Two Great Escapes

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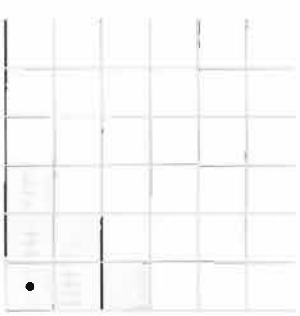
The Great Amoeba Escape

The world of the amoeba consists of the first quadrant of the plane divided into unit squares. Initially, a solitary amoeba is imprisoned in the square in the bottom left corner. The prison consists of six shaded squares, as shown in Figure 1. It is unguarded, and the Great Escape will have succeeded when the entire prison is unoccupied.

Figure 1



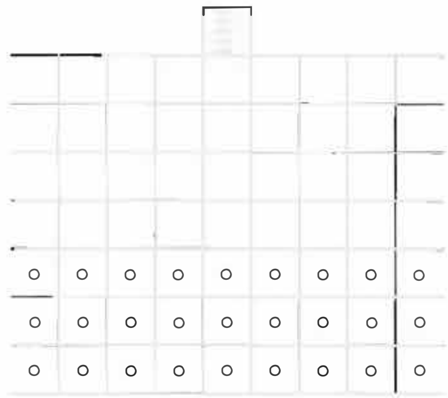
In each move, an amoeba splits into two, with one going to the square directly north and one going to the square directly east. However, the move is not permitted if either of those two squares is already occupied.

Can the Great Escape be achieved?

The Great Beetle Escape

The world of the beetle consists of the entire plane divided into unit squares. Initially, all the squares south of an inner wall constitute the prison, and every square is occupied by a beetle. Freedom lies beyond an outer wall four rows north of the inner wall. If any beetle reaches any square outside the unguarded prison, such as the shaded one in Figure 2, it will trigger the release of all the surviving beetles. Then the Great Escape will have succeeded.

Figure 2



In each move, a beetle can jump over another beetle in an adjacent square and land on the square immediately beyond. However, the move is not permitted if that square is already occupied. The beetle that is jumped over is removed, making a sacrifice for the common good. The jump may be northward, eastward or westward.

Can the Great Escape be achieved?

REMARK 1. *The reader may wish to attempt to solve these two problems before reading on. At the least, the reader should delay reading beyond Strategies.*

Strategies

In both problems, the configuration keeps changing, with more and more amoebas in one case and fewer and fewer beetles in the other. The changes must be carefully monitored before things get out of hand. What we seek is a quantity that remains unchanged throughout. Such a quantity is called an invariant.

In the amoeba problem, the situation is simpler at the start, with only one amoeba. After one move, we have two amoebas. However, each is less than one full amoeba. Suppose we assign the value 1 to the initial amoeba, x to the one going north and y to the one going east. After the move, the initial amoeba is replaced by the other two. If we want the total value of amoebas to remain 1, we must have $x + y = 1$. By symmetry, then, $y = x$.

In the beetle problem, the situation is simpler at the end, with one beetle beyond the outer wall. Let's assign the value 1 to that beetle. It has reached its current position by jumping over another beetle. Let's assign x to the jumped-over beetle and y to the beetle before making the jump. After the move, the final beetle replaces the other two. For the total value of the beetles to be invariant, we must have $x + y = 1$, as in the amoeba problem.

A beetle with value z could jump over the beetle with value y to become the beetle with value x. If we choose $y = x$, as in the amoeba problem, then we must make $z = 0$ to maintain $z + y = x$. This is undesirable. A better choice is $y = x^2$. Then we can make $z = x^3$. Since $x^2 + x = 1$, we indeed have $z + y = x^3 + x^2 = x(x^2 + x) = x$.

The idea of an invariant is an important problem-solving technique. For further discussion and practice, see Fomin, Genkin and Itenberg (1996, 123–33, 254–57) and Tabov and Taylor (1996, 93–109).

Solution to the Amoeba Problem

We now put into practice the strategy discussed earlier. Clearly, the value of an amoeba is determined by its location. So we may assign values to the squares themselves, as shown in Figure 3.

Figure 3

$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$
$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$
$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$
1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$

The total value of the squares in the first row is

$$S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots.$$

Then,

$$2S = 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots.$$

Subtracting the first equation from the second, we have $S = 2$. Since each square in the second row is half the value of the corresponding square in the first row, the total value of the squares in the second row is 1. Similarly, the total values of the squares in the remaining rows are $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, ... Hence, the total value of the squares in the entire quadrant is 4.

