Communication
- Connections
- Representation

One of the process standards, connections, puts special emphasis on understanding how mathematical ideas interconnect and build on one another to produce a coherent whole. The question is how to help high school students make these connections, particularly students who do not always understand the relationships between the concepts they studied in junior high. This article addresses this question using concept and task analysis and its role in planning mathematics teaching.

Concept and Task Analysis

Concept and task analysis, taken from Gagne's (1970) learning hierarchies and Ausubel's (1968) theory of advance organizers, is a widely accepted idea that is used to inform instruction and facilitate meaningful learning. The purpose of concept and task analysis, or decomposition, is to gain insight into the structure of a given concept or task that students must learn or perform. This is done by breaking down superordinate concepts or tasks into their constituent subconcepts or subtasks (also known as decomposing). This analysis helps classify and categorize students' cognitive processes and actions (Gagne 2001). Many reasoning skills and tasks in which students are engaged are quite complex. They involve various relatively small and distinct subconcepts and subskills that students are expected to learn. For example, operating with positive and negative numbers is a prerequisite for the concept of collecting like terms, which in turn is a prerequisite for solving multistep equations. The flowchart in Figure 1 represents the organization of the subconcepts that constitute a superordinate concept, in this case, the concept of the product of two negative numbers. It also represents the hierarchy of the subconcepts involved in the superordinate concept of the product of two negative numbers. This classification can help the educator grasp the big picture.

Concept and task analysis is the process of progressing from abstract to concrete, and from complex to simple[1]. Students who reason at the concrete or preoperational levels (Piaget 1970) are most successful when they advance from concrete to abstract, as shown in Figure 2a. For these students, if the concept is presented gradually through three different modes, they are able to grasp the relations among the concepts and enhance their thinking skills. The three modes are: concrete (working and operating with real objects, geometrical shapes and simple, hands-on experiments), mental imagery (pictures or diagrams that summarize and represent properties) and abstract (or symbolic) representation (formulas and terminology). Figure 2 illustrates the representations from concrete to abstract (Figure 2a) and from simple to complex (Figure 2b) as two parallel, top-to-bottom processes. At the same time, Figure 2b, when viewed from bottom to top, shows concept and task analysis as a reciprocal process. For example, one way to present a complex concept (such as factoring the binomial $2x + 6$) from concrete to abstract and from simple to complex (a top-to-bottom process) is to decompose or subdivide it into subconcepts (a bottom-to-top process). The binomial $2x + 6$ is factored by applying the distributive property, an abstract concept that is not likely to be understood and successfully applied without knowing the underlying concepts and meaning.

The first subtask (see Figure 2b) might involve multiple subconcepts. For example, it may be necessary to write each term of the binomial as a product to determine the common factor, which in turn requires understanding that multiplication can be represented as repeated addition (for example, $2x = x + x$). The product of two positive quantities can be represented as the area of a rectangle, the adjacent sides of which have the measure equal to the values of the factors of the product. Knowledge of coefficient and of prime and composite numbers may also be required.

Figure 2b also illustrates the inverse operation of factoring [such as presenting the product $2(x + 3)$ as a polynomial], which in turn relates to the concept of order of operations. Students may assume that work inside parentheses should be done first, such as with the example $2(x + 3)$. This would indicate two things. First, those students do not fully understand the nature of variables (x and 3 can be added to obtain a binomial, but until the variable is involved, the binomial cannot be converted to a monomial). Second, those students do not understand the connection between the order of operation and the properties of operations, nor that some problems can be calculated by integrating several concepts (Panasuk, in press). For example, when calculating the value of the numerical expression $\frac{7}{4}\times(0.5+\frac{4}{7})$, $\frac{7}{4}$ can be distributed through through 0.5 (that is, $\frac{1}{2}$) and $\frac{4}{7}$, and then the two products $\frac{7}{4}\times\frac{4}{7}$ and $\frac{7}{4}\times\frac{1}{2}$ can be added to produce $\frac{7}{8}+1=1\frac{7}{8}$.

Understanding the subconcepts and subskills listed above is the easiest way to successfully complete a problem involving a complex concept. Identifying and categorizing these subconcepts and subskills in a hierarchy of knowledge helps teachers make instructional decisions when planning lessons.

Figure 1
Flowchart of the product of negative numbers subconcepts

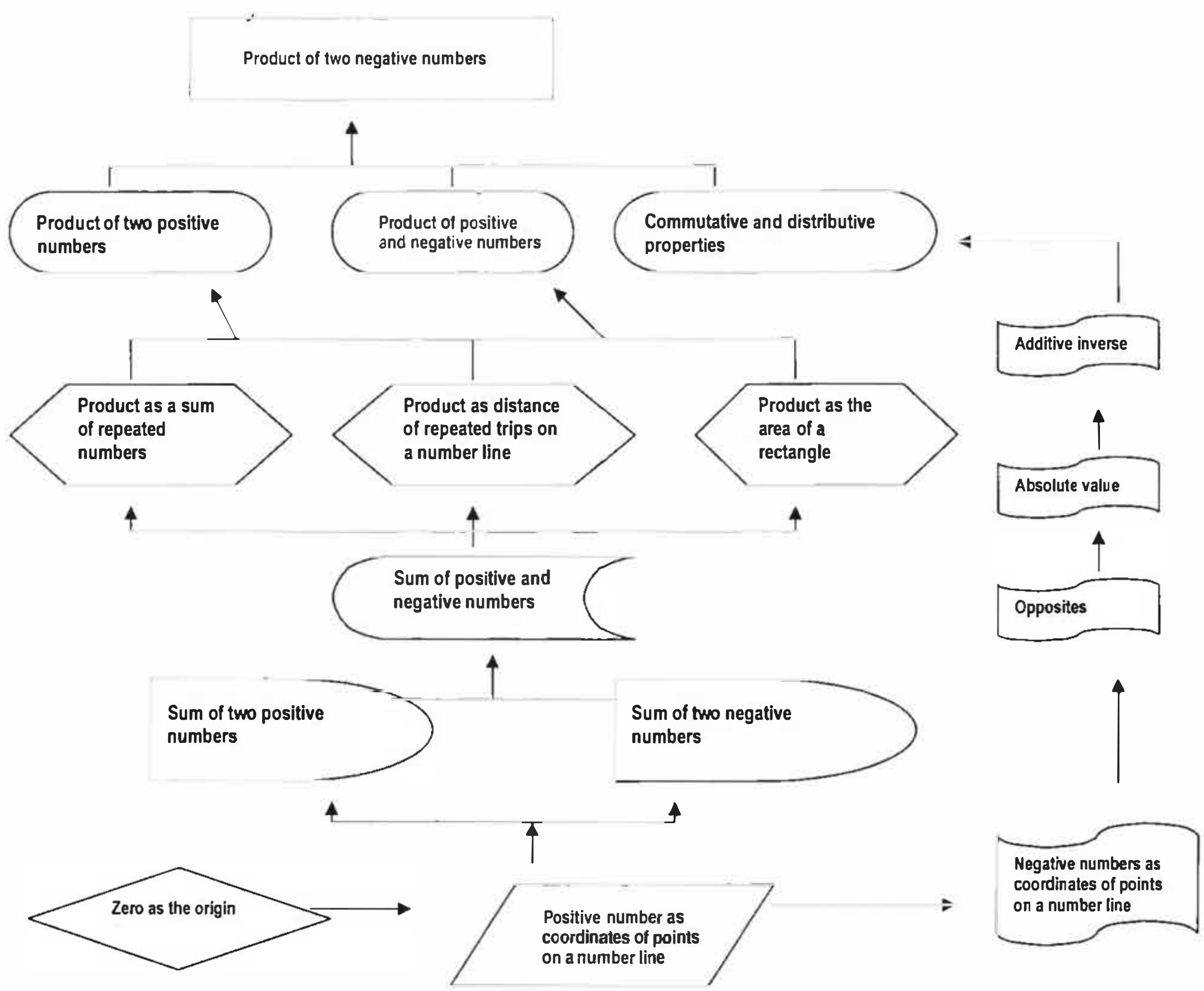
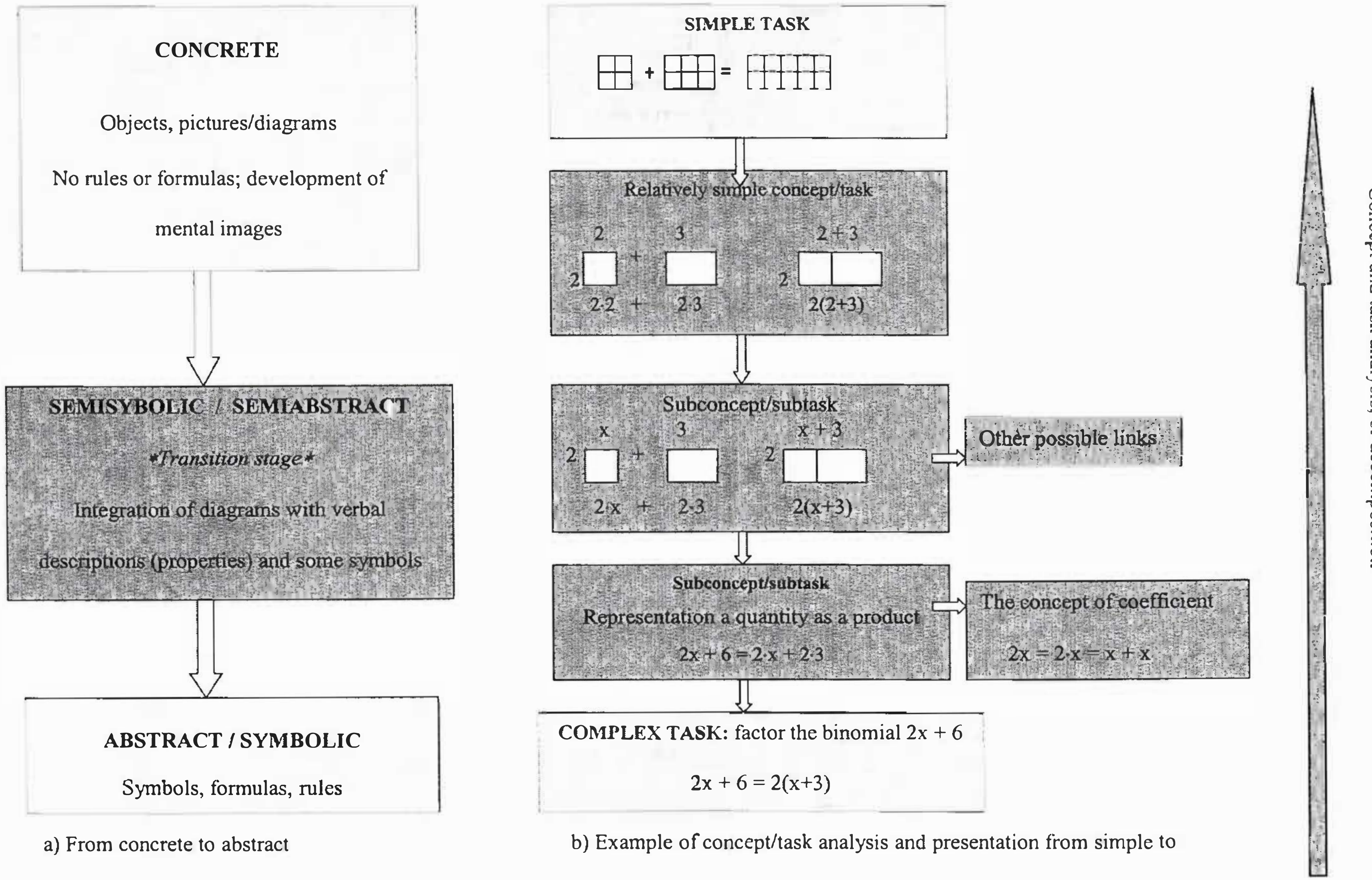


