Alberta High School Mathematics Competition 2005: Part I

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The Alberta High School Mathematics Competition (AHSMC) is open to students in Alberta and the Northwest Territories. It is designed for high school students, but occasionally students in the earlier grades also take part. The prizes are books and scholarships ranging in value from \$50 for some of the Part I prizes to \$1,500 for the major prize of Part II.

For more information, visit the AHSMC website at www.math.ualberta.ca/~ahsmc/index.html.

Part I of the 2005 competition (with solutions) follows.

1. What is the value of $2{,}005 \times 20{,}042{,}004 - 2{,}004 \times 20{,}052{,}005$?

   (a) 0 (b) 10,000 (c) 2,003,000 (d) 2,005,000 (e) none of these

2. An ice cream store has 20 kinds of ice cream. A customer may get one or two scoops of ice cream. If she gets two scoops, the scoops can be the same or different, and the order of the scoops does not matter. How many different cones are possible?

   (a) 90 (b) 99 (c) 100 (d) 230 (e) none of these

3. In triangle ABC, let D be the midpoint of BC, and let E be on AD such that $ED = 2AE$. If the area of triangle ABC is 150, what is the area of triangle ABE?

   (a) 25 (b) 32.5 (c) 50 (d) 75 (e) none of these

4. Let a, b and c be real numbers and $x = 11c - a - b$. If $b - a - 3c \le -2$ and $b - 2a + c \ge 3$, then

   (a) $x \in [0,1]$ (b) $x \in [2,5]$ (c) $x \in [6,9]$ (d) $x \in [10,11]$ (e) $x \in [12,\infty]$

5. Penelope has three red socks, three yellow socks and three blue socks. If she picks socks without looking, what is the smallest number of socks she must pick to guarantee that she has four socks that form two pairs of socks of matching colours?

   (a) 4 (b) 5 (c) 6 (d) 7 (e) 8

6. A building consists of a square-based pyramid on top of a square-based prism. All vertical edges of the prisms have lengths of 2 m. All other edges have lengths of 3 m. What is the height, in metres, of the top of this building from the ground?

   (a) $2 + \dfrac{\sqrt{2}}{2}$ (b) $2 + \dfrac{3\sqrt{2}}{2}$ (c) $\sqrt{20}$ (d) $\sqrt{21}$ (e) none of these

7. If f is a function defined on the positive real axis and

   $$f(x) + 3f\left(\frac{1}{x}\right) = x - \frac{5}{x} + 4 \quad \text{for all } x > 0,$$

   then $f(½)$ is

   (a) $\dfrac{1}{2}$ (b) 1 (c) $\dfrac{3}{2}$ (d) 2 (e) 4

8. In the sequence obtained by omitting the squares and the cubes from the sequence of positive integers, 2,005 sits on which position?

   (a) 1,950 (b) 1,951 (c) 1,952 (d) 1,953 (e) none of these

9. The positive integer a is such that the inequality $2a + 3x \le 101$ has exactly six solutions in positive integers x. How many possible values of a are there?

   (a) 1 (b) 2 (c) 3 (d) 4 (e) 5

10. P is a point inside a parallelogram $ABCD$. If the area of triangle PAD is one-third that of $ABCD$, and the area of triangle PCB is 6 cm^2, then what is the area, in square centimetres, of the parallelogram?

    (a) 24 (b) 36 (c) 48 (d) 60 (e) 72

11. There are three problems in a contest. Students win bronze, silver or gold medals if they solve one, two or three problems, respectively. Each problem is solved by 60 students, and there are

100 medallists. What is the difference between the number of bronze medallists and the number of gold medallists?

(a) 0 (b) 10 (c) 20 (d) 30
(e) not uniquely determined

12. The positive numbers a, b and c are such that $a + b + c = 7$ and

$$\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}=\frac{10}{7}.$$

What is the value, then, of

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}?$$

(a) 1 (b) 3 (c) 7 (d) 10 (e) dependent on a, b and c

13. A tile is obtained from a 3×3 square by first removing two squares in opposite corners and then removing one square adjacent to each of those squares, so that the remaining five squares form a connected piece. The tile is placed on an infinite piece of graph paper so that it covers exactly five squares. It may be turned over or rotated. If we wish to paint the squares of the graph paper in such a way that no matter where the tile is placed, it never covers two squares of the same colour, what is the smallest number of colours needed?

(a) 5 (b) 6 (c) 7 (d) 8 (e) 9

14. The number of ordered pairs of integers (x,y) such that $xy = x^2 + y^2 + x + y$ is

(a) 3 (b) 4 (c) 6 (d) 8 (e) none of these

15. The positive integer n that satisfies the equation $\log_2 3 \log_3 4 \cdots \log_n(n + 1) = 2{,}005$ should be a multiple of

(a) 2 (b) 3 (c) 5 (d) 31 (e) none of these

16. Three spheres of unit radius sit on a plane and are tangent to one another. A large sphere with its centre in the plane contains all three unit spheres and is tangent to them. The radius of the large sphere is

(a) $\sqrt{3}-1$ (b) $\sqrt{\frac{5}{3}}+1$ (c) $\sqrt{\frac{7}{3}}+1$

(d) $\sqrt{3}+1$ (e) none of these

Solutions

1. The answer is (a). We have $2{,}005 \times 20{,}042{,}004 - 2{,}004 \times 20{,}052{,}005 = 2{,}004 \times 2{,}005(10{,}001 - 10{,}001) = 0$.

