Root Multiples and Polynomial Coefficients

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Teachers are always looking for situations in which their algebra students can investigate numerical/symbolic patterns. Polynomial equations provide a setting for such processes.

Consider the following quadratic equation: $x^2 + 3x - 4 = 0$. Either by factoring or by using the quadratic formula, most algebra students would readily conclude that the roots are $x = 1$ and $x = -4$.

Could we write a quadratic equation whose roots are twice those of the original equation? Since this new equation would have roots of $x = 2$ and $x = -8$, the new equation would be

$$(x - 2)(x + 8) = 0,$$
$$x^2 + 6x - 16 = 0.$$

Comparing this new equation to the original equation ($x^2 + 3x - 4$), we note that the x-coefficient has been multiplied by 2 and the constant term has been multiplied by 4.

Does this pattern hold for any quadratic equation? Consider the equation $(x - a)(x - b) = 0$, which has roots of $x = a$ and $x = b$. In expanded form, this equation would be

$$x^2 - (a + b)x + ab = 0.$$

To find a quadratic equation with roots $x = 2a$ and $x = 2b$, we proceed as follows:

$$(x - 2a)(x - 2b) = 0,$$
$$x^2 - (2a + 2b)x + 4ab = 0,$$
$$x^2 - 2(a + b)x + 4ab = 0.$$

The x-coefficient of this new equation is 2 times the x-coefficient of the original equation, and the original constant term has been multiplied by 4. The pattern does indeed hold generally.

This pattern generalizes to any higher-degree polynomial equation and to any multiple of the original roots. To see its extension to the cubic case, suppose that the roots of the original cubic equation are r_1, r_2 and r_3. The cubic equation is then $(x - r_1)(x - r_2)(x - r_3) = 0$. (If the leading coefficient is not 1, divide both sides by that coefficient to simplify the problem.) When the left side of the equation is expanded using the distributive property, we obtain the following equation:

$$x^3 - (r_1 + r_2 + r_3)x^2 + (r_1r_2 + r_1r_3 + r_2r_3)x - (r_2r_2r_3) = 0.$$

Suppose that we want a cubic equation whose roots are k times those of the original cubic—namely kr_1, kr_2 and kr_3. Such a cubic can be written in factored form as $(x - kr_1)(x - kr_2)(x - kr_3) = 0$. Expanding the left side of the equation (with repeated use of the distributive property), we obtain

$$x^3 - k(r_1 + r_2 + r_3)x^2 + k^2(r_1r_2 + r_1r_3 + r_2r_3)\,x - k^3(r_1r_2r_3) = 0.$$

Note the presence of k in increasing powers in the coefficients of the new equation.

Suppose that we want to find a quartic (fourth-degree) equation whose roots are 3 times the roots of an original quartic equation. We verify symbolically that the coefficients of the original quartic equation should be multiplied by 3^0, 3^1, 3^2, 3^3 and 3^4, respectively. Large numbers result!

Challenge for the Reader

Construct a general proof for the nth-degree equation. You will need to determine a way of writing coefficients in terms of the roots.

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