Figure 2

Comparison of the relationship between concept/task analysis and presentation from concrete to abstract and simple to complex



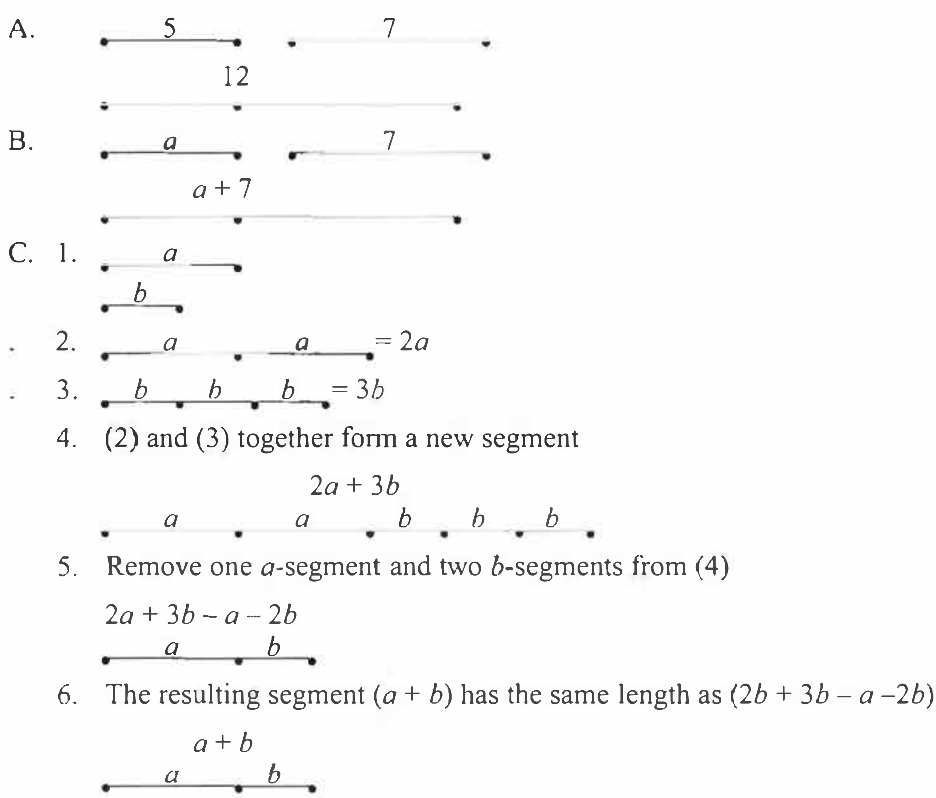
a) From concrete to abstract

b) Example of concept/task analysis and presentation from simple to complex

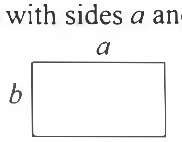
How do teachers identify smaller connections within a subconcept or subtask? What process do teachers go through while analyzing a concept or task and devising a list of subconcepts and subtasks that students are expected to know? How infinitesimal should the links between the subordinates be? To answer these questions and to address concept and task analysis in more detail, consider the concept of solving linear equations with one variable. Assume that the equation $2(3x - 7) - 2(4 + x) = 2$ is a superordinate concept that must be studied during the upcoming curriculum unit. The decomposition of this high-level task into subtasks and operations, and naming the subconcepts underlying each of the subtasks helps teachers comprehend the nature of the concept. This in turn helps order student experience in a logical and coherent manner. The analysis enables teachers to select and sequence the activities required by ordering those activities from simple and less inclusive to complex and more inclusive.

Concept mapping, as shown in Figure 1, can help students record and organize subordinate ideas. Of course, teachers know how to solve the equations they present in class, and concept mapping may be considered unnecessarily utilitarian. The point, however, is not to find the answer but to scrutinize each step, observe whether the equation can be broken into subtasks, recognize the subconcepts involved in each

Figure 3
Using the line segment to represent a variable



Note: The popular student misconception $2a + 3b = 5ab$ can be treated using the variable-segment idea as well. The product of two positive quantities a and b can be illustrated as the area of a rectangle with sides a and b.