Note that the total value of the six prison squares is $2\frac{3}{4}$. Remember that the total value of the amoebas is the invariant 1. If the Great Escape is to be successful, the amoebas must fit into the non-prison squares with total value $1\frac{1}{4}$. Though there is no immediate contradiction, we do not have much room to play about.

Each of the first row and the first column holds exactly one amoeba at any time. If the amoeba on the first row is outside the prison, its value is at most $\frac{1}{8}$. The remaining space with total value

$$\frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots = \frac{1}{8}$$

must be wasted. Similarly, we must leave vacant squares in the first column with a total value of at least $\frac{1}{8}$. Since

$$1\frac{1}{4} - 2 \times \frac{1}{8} = 1,$$

we have no room to play at all.

For the Great Escape to be successful, all squares outside the prison and not in the first row or first column must be occupied. However, this requires that the number of moves be infinite. Hence, the Great Escape cannot be achieved in a finite number of moves.

Solution to the Beetle Problem

As in the amoeba problem, the value of a beetle is determined by its location. So we may assign values to the squares themselves, as shown in Figure 4.

Figure 4

			1			
x^4	x^3	x^2	x	x^2	x^3	x^4
x^5	x^4	x^3	x^2	x^3	x^4	x^5
x^6	x^5	x^4	x^3	x^4	x^5	x^6
x^7	x^6	x^5	x^4	x^5	x^6	x^7
x^8	x^7	x^6	x^5	x^6	x^7	x^8
x^9	x^8	x^7	x^6	x^7	x^8	x^9

The total value of the squares in the central column in the prison is

$$S = x^5 + x^6 + x^7 + x^8 + \cdots.$$

Then,

$$xS = x^6 + x^7 + x^8 + x^9 + \cdots.$$

Subtracting the second equation from the first, we have

$$S = \frac{x^5}{1-x}.$$

Since each square in the adjacent column on either side is x times the value of the corresponding square in the central column, the total value of the squares in either column is

$$\frac{x^6}{1-x}.$$

Similarly, the total values of the squares in the remaining columns on either side are

$$\frac{x^7}{1-x}, \frac{x^8}{1-x}, \frac{x^9}{1-x}, \ldots.$$

The total value of the squares in the prison east of the central column and including this column is

$$\frac{1}{1-x}(x^5 + x^6 + x^7 + x^8 + x^9 + \cdots) = \frac{x^5}{(1-x)^2}.$$

Similarly, the total value of the squares in the prison west of the central column but excluding this column is

$$\frac{x^6}{(1-x)^2}.$$

Hence, the total value of the squares in the entire prison is

$$\frac{x^5 + x^6}{(1-x)^2}.$$

Recall that $x^2 + x = 1$, so $1 - x = x^2$. Hence, the denominator of the total value is $(1 - x)^2 = (x^2)^2 = x^4$. The numerator of the total value is $x^6 + x^5 = x^4(x^2 + x) = x^4$ also, so the total value is exactly 1. Thus the Great Escape can succeed only by sacrificing all but one beetle, and it cannot be achieved in a finite number of moves.

Remark 2. *Everything up to this point has been adapted from material in existing literature. The Great Amoeba Escape is from Kontsevich (see Taylor 1993, 31, 37–39), and the Great Beetle Escape is from Conway (see Honsberger 1976, 23–28). What follows is largely my own contributions.*

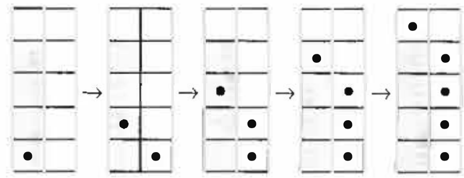
Further Amoeba Problems

We define a prison in the amoeba world as a set of squares consisting of the southernmost a_i squares in the i-th column for $1 \leq i \leq n$ such that $a_1 \geq a_2 \geq \cdots \geq a_n$. Such a prison is denoted by $(a_1, a_2, \ldots, a_n)$. We wish to determine all prisons from which the Great Escape is achievable. We consider the following cases.

Case 0. $a_2 = 0$.

The Great Escape from all such I-shaped prisons is easily achieved. Figure 5 illustrates the Great Escape from the prison (4) in $a_1 = 4$ moves.

