Context

Indy Lagu

The definition of a good mathematical problem is the mathematics it generates rather than the problem itself.

—Andrew Wiles

Growing up in the Calgary school system, I had a pretty dim view of mathematics. I never had any desire to become a mathematician, and I would not have believed you if you had told me that I someday would. It is now clear to me that I had no idea what a mathematician did. All right, I *thought* I knew what a mathematician did, but my speculations could not have been further from the truth. Recently, I began to wonder why that was, and I have come to some partial conclusions. I think that, given the rotten attitudes toward mathematics of the high school graduates I encounter, the reasons have not changed at all; the reasons the students in my class hate mathematics are the same reasons I hated it. One reason is that we have taken an essential element out of mathematics classes: context.

Context, in this context, does not mean merely putting a student's name into a question, *à la*

Johnny has three quarters, two dimes and four pennies. How much money does he have?

Nor does it mean succumbing to an idiotic political correctness and changing the name Johnny to Jane, Isaac, Sunil, Ahmed or any of a host of wonderful names. What I mean by *context* is a situation that allows for exploration. Now, such situations do not have to be real-life situations, whatever that means. Johnny's financial woes may be a real-life situation, but they do not lead to a question that is interesting (mathematically or otherwise).

I do not understand how or why we have lowered mathematics to this level, but I do know that questions of this type are what most of my students believe mathematics is all about. They also have a misguided belief that the answer is the most important thing in mathematics and that communicating ideas and clear exposition are a nuisance.

We often encourage these destructive and insulting beliefs. Why destructive? Because they relegate mathematics from a useful way of thinking to a form of *Jeopardy*. Why insulting? What other word is there to describe such a denigration of the enormous accomplishments of Gauss, Euler, Newton and Hopper, to name a few?

One way we encourage these beliefs is by making it appear to our students that only the answers are important. This is most easily accomplished through giving our students multiple-choice tests.

Another way we encourage these beliefs is by presenting mathematics as an unrelated string of facts. I call this unit mentality—the belief that factoring, graphing and finding the roots of a polynomial are three distinct activities (units) that come with three different exams (unit tests). We must divorce ourselves from the notion that mathematics comes in discrete and disparate pieces. To that end, I offer the following problem.

In the old days of college basketball, before the shot clock and the three-point line, a team could score only one or two points at a time. For example, if a team had five points, those points must have been obtained in one of the following eight ways:

$$1 + 1 + 1 + 1 + 1,$$
$$2 + 2 + 1,$$
$$2 + 1 + 1 + 1,$$
$$1 + 2 + 1 + 1,$$
$$1 + 1 + 2 + 1,$$
$$1 + 1 + 1 + 2,$$
$$1 + 2 + 2 \text{ or}$$
$$2 + 1 + 2.$$

How many ways could the team score n points?

If your first instinct is to see the words *how many ways* and think, *This is a question about perms and combs,* forget that instinct. The expression *perms and combs* belongs in a hair salon, not in mathematics. Mathematicians do not use those terms, so why do we? And why do we expose our students to that false terminology?

Now, why is the basketball problem a good one? Well, basketball teams likely do not care about the answer, and therefore it is certainly not a real-life problem. However, it is a good problem because of where it leads—to exploration. We can play with the problem.

If we let w_n be the number of ways to score n points, it is not too difficult to see that $w_0 = 1$ and $w_1 = 1$, because there is only one way to have zero points (score no points) and only one way to have one point (score only one point). Since there are two ways to get two points (1 + 1 and 2), $w_2 = 2$. And so on. Play with this problem some more before reading on, trying to generate the sequence.

The sequence generated is as follows:

$$w_0 = 1, w_1 = 1, w_2 = 2, w_3 = 3, w_4 = 5, w_5 = 8, \ldots.$$

Do you see a pattern? After w_1, each number is the sum of the two that came before. Is that really the pattern? Why? (Astute readers will recognize this as the Fibonacci sequence.) Notice that the basketball problem has led us to sequences. In particular, it has led us to the important fact that a sequence is defined not by a few terms but, rather, by its context. If you are not convinced, try the following sequence[1]:

$$x_n = \left\lceil \sqrt{e^{n-1}} \right\rceil.$$

You will get $x_0 = 1$, $x_1 = 1$, $x_2 = 2$. As it turns out, x_n and w_n agree for $n = 0, 1, 2, \ldots, 9$. However, for $n = 10$, we get $w_{10} = 89$, whereas $x_{10} = 91$. Thus, it is not sufficient to look at a list of numbers.

For the basketball problem, the pattern wherein each term is the sum of the two previous terms does hold, and I will leave it to you to discover why. Algebraically, $w_0 = w_1 = 1$, and

$$w_{n+2} = w_{n+1} + w_n \qquad \text{for } n \geq 0.$$

Typically, people will ask for an explicit formula for w_n. Try this: assume that $w_n = x^n$ for some (non-zero) value of x. Then,

$$x^{n+2} = x^{n+1} + x^n.$$

Hence, we must have

$$x^2 - x - 1 = 0,$$

and we have reduced the original question to finding the roots of a quadratic. This leads to factoring, working with the quadratic formula and graphing.

So we see that a good problem—that is, a problem with a context—can lead us to many of the mathematical places our curriculum mandates: sequences, polynomials, factoring, graphing and combinatorics, to name a few. The basketball problem also leads to the theory of partitions, difference equations, the golden rectangle, phyllotaxis (the pattern of bumps on a pineapple) and the solution of certain differential equations.

What we need is not only a bank of such problems but also teachers who have the ability and desire to investigate good problems, and the belief that finding the answer is not the reason for trying to solve a problem.

Note

1. The symbol $\lceil x \rceil$ means "round up," so that $\lceil 1.1 \rceil = 2$, $\lceil \pi \rceil = 4$ and so on. It is called the ceiling function.

Indy Lagu completed his PhD in numerical analysis at the University of Calgary in 1996 and is currently the chair of the Department of Mathematics, Physics and Engineering at Mount Royal College in Calgary. He has served on the MCATA executive for five years and has been working with K–12 teachers and students for nearly 10 years. Outside of mathematics, his interests include playing guitar, cooking, golfing and wondering when the Montreal Canadiens will win another Stanley Cup.