2. The answer is (d). The number of different single cones possible is 20, as is the number of different double cones with identical scoops. The number of different double cones with different scoops is $(20 \times 19)/2 = 190$. Hence, the total is $20 + 20 + 190 = 230$.

3. The answer is (a). Since $BD = CD$, the area of triangle BAD is 75. Since $ED = 2AE$, the area of triangle ABE is 25.

4. The answer is (e). The first given inequality may be rewritten as $a - b + 3c \geq 2$. Adding three times this to two times $b - 2a + c \geq 3$, we have $x \geq 12$.

5. The answer is (c). If Penelope picks only five socks, she may get three red socks, one yellow sock and one blue sock, forming only one matching pair. If she leaves out three socks, she can leave out at least two socks of at most one colour. Hence, if she picks six socks, she can be guaranteed to have two matching pairs.

6. The answer is (b). Half the diagonal of the square base has length

$$\frac{3\sqrt{2}}{2}.$$

Since the pyramid is half of a regular octahedron, its height has the same length. Hence, the height of the top of the building from the ground is

$$2+\frac{3\sqrt{2}}{2}.$$

7. The answer is (d). We have

$$f(2)+3f\left(\frac{1}{2}\right)=\frac{7}{2} \text{ and } f\left(\frac{1}{2}\right)+3f(2)=-\frac{11}{2}.$$

Subtracting the second equation from three times the first, we have

$$8f\left(\frac{1}{2}\right)=16, \text{ so that } f\left(\frac{1}{2}\right)=2.$$

8. The answer is (c). Since $44^2 < 2{,}005 < 45^2$, $12^3 < 2{,}005 < 13^3$ and $3^6 < 2{,}005 < 4^6$, the number of squares, cubes and sixth powers less than 2,005 are 44, 12 and 3, respectively. Hence, 2,005 sits in position $2{,}005 - 44 - 12 + 3 = 1{,}952$.

9. The answer is (a). The six solutions for x must be 1, 2, 3, 4, 5 and 6. If $a = 42$, then $3x \leq 17$ and $x = 6$ is not a solution. Larger values of a eliminate further solutions. If $a = 40$, then $3x \leq 21$ and $x = 7$ is also a solution. Smaller values of a allow for further solutions. Hence, we must have $a = 41$. Then $3x \leq 19$, and we have the six solutions above.

10. The answer is (b). The total area of triangles *PAD* and *PCB* is half that of the parallelogram. Hence, $1/2 - 1/3 = 1/6$ of the area of the parallelogram is 36, and its area is 36.

11. The answer is (c). If we have a gold medallist and a bronze medallist, we can convert them to two silver medallists. This way, we will end up with either no gold medallists or no bronze medallists. Altogether, $3 \times 60 = 180$ correct solutions are received. If all students were silver medallists, they would have turned in $2 \times 100 = 200$ correct solutions. Since we are $200 - 180 = 20$ short, we have 20 bronze medallists and 0 gold medallists. This difference is not affected by our conversions.

12. The answer is (c). We have

$$\begin{aligned}&\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\\ &=\frac{7-(b+c)}{b+c}+\frac{7-(c+a)}{c+a}+\frac{7-(a+b)}{a+b}\\ &=7\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)-3\\ &=7.\end{aligned}$$

13. The answer is (e). Divide the infinite piece of graph paper into 3×3 regions. If we paint the nine squares in each region in nine different colours in exactly the same way, the tile cannot cover two squares of the same colour. If we use fewer than nine colours, two squares in the same region will have the same colour. Both can be covered by a suitably placed tile.

14. The answer is (c). Adding $x + y + 2$ to both sides of the given equation, and letting $u = x + 1$ and $v = y + 1$, the equation becomes $u^2 + v^2 = uv + 1$. If $uv = 0$, the solutions are $(u,v) = (0, \pm 1)$ or $(\pm 1, 0)$. If $uv > 0$, then $(u - v)^2 = 1 - uv$, and we must have $uv = 1$. The solutions are $u = v = \pm 1$. If $uv < 0$, let one of them be a and the other be $-b$, where a and b are positive. Then the equation becomes $a^2 + ab + b^2 = 1$, which has no solutions. Hence, there are six solutions for (u,v), and six corresponding solutions for (x,y).

15. The answer is (d). Through changing bases, the given equation becomes $\log_2(n + 1) = 2{,}005$. Hence $n = 2^{2{,}005} - 1$. It leaves a remainder of 1 when divided by 2, 3 or 5 but is divisible by $2^5 - 1 = 31$.

16. The answer is (c). Since the three equal spheres are pairwise tangent, their centres form an equilateral triangle of side 2. The distance of the centre of this triangle from any vertex is

$$\frac{2\sqrt{3}}{3}.$$

The projection of this centre onto the plane is the centre of the large sphere. Its distance from the centre of one of the spheres is

$$\sqrt{1+\frac{2\sqrt{3}}{2}}=\sqrt{\frac{7}{3}}.$$

Since the large sphere is also tangent to the others, its radius is

$$\sqrt{\frac{7}{3}}+1.$$