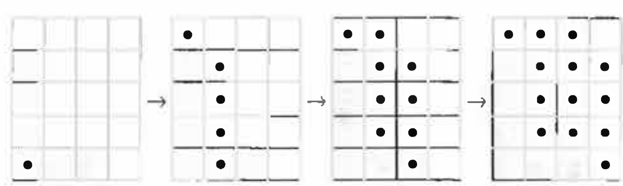
Figure 5



Case 1. $a_2 = 1$.

The Great Escape from all such L-shaped prisons is achievable in two stages. Figure 6 illustrates the Great Escape from the prison (4,1,1) in 12 moves. The first stage is the northward breakout in $a_1 = 4$ moves, exactly as in Case 0. The second stage is the eastward breakout in $n - 1 = 2$ phases, each involving $a_1 = 4$ moves.

Figure 6

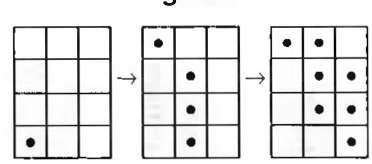


Case 2. $a_2 = 2$.

By symmetry, we may assume that $a_1 \geq n$. Since the Great Escape from the original (3,2,1) prison is not achievable, we may assume that $a_3 = 0$. The principal result is that the Great Escape from the prison (3,2) is not achievable. It then follows that it is not achievable from any P-shaped prisons $(a_1,2)$ where $a_1 \geq 3$.

Suppose the Great Escape from (3,2) is achievable. The order of the moves is irrelevant, as long as we allow temporary multiple occupancy of squares. Thus, there is essentially one escape plan, if any exists. So we may begin an attempt by making a three-move northward breakout followed by a three-move eastward breakout, as shown in Figure 7.

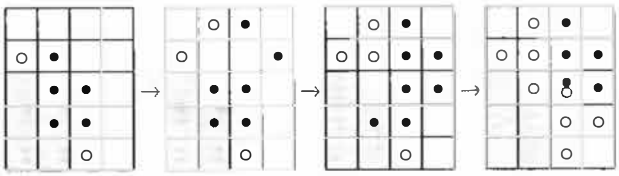
Figure 7



At this point, note that the amoeba on the first column and the one on the first row should not be moved any further, since they are outside the prison and not blocking the escape paths of any other amoebas.

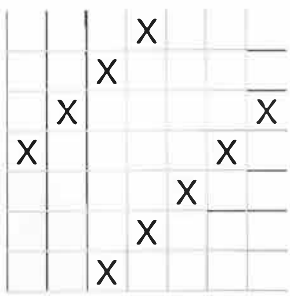
We mark them with white circles. We now move the other five amoebas one row at a time, as shown in Figure 8.

Figure 8



We have five more amoebas to move, and they form the same configuration as before except shifted one square diagonally in the northeast direction. It follows that in the Great Escape from (3,2) the amoebas do not venture outside the two diagonals of squares, as indicated in Figure 9.

Figure 9



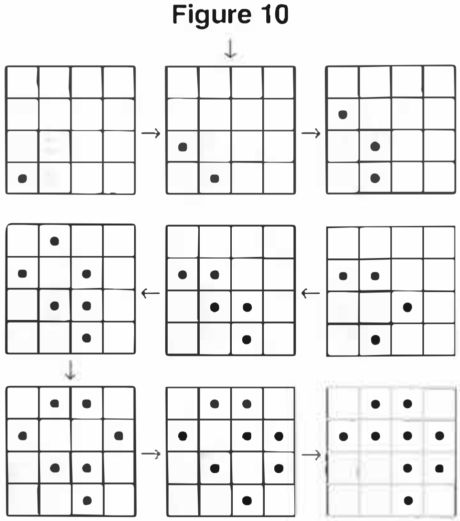
The total value of the squares between and including these two diagonals but outside the prison is

$$\frac{1}{4}+3\left(\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots\right)=\frac{1}{4}+\frac{3}{4}=1.$$

Hence, the Great Escape cannot be achieved in a finite number of moves.

Finally, the only prison for which $a_2 = 2$ and from which the Great Escape is achievable is (2,2), in eight moves, as shown in Figure 10.

Figure 10



Case 3. $a_2 \geq 3$.

Such a prison contains the prison (3,2) as a subset. By Case 2, the Great Escape from (3,2) is not achievable. Hence, it is also not achievable for any prison with $a_2 \geq 3$.

