Euclid's Algorithm: Revisiting an Ancient Process

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Believed to be one of the oldest algorithms, Euclid's algorithm (also called the Euclidean algorithm) was presented in Proposition 2, Book VII of Euclid's *Elements* as a method for finding the greatest common factor (GCF) of two integers. To conventionally determine the GCF of the integers a and b, let $a = k \cdot a_1 \cdot a_2 \cdot a_3 \cdot \ldots \cdot a_n$ be a prime factorization of a, and let $b = k \cdot b_1 \cdot b_2 \cdot b_3 \cdot \ldots \cdot b_n$ be a prime factorization of b. Then, if none of the factors $a_1, a_2, a_3, \ldots, a_n$ are equal to any of the factors $b_1, b_2, b_3, \ldots, b_n$, we know that GCF$(a,b) = k$. However, if we also have $a_1 = b_1$, then GCF$(a,b) = k \cdot a_1$, and so on. That is, the GCF of two numbers will contain all factors common to both numbers, as the name suggests. In this article, I present two ways to apply this algorithm in the secondary mathematics classroom.

GCF: The Silent Partner

Why can't we reduce 3/7? Why do we multiply 2×3 to determine the lowest common multiple (LCM) of 1/2 and 1/3, but the product of 2×6 does not give us the LCM of 1/2 and 1/6? Why can't we combine $\sqrt{3} + \sqrt{7}$? Why does rationalizing

$$\frac{1}{\sqrt{6}} = \frac{\sqrt{6}}{6}$$

immediately give us a radical expression in lowest terms, but

$$\frac{2}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} = \frac{2\sqrt{6}}{6}$$

is not completely reduced? Why is the LCM of $\sin x$ and $\cos x$ their product? Why can we use a cross-multiplication rule to obtain an equivalent expression for

$$\frac{1}{x+h} - \frac{1}{x} \text{ as } \frac{x-(x+h)}{x(x+h)} = \frac{-h}{x(x+h)}?$$

The answer to all of these questions involves a GCF of 1. Like a silent partner, the GCF is always there when we need it, but it is never in the forefront. A GCF of 1 is almost always taken for granted, for without it we could not justify the steps of our procedure.

A GCF of 1 is the reason we can or cannot proceed with some of the most fundamental procedures in mathematics. It is our rationale for the answers we give to the questions raised above. Quite simply, we cannot reduce 3/7, since GCF(3,7) = 1. We perform $2 \times 3 = 6$ to obtain the lowest common denominator of 1/2 and 1/3 because GCF(2,3) = 1, but performing $2 \times 6 = 12$ does not give us the lowest common denominator of 1/2 and 1/6 because GCF(2,6) $\neq$ 1. We cannot combine $\sqrt{3} + \sqrt{7}$, since GCF(3,7) = 1, and we rationalize $1/\sqrt{6}$ immediately, since GCF(1,6) = 1. Rationalizing the denominator of $2/\sqrt{6}$ does not immediately give us a radical in simplest form, because GCF(2,6) $\neq$ 1. The same holds true for common denominators in the mathematics of trigonometry and calculus. For example, GCF($\sin x$,$\cos x$) = 1, and therefore the LCM is their product, $\sin x \cos x$. Finally, a cross-multiplication rule can be used to determine an equivalent expression (reduced to lowest terms) for

$$\frac{1}{x+h} - \frac{1}{x},$$

since GCF$(x + h, x) = 1$.

The nature and calculation of the GCF should take a more prominent role in the curriculum and, hence, in the classroom because it is so fundamental. It is time to reinforce basic skills so that students can understand the logic behind the mathematical procedures they discover and are taught. Understanding and basic skills, such as the division algorithm, will greatly enhance students' mathematical literacy.

Now, let's pursue our discussion of the GCF by examining its role in reducing fractions and rational expressions. Consider the task of reducing 57/95 to lowest terms without using a calculator. Many students would first wonder if it was possible and then how to proceed. The quickest method is to calculate GCF(57,95) using Euclid's algorithm.