While $2a + 3b$ represents a line (a one-dimensional figure), $5ab$ would represent a rectangle (a two-dimensional figure).

subtask and delineate the subconcepts by developing a hierarchy of prerequisite knowledge that students must be able to combine in a new structure to master higher-level concepts. The number of steps, the range of detail into which to decompose, and the hierarchy of the subconcepts and subtasks can vary, depending on the teacher's best judgment of the students' prior knowledge, needs, ability to execute a chain of simple tasks and ability to explain their actions. Some students display the desired mathematical skills with minimum guidance and need few ideas to grasp the concept. Others require more support and become competent from seeing more increments. For those students, in particular, missing some steps could contribute to the development of misconceptions.

Solving a linear equation involves executing a set of distinct, relatively simple and sequential tasks. The following is an example of how the equation $2(3x - 7) - 2(4 + x) = 2$ could be solved:

$2(3x - 7) - 2(4 + x) = 2$

$6x - 14 - 8 - 2x = 2$ (application of the distributive property, and operations with positive and negative numbers)

$4x - 22 = 2$ (collecting like terms by applying associative and commutative properties)

$4x = 24$ (addition of 22 to each side of the equation using additive inverse $[a + (-a) = 0]$)

$x = 6$ (dividing both sides of the equation by 4, using multiplicative inverse $[a \times \frac{1}{a} = 1; a \neq 0]$)

These steps, described using minimal language, are important in the process of solving equations and are a well-defined routine that helps establish and maintain certain order and parsimony. However, many students might view this transformation as a mysterious procedure that cannot be understood and must be memorized and imitated. Because of its refined form, the procedure has a specific logic, which in turn makes it difficult to understand when it is presented in a concise way. Presenting a brief and refined procedure can be a barrier to learning for many students. The procedure makes sense when it is part of a system of related concepts. Unfortunately, accurately performing a procedure will not guarantee that a correct solution to the problem will be found.

A close look at each of the four steps will help reveal the underlying subconcepts that make up the procedure.

Distributive Property

The distributive property is a complex concept, particularly when used with expressions with variables. Errors are often made when the distributive property is applied, perhaps because students have not yet developed a solid visual image of the property (see Figure 2b). It is common for students to forget to distribute a negative sign and/or view the expression $(x - y)$ exclusively as subtraction. Some have difficulty applying the difference property, $(x - y) = x + (-y)$, which encourages thinking of subtraction as addition. To help students understand that they are actually distributing –2, for example, they can be encouraged to perform the expansion using smaller increments, such as $2(3x) + 2(-7) + [-2(4)] + (-2)x = 2$. The transition from this expression to $6x + (-14) + (-8) + (-2)x = 2$ may seem trivial at first glance, but it involves several subconcepts that must not be overlooked. These subconcepts are the order of operation that was discussed earlier, the multiplication of positive and negative numbers, the concept of coefficient, and the rule that multiplication does not distribute over multiplication or division but only over addition and subtraction. Students must know that $(2)(3x) = (2 \times 3)(2 \times x)$ and be able to draw a chart similar to that of Figure 2b. Grades 6, 7 and 8 mathematics teachers who were involved in a study of lesson planning and concept and task analysis indicated that they "entered the study with a wrong assumption." They had been convinced that proceeding with "baby steps" was "too elementary" for the students (Panasuk, Stone and Todd 2002, 821).

Collecting Like Terms

The concept of collecting like terms can be dissected into relatively simple subconcepts, including coefficient; variable; addition and subtraction of positive and negative numbers; and commutative, associative and distributive properties. Each of these subconcepts can be interpreted through different representations. For example, a variable can be represented as a line segment—the method invented and used by ancient mathematicians (and a shift from abstract to concrete). This can help students visualize and operate with the terms of the expressions (see Figure 3). Representing the terms $2a$ and $3b$ in the expression $2a + 3b$ as two segments of length a and three segments of length b may help some students manipulate terms of polynomial expressions. The segments are easily drawn or diagrammed, which makes it easy to see the resulting segment and name its length. A different colour can be used to represent a negative number. To solve the equation $2(3x - 7)$

– 2(4 + x) = 2, for example, $6x$ and $-2x$ will have to be added. Set up a diagram with six segments of the same length and colour (representing $6x$) and two segments of the same length but different colours (representing $-2x$). Pairs of segments of the same length but different colour behave as additive inverse and cancel each other, leaving a sum of $4x$.

When students lack basic knowledge, teachers sometimes have to revert to the rudimentary level. For example, the expression $6a - 2a$ could be rephrased as "6 apples – 2 apples = 4 apples" and then converted back to $6a - 2a = 4a$. (Dollars could be used to illustrate negative difference: \$2 – \$4 = –\$2.)

Further Decomposition

Further decomposition of the one-step linear equation can also incorporate visual representations of the unknown,[2] be replaced by a word that represents the object or be represented as a line segment. When solving one-step linear equations becomes an obstacle, referencing diagrams like those in Figure 4 could be helpful.

Incorporating Concept and Task Analysis into Planning and Teaching

How can a teacher integrate the knowledge gained from concept and task analysis into planning and teaching? Concept and task analysis is one of the most important ingredients in lesson planning (Panasuk 2002; Panasuk, Stone and Todd 2002). It helps teachers make instructional decisions and helps students recognize and build connections among subconcepts. Teachers who have worked with concept and task analysis possess a better understanding of the prerequisite knowledge and skills that are needed. Teachers are better prepared to identify students' prior knowledge and facilitate learning by choosing and sequencing the activities, problems and/or exercises that integrate the prerequisite knowledge.

Although task analysis is important for each element of planning—including objectives, homework, developmental activities and mental mathematics

Figure 4
Representations of the concept of solving linear equations

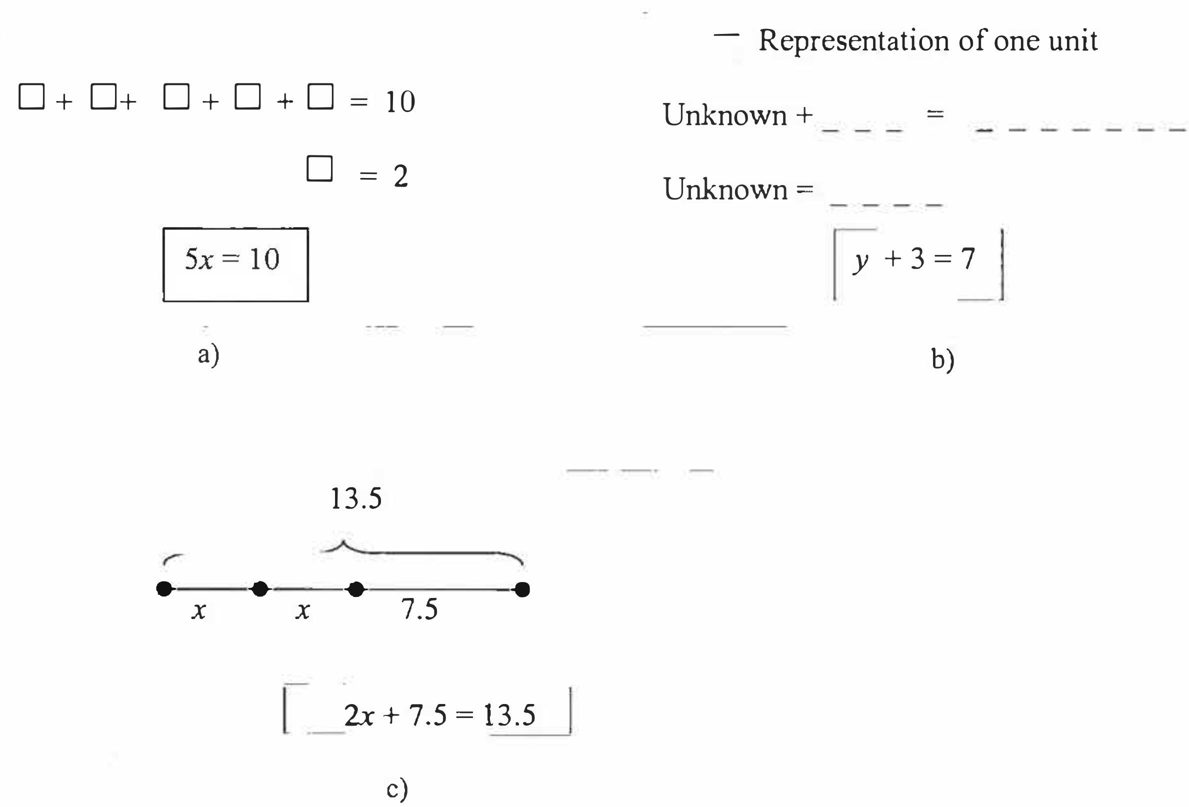


Figure 5
Possible mental mathematics problems for the unit on like terms and operations