Further Beetle Problems

We have already shown that the Great Escape from the original prison in the beetle world is not achievable. We modify the prison by reducing the distance d between the outer wall and the inner wall. It turns out that for $t \leq 3$ the Great Escape can be achieved in a finite number of moves. Thus, it involves a team of beetles, all but one of which will be sacrificed. What we want is to minimize the size of the team. We consider the following scenarios.

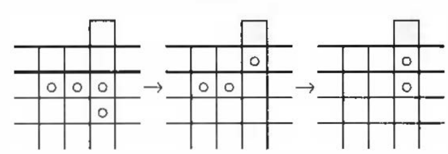
Scenario 0. $d = 0$.

Clearly, two beetles lined up directly in front of the target square can serve as the escape team. A team of size one is insufficient, because the maximum value of the lone beetle is x, and $x < x + x^2 = 1$.

Scenario 1. $d = 1$.

Four beetles positioned as shown in Figure 11 can serve as the escape team. After the first two moves, we can continue as in Scenario 0. A team of size three is insufficient, because the maximum total value of the beetles is $x^2 + 2x^3 < 2x^2 + x^3 = x + x^2 = 1$.

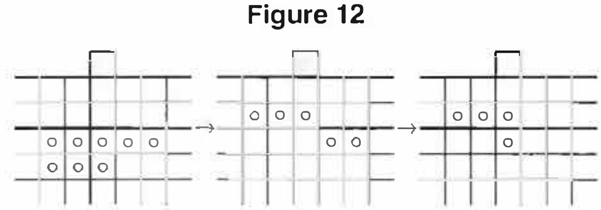
Figure 11



Scenario 2. $d = 2$.

Eight beetles positioned as shown in Figure 12 can serve as the escape team. After the first four moves, we can continue as in Scenario 1. A team of size seven is insufficient, because the maximum total value of the beetles is $x^3 + 3x^4 + 3x^5 < x^3 + 4x^4 + 2x^5 = 3x^3 + 2x^4 = 2x^2 + x^3 = 1$.

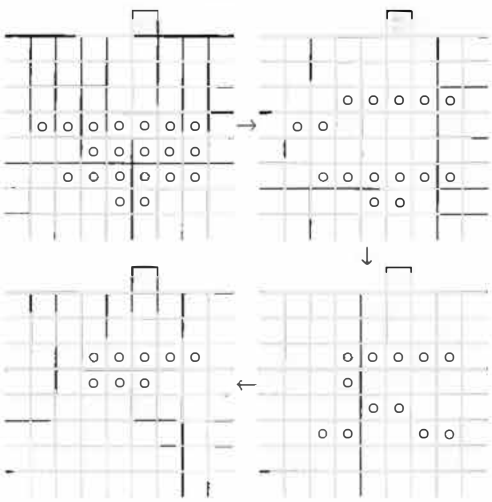
Figure 12



Scenario 3. $d = 3$.

Twenty beetles positioned as shown in Figure 13 can serve as the escape team. After the first 12 moves, we can continue as in Scenario 2.

Figure 13

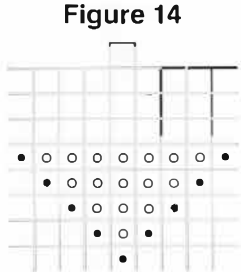


An escape team of size 19 may just be sufficient, because the maximum total value of the beetles is

$$\begin{aligned} & x^4 + 3x^5 + 5x^6 + 7x^7 + 3x^8 \\ &= x^4 + 3x^5 + 8x^6 + 4x^7 \\ &= x^4 + 7x^5 + 4x^6 \\ &= 5x^4 + 3x^5 \\ &= 3x^3 + 2x^4 \\ &= 1. \end{aligned}$$

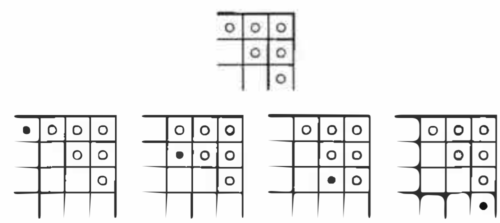
If this is the case, the escape team must consist of the 16 beetles in Figure 14, plus three more on the squares marked with black circles.

Figure 14



By symmetry, we may assume that at most one of the three additional beetles appears to the left of the central column. In each of the five cases shown in Figure 15, it is easy to verify that at least one beetle will remain to the left of the central column. This means that an escape team of size 19 is insufficient.

Figure 15



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