Step 1. Divide the smaller number into the larger, keeping track of the remainder.

$$\begin{array}{r} 1 \\ 57\overline{)95} \\ \underline{57} \\ 38 \end{array}$$

Step 2. Divide the remainder into the previous divisor, again keeping track of the remainder.

$$\begin{array}{r} 1 \\ 38\overline{)57} \\ \underline{38} \\ 19 \end{array}$$

Step 3. Repeat Step 2 until the remainder is 0.

$$\begin{array}{r} 2 \\ 19\overline{)38} \\ \underline{38} \\ 0 \end{array}$$

Step 4. The divisor that yields a remainder of 0 is our GCF.

Here, GCF(57,95) = 19, so to reduce 57/95, we simply divide both top and bottom by 19 to obtain

$$\frac{57 \div 19}{95 \div 19} = \frac{3}{5}.$$

This skill may also be applied to a task such as reducing the rational expression

$$\frac{x^2 - x - 6}{x^2 + x - 12}$$

to lowest terms, as is required in Grades 10, 11 and 12. Students are usually instructed to factor both top and bottom and then reduce. Logically, it would be better to first answer the key question, Will this rational expression reduce at all? If GCF $(x^2 - x - 6, x^2 + x - 12) = 1$, then the answer is no. However, if the GCF is not 1, then we must proceed.

Applying Euclid's algorithm, we get

$$\begin{array}{r} 1 \\ x^2 + x - 12\overline{)x^2 - x - 6} \\ \underline{x^2 + x - 12} \\ -2x + 6 \end{array}$$

$$\begin{array}{r} -\tfrac{1}{2}x - 2 \\ -2x + 6\overline{)x^2 + x - 12} \\ \underline{x^2 - 3x} \\ 4x - 12 \\ \underline{4x - 12} \\ 0 \end{array}$$

Here, $-2x + 6 \neq 1$ necessarily, and since $-2x + 6 = -2(x - 3)$, the binomial factor $(x - 3)$ is common to both $x^2 - x - 6$ and $x^2 + x - 12$. For students who have difficulty with factoring, half the work of reducing is now already done. That is, we know that the expression can be reduced, and we know the factor needed to begin the process.

The above procedure is another way to approach a common algebra problem, but it is not helpful for every student. Some students may have forgotten the division algorithm altogether, and others would rather try their luck at factoring. On the other hand, through this approach, some students will add to their understanding of mathematical process and become more independent learners. They may even recognize the power of division and Euclid's algorithm. A high-energy honours class may appreciate the power of reducing fractions without a calculator and how the approach serves as a natural lead-in to calculating LCMs using

$$\text{LCM}(a,b) = \frac{ab}{\text{GCF}(a,b)},$$

which is the method used in Asia. A teacher who is going back to the basics may also appreciate this application of the division algorithm.

In short, the calculation of the GCF is simple and direct, and an understanding of the significance of 1 as the GCF of two numbers or expressions may help students understand the logic behind many mathematical processes. This silent partner need not be silent anymore!

Radical Radicals

Today's math student does not like radicals any more than yesterday's math student did. Although the modern student is usually armed with a calculator, the process of guessing how to break down radicals or when to rationalize the denominator still dominates the thinking and strategy processes. Also, students are not always sure that their final answer is in lowest terms, especially when dividing. There has to be a set approach to all operations involving radicals that students can use to resolve these issues.

Consider $\sqrt{75} + \sqrt{27}$. We suggest to our students that they reduce the radicands before combining terms. The logical question, of course, is whether these terms can be combined and, if they can, how to proceed. In this case, how are numerically challenged students supposed to know that they should start with 3, especially if we have taught them to extract perfect squares from each of the given radicals? How is the student to proceed with confidence and certainty from the outset?

We may tell our students that we cannot add or subtract the terms if the GCF of the radicands is 1. In simplifying $\sqrt{75} + \sqrt{27}$, the trained eye observes that GCF(27,75) is not 1 but 3; thus, it may be possible to simplify the expression. Since 3 is a common factor of both 75 and 27, we can now express $\sqrt{75}$ and $\sqrt{27}$ in terms of $\sqrt{3}$. Instead of having to guess how to break down both 75 and 27, we already have one of the key factors. The student then has the simple task of dividing 3 into both 27 and 75 before simplifying the terms to obtain

$$\begin{aligned}\sqrt{75}+\sqrt{27} &= \sqrt{3\cdot 25}+\sqrt{3\cdot 9}\\ &= 5\sqrt{3}+3\sqrt{3}\\ &= (5+3)\sqrt{3}\\ &= 8\sqrt{3}.\end{aligned}$$

The expression $\sqrt{75}+\sqrt{27}$ can be simplified, since GCF(27,75) is 3 and removing a GCF of 3 from each radicand (that is, a common factor of $\sqrt{3}$ from each radical) reveals a factor that is a perfect square in each radicand.