Level 1

1. Combine together like (similar) shapes

▲ + ▲ + ▲ + ▲ + ■ + ■ + ■

♥ + ♥ + ♥ + ○ + ○ + △

2. Combine similar items:

2 donuts + 3 sodas + 2 donuts + 3 sodas

5 baseballs + 3 clubs + a baseball

3. True or false:

2 girls + 3 girls = 5 girls

$2g + 3g = 5g^2$

$2g + 3g = g \times (2 + 3)$

Level 2

4. Name the coefficient of each term:

$5d,\ -3x,\ y,\ -mn$

5. Name the coefficient of each term:

$5d - 3x + y - mn$

6. True or false:

$5d - 6y - 3d = 2d - 6y$

7. Find the perimeter of the following figures:

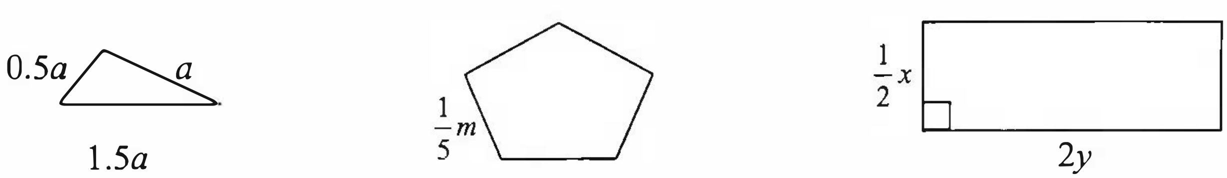


Figure 6
Example of mental mathematics problems to precipitate solving multistep linear equations

See Figure 2b

Operations with positive and negative numbers; the commutative and associative properties

See Figure 4

Possible links

a) Distributive property:

True or false:

$2(x + 3) = 2x + 6$

$3(2x - 1) = 6x - 1$

$-2(4 + x) = -8 + 2x$

b) Collect like terms:

$3s + 4s - 2s + 12$

$6x - 14 - 8 - 4x$

$7p + p + 3r - 2p - r - 5p$

c) Solve one-step equations:

$5x = 10$

$y + 3 = 7$

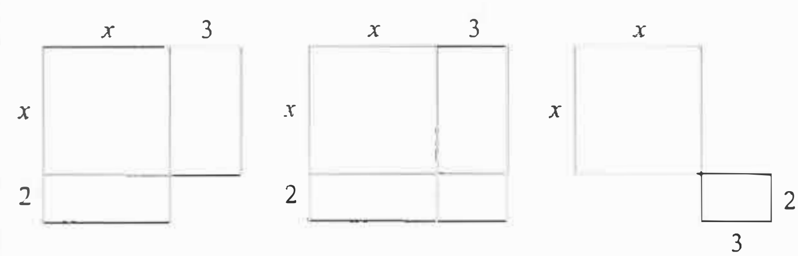
$4t = 13$

Big concept: The students must be able to solve equations such as

$$2(3x - 7) - 2(4 + x) = 2$$

(Panasuk, Stone and Todd 2002; Panasuk and Todd, in press)—the following will only address its application to mental mathematics. Mental mathematics is an effective teaching tool that can test and advance both basic skills and higher-order thinking (Reys, Reys, Nohda and Emori 1995; Panasuk 2002). Unfortunately, mental mathematics is often associated with elementary-level grade mental computations that are not necessarily connected to the lesson. Mental mathematics can be designed for any topic, concept or grade, and can effectively meet different instructional purposes. The usual recommendation is that mental mathematics activities not exceed 7–10 minutes and be fast-paced and simple enough for all students to be able to respond to them. Figure 5 lists mental mathematics problems that can be used to teach about collecting like terms. Students are encouraged to complete a set of relatively simple tasks that are organized from simple to more complex. Students who have achieved a higher level of abstract reasoning will have outgrown examples 1, 2, and 3 in Figure 5. However, students who have not fully grasped the concept of like terms would still benefit from this rudimentary representation.

Table 1
Connecting mental mathematics and homework problems

Examples of homework problems (not for the same lesson)	Next class mental mathematics problem
Solve the equation: $3x + 7 = -2$	Check whether $x = 3$ is the solution of the equation $3x + 7 = -2$
Collect like terms: $-3x - y + 2x + 7y - x$	True or false: $-3x - y + 2x + 7y - x = (2x - 4x) + (7y - y)$
Solve the inequality: $-12y < 3$	Given the inequality $-12y < 3$, is $y < -4$ true or false?
Represent the product as a trinomial: $(x + 2)(x + 3)$	Which figure correctly represents the product $(x + 2)(x + 3)$? 

For a lesson in solving linear equations, such as the earlier example of $2(3x - 7) - 2(4 + x) = 2$, a set of mental mathematics activities should be created to reflect the key subconcepts that are linked together into a larger concept later on in the lesson. These mental mathematics problems can include those shown in Figures 6b, c and d. Again, if further reduction of the complexity of the problems and level of presentation is needed, the examples given in Figure 2b and Figure 4 can be used. Mental mathematics activities are best used when they are prepared in advance and displayed to the whole class, such as on an overhead transparency. Different aids, such as individual white boards, are helpful in engaging all of the students. This instructional efficiency creates more time for students to think, communicate and explain.

To make a transition from relatively simple tasks that can be done mentally (see Figures 6b, c and d) to another context (such as solving the equation), students could be encouraged to combine their prior knowledge and skills into a new strategy (see Figure 6a). This instructional decision can be more effective than simply giving the steps of the procedure and working through some examples, which is rarely effective and contributes to a lack of understanding of subconcepts, which is compensated for through imitation and/or rote memorization (NCTM 2000).

When appropriate, mental mathematics can be a way to review homework to ensure that the association between lesson components has been made. Table 1 displays examples of homework problems and their mental mathematics counterparts. When combining their homework review with mental mathematics, students are encouraged to communicate, analyze and evaluate their thinking, and to compare their strategy with other approaches. Students are also encouraged to reveal their misconceptions, if any.

After completing concept and task analysis, teachers can provide students with the most useful experiences for meaningful learning. Concept and task analysis helps teachers plan gradual progression from one level of representation to another, ultimately helping students develop an understanding of the concepts and learn the skills to become confident when solving other problems.

Notes

1. Simple does not always mean concrete. The concept $(a + b)^2 = a^2 + 2ab + b^2$ can seem simple, but it is actually an abstract (symbolic) representation of a square of binomial.

2. The notions of *unknown* and *variable* are often used interchangeably, but there is a distinction between the two. A variable, represented by a letter, is a cumulative representation of a set of numbers, such as $a + b = b + a$, where a and b are any real number. An unknown, also represented by a letter, is a number that must be identified, such as when solving the equation $x + 2 = 3$. In this equation, x stands for a unique number that would make the equation a true numerical statement.

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Selective Testy Comments About Testing, Tests and Test Items

Werner Liedtke

Tests and testing are an integral part of teaching and learning. The results from tests are used for more than making instructional decisions and adjustments. Data from tests are used to rank schools and compare achievements in school districts, as well as provinces/territories and countries. Responses on tests are used to generate and publish statements about the development of student numeracy. These are important decisions, and testing and tests are therefore also important—the testing settings, the tests and the items on the tests yield the information that is both necessary and appropriate for these types of tasks.

We often assume that the tests we use provide us with the information we think they do, and that the items on these tests assess what we think they assess. Several examples will be presented in this article to illustrate that such an assumption may not always be true, and that greater care needs to be given to selecting test settings and designing test items. Without this care, the conclusions that are based on the results of tests may not be as meaningful and may come into question.

Test Settings

A valid comparison of test results is based on the premise that the tests were administered under similar circumstances for all or most students. However, this may not always be the case. Published rankings of schools by newspapers, and parents in British Columbia being allowed to send their children to the school of their choice puts a lot of pressure on having students do well on tests. The following examples indicate this.

A report from the United States included in a recent issue of *Maclean's* magazine mentioned that young students in a school were asked to change their answers on a test they just had written.

During the second part of a final practicum, one of my student teachers came to me to say that she was unable to teach a mathematics unit she had planned for her Grade 4 class. The school had decided to set aside three weeks to prepare these students for the topics that were part of the upcoming provincial test. Another student teacher told me that the students in her school were receiving specific instructions about how to write the provincial examination. I am sad to say that I was told of a school where someone looked at the actual test before it was administered and shared some information with students. This was done because of the low ranking from the previous year.

Imagine if a teacher takes the questions from the previous year's examination and displays them on the walls of the classroom at the beginning of the school year. These questions will become not only the focus of instruction but the year's curriculum. The students will score well on the tests, but will they be "Wise/Numerate and/or Test-wise and/or Otherwise" (Liedtke 2003)?

Some of these scenarios remind me of my Grade 12 experience in Alberta. The final mark was entirely based on the departmental examinations score. After a mad rush through the content, the last three months were spent writing old examinations.

Differences in settings can bring conclusions and comparisons into question. Settings that focus on test writing do not have any instructional value and do not contribute to fostering the development of number sense and numeracy.

Tests and Test Items

Tests Prepared by Teachers

No doubt many of us have been surprised, amused or perhaps a little disturbed at a student's response to an item on one of our tests. How could this happen after the great care we had taken to construct each item? No matter how much time and care are taken or how many times an item is revised, unpredictable responses will surface from time to time.

To collect assessment information about the strategies, ideas and procedures that students have learned and the conceptual understanding they have, appropriate test items need to be prepared or selected. This task requires time, care and reflection. Items should be free of mathematical terminology, and instructions should be clearly identified and succinctly stated. Meeting these conditions can be quite a challenge. Without appropriate test items, student responses—correct, partially correct and incorrect—will not yield any meaningful information about student knowledge.

Some of the student teachers I work with are surprised when I express concern about some of the assessment items they have included in their first unit plan. They had used student textbooks as references, and many of the examples were selected from chapter and unit tests in these books. The fact is, I think that many items in these references are not appropriate and do not yield valuable assessment information for appropriate conclusions about students.

The following are examples of assessment items that were part of unit tests on measurement prepared by student teachers. Key questions that were posed and some of the points that were raised during the discussions about these items are identified.

The following examples illustrate the challenges of preparing and selecting test items, and raise awareness of inappropriate test items that sometimes appear in published references.

For figures showing six segments and unions of segments, the directions were: "Measure to the nearest centimetre. Record the results." Why were there so many items? What new information could become available about a student from the repetition of these tasks? How many items are needed to find out whether students know how to measure to the nearest unit? How can three examples tell us everything we need to know about a student's ability to round up and round down? Are complex figures that show the union of segments required? Why or why not?

The instructions for several segments were: "Estimate the length of each path. Record your estimate." What marking scheme is used for the response? What, if anything, does a student's response tell us about her ability to estimate or about the estimation strategies she has at her disposal? During discussions about estimation, the word *reasonable* is almost sure to surface. How is *reasonable* defined? Different levels of number and measurement sense exist, and if students estimated, should not all of their responses be considered reasonable?

The instructions for several segments were: "First record an estimate. Measure and record the results." How many students will complete the task in this order? If students do follow the instructions, how can we know if they changed their estimates after carrying out the measurements? What, if anything, can we learn about students from items of this type?

For five irregular figures of different shapes and with straight sides, enough information was provided to determine the dimensions of each of the sides. For one of the figures, two different units are used for the dimensions. The instructions were: "Find and record the perimeter for each." What is assessed? Why are so many items used? What do we find out about a student's knowledge of perimeter if they answered incorrectly? Why are two different units used for one of the figures? Would anyone ever use different units for sketches or diagrams? If students answer incorrectly, what can we possibly say about their knowledge of perimeter? What types of questions would give us insight into students' conceptual understanding of perimeter?

The same or similar questions can be posed for assessment items for other areas of measurement and other topics as well. These examples and questions illustrate that appropriate assessment items are required to make meaningful evaluative statements about students. Without appropriate items, it may not be possible to plan for effective instruction or intervention (individualized education plans [IEPs]).

As one might expect, one of the major goals of the Diagnosis and Intervention course (available from the division of continuing studies at the University of Victoria) is teaching the creation and selection of effective assessment. Effective intervention is not possible if we do not know what students know and do not know, or how they think. A major part of the course consists of designing questions and assessing questions designed by others. In one assignment, teachers enrolled in the course take seven sample items from one of the readings for the course (Charles and Lobato 1998) and collect reactions from their students. I was amazed, to say the least, to see the results. Three are described below.

The first item:

Circle the number that is closest to ¼:

a. 0.4	b. 1.4
c. 0.14	d. 0.25
e. 0.5	

Two things became obvious. Many students did not know how to deal with 0.25. Should it be one of the choices? If so, why? Some students chose 1.4 because it was "closest" to ¼. How can one argue with that response? How many markers would think of that response?

The importance of using correct language is illustrated in the following example:

Lakiesha says that you can add zero to a decimal number (like 0.2) without it changing the value of the number. Is she correct? Explain why or why not.

Were the authors thinking about adding a zero (0.2 + 0)? I have my doubts, but that is what the item calls for. Many of the students understood the item this way.

One item was about two boys fishing. A couple of girls asked why boys were fishing and not girls.

If these sorts of things can appear in a monograph published by the National Council of Supervisors of Mathematics, then my concerns for greater care in designing test items are warranted and reinforced.

Published Tests

For the sake of discussion, let's imagine that a test to establish provincial comparisons, establish national or provincial norms, rank schools or make international comparisons is about to be administered to students who have been exposed to a sequential, research-based curriculum. If topics or items are included that are not part of this curriculum, such items should be blacked out. No matter how well-known or important a test, its items should not be a basis for curriculum revision. It may seem a little far-fetched for this to happen, but I have heard the suggestion many times for introducing topics simply because they are part of a well-known or important test. Tests should not determine curriculum; it should be the other way around.

The many tests I have examined over the years have led me to conclude that people in measurement and testing who design and construct these instruments need to take much greater care. Not only are editors required, referees are needed as well. Without appropriate test items, interpretations of the results come into question.

The mathematical language used in tests has to be correct. *Number* is too often confused with *numeral*, and *more* is used when *greater than* is correct. The language is often unclear, uncommon or even incorrect.

The following examples illustrate some of the reasons for my concerns. (More examples could have been included. At least half of the test items could be edited, especially for inappropriate spelling or representation.)

> Trends in International Mathematics and Science Study (TIMSS) sample elementary school mathematics test (Grades 3 and 4):
>
> Question 6) 25 × 18 is more than 24 × 18. How much more?
>
> A) 1   B) 18   C) 24   D) 25

Not only is the language incorrect, but no one actually talks like that, especially not Grade 3 or 4 students. Several items on this test have sentences that are too long and complex. However, this item requires a few more appropriate words. As the item stands, there are at least two possible interpretations. The answers could be 1 (1 more group) or 18 (18 more items). The instructions need to tell students to consider the answer or product, and the comparison should be discussed in terms of *greater*.

> Question 7) The given numerals in column A are 10, 15, 25 and 50. Corresponding numerals in column B are 2, 3, 5 and 10, respectively. The question posed is: "What do you have to do to each number in column A to get the number next to it in Column B?"
>
> A) Add 8 to the number in column A.
> B) Add 8 to the number in Column A.
> C) Subtract 8 from the number in Column A.
> D) Multiply the number in Column A by 5.
> E) Divide the number in Column A by 5.

The first two choices are the same—well, almost. In the first choice, the *c* in *column* is lower case. Also, each choice uses the singular form of *number*. This could make choice C correct because 10 – 8 = 2. Even if this problem can easily be solved, the test in the present form should never be displayed for anyone or, even worse, used by teachers.