We might also suggest that students not multiply or divide until they have determined the GCF of the radicands. Consider the conventional way of multiplying two radicals, such as $\sqrt{28}$ and $\sqrt{63}$. Many calculator-oriented students would perform $28 \times 63 = 1{,}764$ and then try to simplify ($\sqrt{1{,}764} = 42$). If we suggest that students simplify the radicals before multiplying, a guessing or guess/estimating process begins in an attempt to determine what numbers go into both 28 and 63. It is discovered that $\sqrt{28} = 2\sqrt{7}$ and that $\sqrt{63} = 3\sqrt{7}$, but only after dealing with the radicands one at a time.

The process of Radical Radicals involves considering both radicands at the same time by finding their GCF using Euclid's algorithm (without using a calculator).

For example, find GCF(28,63).

Step 1. Divide the larger number by the smaller number, keeping track of the remainder.

$$\begin{array}{r} 2 \\ 28\overline{)63} \\ \underline{56} \\ 7 \end{array}$$

Step 2. Divide the remainder into the previous divisor.

$$\begin{array}{r} 4 \\ 7\overline{)28} \\ \underline{28} \\ 0 \end{array}$$

Step 3. Continue Step 2 until the remainder is 0.

Step 4. The divisor that yields a remainder of 0 is our GCF.

Since the divisor of 7 gives us a remainder of 0, we know that 28 and 63 have 7 as a GCF, so $\sqrt{28}$ and $\sqrt{63}$ have $\sqrt{7}$ as a common factor. This brings about a different way of doing radicals, because we work with two radicands at a time, not one. Also, when it comes time to break down the radicals, we will already know one of the factors and, therefore, half the work will have already been done. Thus, guessing or guess/estimating is reduced dramatically.

Now consider $\sqrt{26}\cdot\sqrt{65}$. Instead of looking at $\sqrt{1{,}690}$, we determine that GCF(26,65) = 13 and express each factor in terms of $\sqrt{13}$. This gives us

$$\begin{aligned}\sqrt{26}\cdot\sqrt{65} &= \sqrt{2}\cdot\sqrt{13}\cdot\sqrt{5}\cdot\sqrt{13}\\ &= \sqrt{13}\cdot\sqrt{13}\cdot\sqrt{2}\cdot\sqrt{5}\\ &= 13\sqrt{10}\end{aligned}$$

The numbers never become large, so simplifying is easier.

The disadvantage of this method is that students must find the GCF of two numbers with little help from a calculator. Its advantages are that once the GCF has been found, half the work is already done; guessing is virtually eliminated; and the numbers never increase in size, which reduces the frustration and errors that come with working with large numbers.

In summary, the steps to this new approach are as follows:

Step 1. Find the GCF of the radicands.

Step 2. Express each radicand in terms of the square root of the radicand.

Step 3. Pair off like radicals.

Step 4. Simplify remaining terms.

If you thought multiplication done in this way was efficient, let's now look at division. This is where this method really shines! We do not rationalize the denominator unless the GCF of both the numerator and the denominator is 1.

Consider $\sqrt{175}/\sqrt{112}$. If students do not recognize that 7 is a common factor, they might multiply top and bottom by $\sqrt{112}$, giving them horrendous numbers to work with. Instead, we use Euclid's algorithm to find that GCF(112,175) = 7. Thus, we have

$$\frac{\sqrt{175}}{\sqrt{112}} = \frac{\sqrt{7\cdot 25}}{\sqrt{7\cdot 16}} = \frac{\sqrt{7}}{\sqrt{7}}\cdot\frac{\sqrt{25}}{\sqrt{16}}.$$

Since $\sqrt{25} = 5$, $\sqrt{16} = 4$ and the radical factors $\sqrt{7}$ divide to give us 1, we are left with the answer 5/4.

What happens if the GCF of the numerator and the denominator is 1? We simply multiply top and bottom by the denominator, knowing that we will not have to reduce the fraction after multiplying the terms. For example,

$$\frac{\sqrt{15}}{\sqrt{7}} = \frac{\sqrt{15}}{\sqrt{7}}\cdot\frac{\sqrt{7}}{\sqrt{7}} = \frac{\sqrt{105}}{7}.$$

We realize that $\sqrt{105}$ cannot be simplified, since GCF(15,7) = 1.