> Question 15) Four children measured the width of a room by counting how many paces it took them to cross it. The chart shows their measurements.

Name	Number of paces
Stephen	10
Erlane	8
Ana	9
Carlos	7

> Who has the longest pace?
>
> A) Stephen
> B) Erlane
> C) Ana
> D) Carlos

Important information is missing, and the language could be much simpler. Should students be told that an attempt was made to take steps of the same size? What exactly is meant by "their measurements?" Is that a common expression?

> Programme for International Student Assessment (PISA)—National Centre for Education Statistics (NCES) (Grade 4):
>
> Question 3) Which of these is largest?
>
> 1 kilogram 1 centigram 1 milligram 1 gram

My curiosity is killing me. What is the person who designed this item trying to assess? What would anyone possibly learn about a student's knowledge if they made a correct or incorrect choice? What about the choice of centigram? Who uses this unit?

One major reason we teach about measurement is that our eyes may deceive us. For example, is it possible for one gram of cotton candy to be larger than one kilogram of lead? Items like these are of as little value as examples that ask students to record an estimate, or even worse, to first record an estimate and then to perform a calculation or measurement.

Question 6) Here is a number sentence 4 × _ < 17
Which number could go in the _ to make the sentence true?
• 4 • 5 • 12 • 13

This is the exact language and punctuation used on the test. Not only is the presentation awkward and somewhat unusual, but the instructions are difficult to follow as well.

Question 14) What is 3 times 23?
• 323 • 233 • 69 26

The last choice lacks an indicator and the question is curt and unusual. Students who have a conceptual understanding of multiplication could supply all kinds of responses to the request. The question needs to be clearer by asking students to think about the answer.

Question 19) In which pair of numbers is the second number 100 more than the first number?
• 199 and 209 • 4236 and 4246
• 9635 and 9735 • 51,863 and 52, 863

Some people will object to the use of numerals or number names in items like these. However, when numerals are compared, the terms *greater* and *less* should be used. The use of commas in the last item looks odd and could confuse. For some students in Canada and in other countries, commas are decimal points. Spaces and half-spaces should be used.

Question 20) Figure 1 - A 1by3 rectangle showing 3 squares
Figure 2 - A 2by3 rectangle showing 6 squares
Figure 3 - A 3by3 rectangle showing 9 squares
Here is the beginning of a pattern of tiles. If the pattern continues, how many tiles will be in Figure 6?
• 12 • 15 • 18 • 21

Test items about patterns require a lot of essential information. Any repeating pattern can be changed into a growing pattern and vice versa. Because growing patterns can be extended in many different ways, students need to be told if the pattern is repeating or growing, and if it is the latter, if it is growing in the same way as indicated by the examples. It is possible, for example, that the given pattern could continue and that 21 could be a logical choice for Figure 6. Would that be marked as incorrect? It likely would, even though the answer can be justified.

Mathematics Assessment (Grade 6)

While my daughter was teaching in northern Manitoba, she shared with me an examination that the Grade 6 students in the district had to write. Quite a few items caught my eye, and I will comment on a couple of examples.

Two items dealt with rolling a die. In both instances, the rather elaborate illustrations show two dice. In one instance, the action diagram indicates that they are being rolled. This can create some interesting scenarios, especially because many people have difficulty distinguishing between *die* and *dice*.

A question labelled "Number Sense" asks students to "Write equations that equal 180." After students are asked to use each of the four operations, they are requested to "use 3 or more numbers in each equation," "use 2 or more operations in each equation" and "use decimals." However, what if students used only zeroes and/or ones for the equations? How might their answers be assessed? How might students define *number sense* and why might they think it was used as the title to this question?

One of the items dealt with patterns. Squares with sides 1, 2 and 3 are shown, and the respective perimeters 4, 8 and 12 are provided. The first request is: "Draw the next 2 models in this pattern." The word *draw* could result in some students attempting to produce some time-consuming constructions. How might responses of drawing the squares with sides 1, 2 and 3, or with sides 3, 2 and 1 be assessed? My concerns are the same as for Question 20 on the PISA, which dealt with pattern. The term *model* seems somewhat unusual.

The item that brought a smile to my face listed four fractions—2/10, 2/3, 2/4 and 2/6. A number line that shows zero, one-half and one-whole is given. The students are requested to "Place the following fractions in order on the number line." How might students interpret "in order," and how will the markers tell if the directions were followed?

I found it somewhat satisfying to see items labelled "Number Sense" and "Mental Computations," even if I was curious about the assessment criteria used for the latter. One item about a menu for lunch puzzled me by being labelled "Nonroutine Problem." Could this identification be considered a contradiction?

Conclusion

I could go on discussing specific items to illustrate my major points, but there is no need. It is sad and sobering that these types of items were or are used to assess students' mathematical ability. Items that lack clarity and are inappropriate are not fair because they do not allow students to show what they know and understand. Much greater care needs to be taken in designing test items. Without that care, the conclusions reached about students and shared with parents lack validity and general statements about students' performance may not be as meaningful as we would like.

After many years of devising test questions, we tend to feel as if we have perfected the craft. But inevitably a student will interpret a question in a way we never anticipated. I am reminded of the sobering comments in a report by Peck, Jencks and Connell (1991). Incorrect answers on tests were followed up with brief interviews, and it was concluded that "52 per cent of a group of students would have been misjudged had test results not been supplemented with these brief interviews." It can be concluded that with poor questions, that percentage will be even higher.

On a lighter note, there are students who do not know the answer to a question but will not hesitate to provide one anyway. I read a newspaper report that included a Grade 4 social studies exam question asking students to state the population of a province. One student wrote, "As for the completely total population of that province I would estimate that I distinctly do not know." The teacher's challenge is to award a mark to this possible future politician.

One of my examination questions asked teachers-to-be to use their own words to define Cartesian product. One lady who had obviously been absent during the discussion of the topic had no idea, but she drew a three-dimensional box with a wide ribbon around it and a big bow. Printed in the corner of the box in bold letters was "Product of Cartesia." I still recall the surprise! It still makes me smile!

Most of the items I designed for one examination must have been appropriate because one creative teacher-to-be wrote the following at the end:

Thank God this test is over
I think I am going to die,
I didn't know the answers
And I couldn't even lie!

Perhaps the creator of the next rhyme could have partially blamed her difficulties on the inappropriateness of some of the examination items. However, with an option like hers, who would want to examine the appropriateness of items?

There was a girl who saw teaching
As a goal for which she was reaching.
After writing a test
Which wasn't her best
She decided, "It's Hawaii for beaching!"

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Mania for Measuring: Testing Is Not Teaching

Eden Haythornthwaite

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Why do provinces make students write standardized progress tests?

At the end of January, Grade 10 students all over this province will be writing newly minted compulsory exams in order to meet graduation requirements. The pressure is on and the stresses are deeply felt. Like so many boxes on an assembly line, the students will be subjected to a quality-control exercise. It's part of the process of manufacturing recruits for the workplace at minimum cost to the public treasury. With the rise of standardized testing across Canada, we are welding our public education system to the business world.

Even though the British Columbia government has ripped hundreds of millions of dollars out of public education, the education ministry is anxious to provide evidence that all is well. Why worry about closing schools and cutting teaching positions and learning supports? Apparently, the students are doing just fine—the tests say so.

Should we now accept that education is about assessing superficial standards and supplying marketable commodities? Does meeting minimum standards for literacy and numeracy replace the joy, community spirit and curiosity of learning to search for answers?

Today's education lingo is permeated by business speak; we must "invest" in our children so they can secure "saleable skills." Our teachers are now frequently referred to as "instructors." Our kids must meet "assessment" regimes that determine "success" and "failure." Teachers are pressured to stick to the curriculum so they can "teach to the test" and get results.

But our kids are not investments like stocks, bonds and real estate.

We want them to be well-rounded, thoughtful human beings. This emphasis on training for the marketplace will only produce efficient little test-takers who compete furiously to meet targets of uniformity—now defined as "excellence."

If testing for success takes over the classroom, it will create a lasting rut in the road for a bland and soulless school experience. Although standardized testing represents a hurdle for the kids, it also stands as a heavy-handed method of surveillance for their teachers.

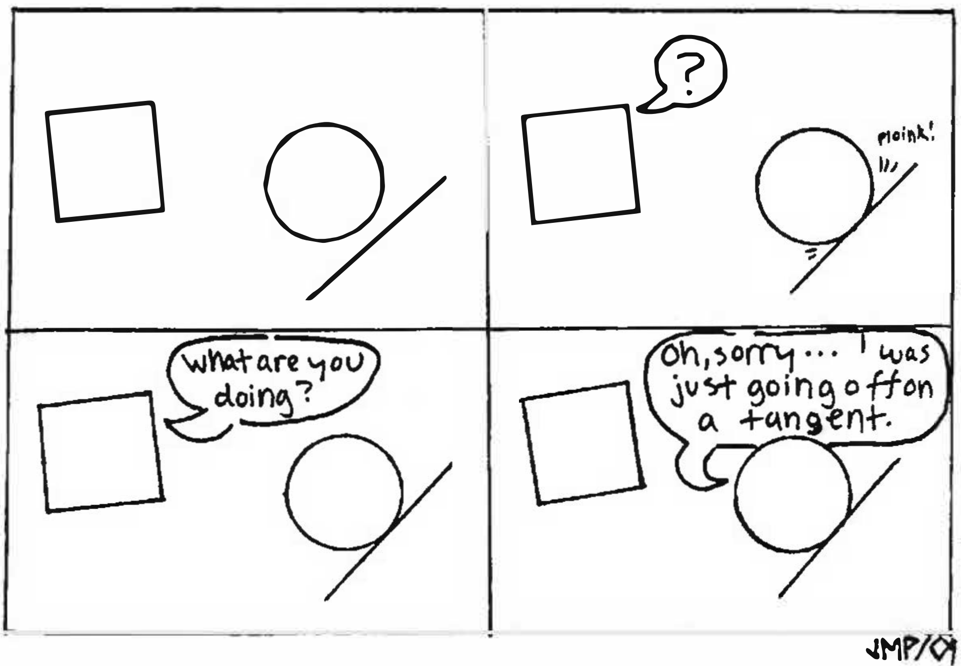
It must disappoint many teachers to see their enthusiasm for entering their profession undermined by an obligatory routine of cold-blooded product delivery. No one is more capable of truly evaluating our kids and their advancement than the teachers they spend their days with. Inspiring kids to love learning, appreciate the endeavour of acquiring knowledge and practice critical thinking is a truly admirable undertaking.

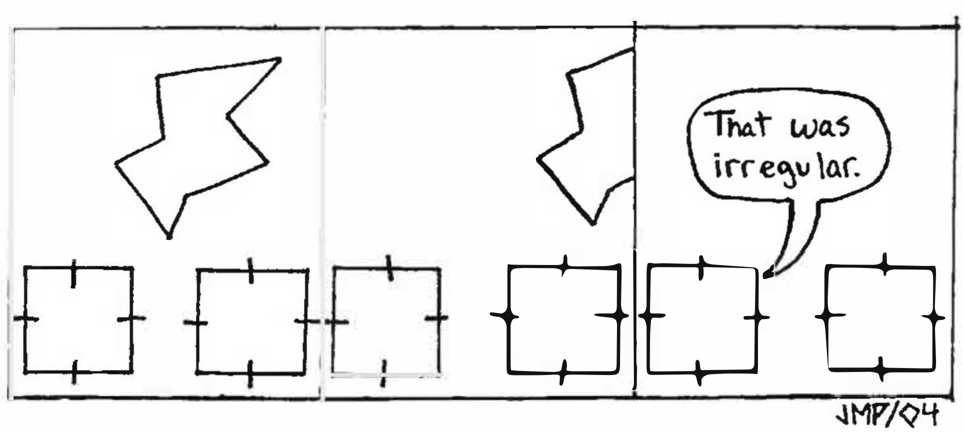
A handful of business loyalists who want to remake our schools in the image of commerce should not dictate our thinking on public education. Why must everything have an economic aim? Testing is not teaching and never will be. Standardized testing must stop. Let our kids learn from their teachers.

Eden Haythornthwaite works for the Community Alliance for Public Education (CAPE) in Cowichan Valley, British Columbia.

360° of Humour

Julie Mallet-Paret





Julie Mallet-Paret is a Grade 12 student at Archbishop Jordan Catholic High School in Sherwood Park, Alberta.

Different Ways to Assess Mathematical Thinking

JoAnn Grand Pooley

I have been a teacher for many years and am constantly trying to improve my mathematics instruction. According to Alberta Education's *Program of Studies*, students are expected to gain facility in comparing fractions through the study of various representations and models of fractions. They should be able to solve problems involving equivalent fractions. New visions of mathematics require new assessment tools. Recently, there has been a big emphasis on assessment in our school division.

While I was teaching a unit on fractions to my Grade 5 class, I began to think about different ways to assess their knowledge. After teaching what a fraction represents, equivalent fractions, mixed numbers and the names of the numerals in a fraction, I asked the students to create their own fraction booklet. This replaced my standard unit exam.

I was teaching 10- and 11-year-olds, so I gave them a lot of guidance on what I expected in the booklet. I distributed the following assignment:

Tell Me What You Know About Fractions

Your booklet should include the following the following:

- Vocabulary words, such as numerator, denominator, mixed number and equivalent fraction
- Comparing fractions
- Fraction problems
- Illustrations that show fractions as part of a shape and as part of a set

Be sure to include how whole numbers can be divided into fractions.

Include anything else you have learned or know about fractions. Cut out images from magazines, add humour, use stickers and have fun.

The booklet is due on ________. (And yes, all of the pages in the booklet should be used.)

I made a booklet myself so that the students would have clear expectations about the product. They were given two class blocks of 45 minutes to create their booklets, and then they were assigned as homework.

I set up a four-point rubric for assessment:

My Fraction Book

4: Meets the standard of excellence

- The fractions and illustrations match and are accurate.
- The illustrations enhance understanding of the ideas.
- The booklet includes all of the suggested elements, as well as additional information.
- The explanations and illustrations use appropriate math terminology accurately.
- All problems are stated clearly and solutions are shown. The problems show a high degree of understanding of the concepts taught.
- The booklet is neat and well organized.

3: Exceeds acceptable standard

- The fractions and illustrations match and are accurate.
- A number of fractions are used.
- The booklet includes most of the suggested elements.
- The explanations and illustrations use math terminology accurately.
- All problems are stated clearly and solutions are given. The problems show a good degree of understanding of the concepts taught.
- The booklet is neat and organized.

2: Meets acceptable standard

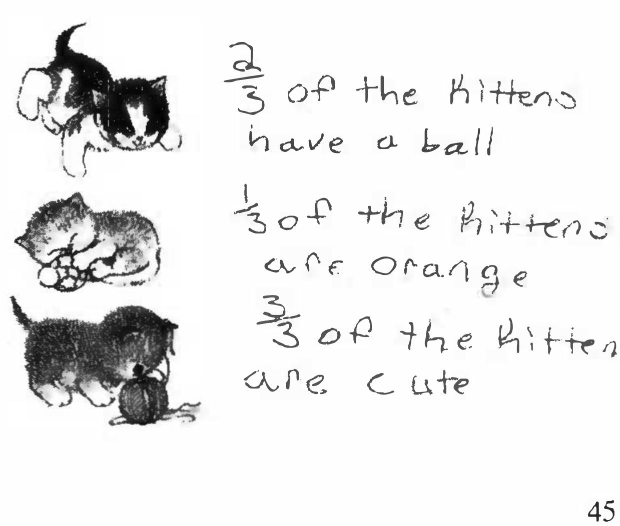
- Fractions and illustrations are generally accurate, although few fractions are used.
- Some fractions are repeated.
- Some of the elements of the assignment are missing.
- The explanations and illustrations use math terminology in a general way.
- Not all problems are clearly stated and no solutions are given.
- The booklet is fairly neat but is poorly organized.