Now, what happens if the numerator is not a radical? Consider $2/\sqrt{10}$. Here, we do not multiply top and bottom by $\sqrt{10}$, since GCF(2,10) = 2 (not 1). Instead, we recall that $\sqrt{a} \cdot \sqrt{a} = a$, and in this case $\sqrt{2} \cdot \sqrt{2} = 2$. Since GCF(1,100) = 1, we express both numerator and denominator in terms of $\sqrt{2}$. This gives us

$$\frac{2}{\sqrt{10}} = \frac{\sqrt{2}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{5}}.$$

The $\sqrt{2} / \sqrt{2}$ divides to give us 1, and we are now left with $\sqrt{2} / \sqrt{5}$. Since GCF(2,5) = 1, we can now multiply top and bottom by $\sqrt{5}$ and not have to worry about reducing the final form of the quotient. Finally, we have

$$\frac{\sqrt{2}}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{10}}{5}.$$

We do not have to go backward or look over our shoulder to see if the quotient can be reduced, since the GCF is 1. Our rule, then, is to multiply top and bottom by the denominator factor only when the GCF of the radicands is 1.

In general, the steps for division are as follows:

Step 1. Find the GCF of both numerator and denominator.

Step 2. Express each radicand in terms of the square root of the GCF.

Step 3. Pair off like radicals and reduce.

Step 4. Simplify remaining terms.

Step 5. Multiply top and bottom by the denominator term only when the GCF of numerator and denominator is 1.

For division, then, we merely add Step 5 to the method used for multiplication.

These new processes for multiplication and division now simplify the processes for addition and subtraction, because we can operate only if the GCF is not 1. Hence, $\sqrt{3} + \sqrt{7}$ cannot be simplified any further, since GCF(3,7) = 1. For $\sqrt{24} + \sqrt{54}$, we find that GCF(24,54) = 6. We then have

$$\begin{aligned} \sqrt{24} + \sqrt{54} &= \sqrt{4} \cdot \sqrt{6} + \sqrt{9} \cdot \sqrt{6} \\ &= \sqrt{6}\,(2+3) \\ &= 5\sqrt{6}. \end{aligned}$$

Again, we look at two radicands at a time. We ascertain that it may indeed be possible to combine terms if their GCF is not 1. If we have more than two terms, we can look for two or more with the same GCF.

For combined operations, we again look at two GCFs as opposed to one. Consider $\sqrt{3}\,(\sqrt{15}+\sqrt{21})$. Since GCF(3,15) = 3 and GCF(3,21) = 3, we have $\sqrt{15} = \sqrt{3} \cdot \sqrt{5}$ and $\sqrt{21} = \sqrt{3} \cdot \sqrt{7}$. Thus, we can write

$$\begin{aligned} \sqrt{3}\,(\sqrt{15}+\sqrt{21}) &= \sqrt{3}\,(\sqrt{3} \cdot \sqrt{5} + \sqrt{3} \cdot \sqrt{7}) \\ &= 3\,(\sqrt{5}+\sqrt{7}). \end{aligned}$$

For division, consider

$$\frac{5}{\sqrt{7} - \sqrt{2}}.$$

Since GCF(5,7,2) = 1, we multiply top and bottom by the conjugate $\sqrt{7}+\sqrt{2}$. This gives us

$$\begin{aligned} \frac{5}{\sqrt{77} - \sqrt{22}} \cdot \frac{\sqrt{7}+\sqrt{2}}{\sqrt{7}+\sqrt{2}} &= \frac{5(\sqrt{7}+\sqrt{2})}{7-2} \\ &= \frac{5(\sqrt{7}+\sqrt{2})}{5} \\ &= \sqrt{7}+\sqrt{2}. \end{aligned}$$

In summary, by finding the GCF of the radicands, we introduce the idea of working with two or more radicands at a time. Once we have the GCF, we cut the work by at least half because we already have one of the factors of the radicand. We eliminate large numbers, the errors caused by large numbers and the frustration that results from guessing. We streamline the process by keeping the numbers simple and neat.

In my experience, student feedback on this process is always the same: "This is easy compared to what I used to do!"

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