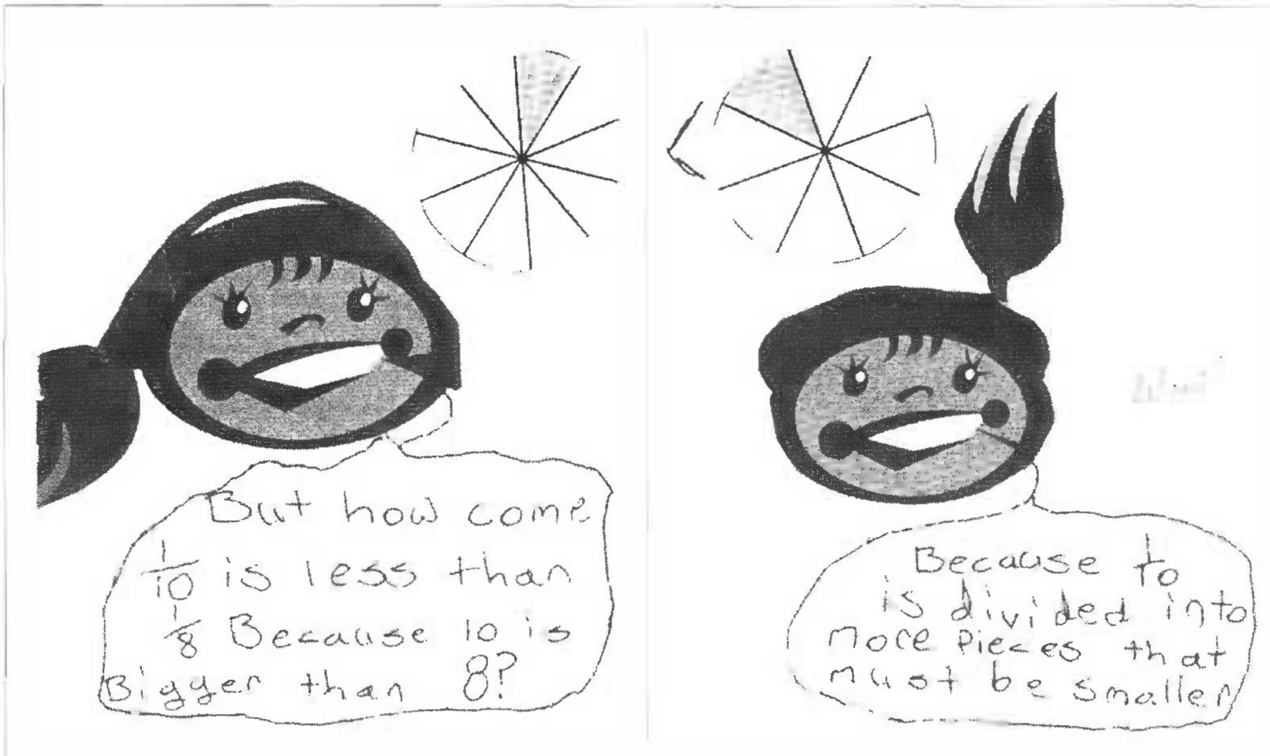
1: Has not yet met the acceptable standard

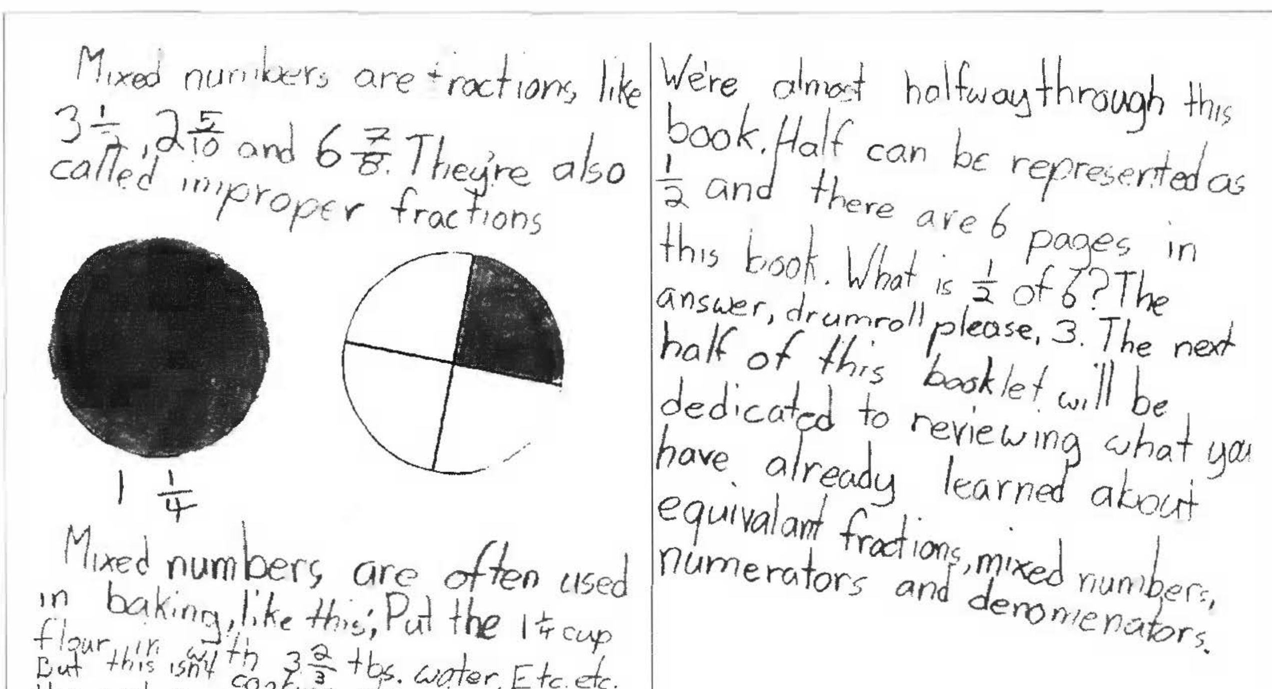
- The fractions and illustrations may not match.
- No examples of equivalent fractions are given.
- Many of the elements of the assignment are missing.
- The explanations and illustrations do not use math terminology, or the math terminology is used incorrectly.
- There is little or no attempt to include a problem.
- The booklet is messy and no care has been taken to complete the task.

As you can see from the following samples, the students demonstrated their understanding of fractions in creative ways.

I was pleased and amazed with the results that I received, and this project gave me a great deal of insight into the students' thinking. Students were able to represent and describe proper fractions and equivalent fractions, both pictorially and symbolically, and they demonstrated their ability to compare and order fractions. Many of the specific outcomes prescribed in the *Program of Studies* were met. Not only were the fraction books a delight to read but they were a great addition to our portfolios as well.







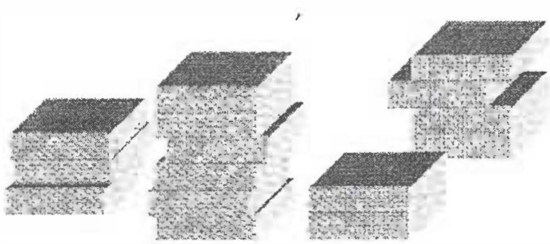
JoAnn Grand Pooley teaches Grade 5 at Graminia Community School in the Parkland School Division. Her interests include science and mathematics curriculum, professional learning communities and authentic assessment.

A Page of Problems

A Craig Loewen

Middle School

Show all the ways that 15 identical objects can be placed into 4 piles so that each pile has a different number of objects in it.



High School

2,985 digits were used to print the page numbers of a book. How many numbered pages are in the book?

Kantecki, C and L E Yunker. 1982. "Problem Solving for the High School Mathematics Student." *Math Monograph* 7: 49–60.

Junior High

You need to roll an even number (or even sum) to win a game. You may choose to roll either one 6-sided die or two 6-sided dice (and add the values rolled).

Which should you choose?

Elementary

A five-volume set of books is lined up on a shelf in order from left to right. Each book has exactly 100 pages. A hungry worm starts eating through the first page of the first volume, and eats her way through to the last page of the last volume. How many pages did she eat through?

Hint: Draw a picture or act it out. The correct answer is neither 250 or 500.

Adapted from www.bettybookmark.com/p/puzzles/puzzle2.htm

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