delta-k

JOURNAL OF THE
MATHEMATICS COUNCIL
OF THE ALBERTA
TEACHERS' ASSOCIATION

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Volume 43, Number 2 June 2006

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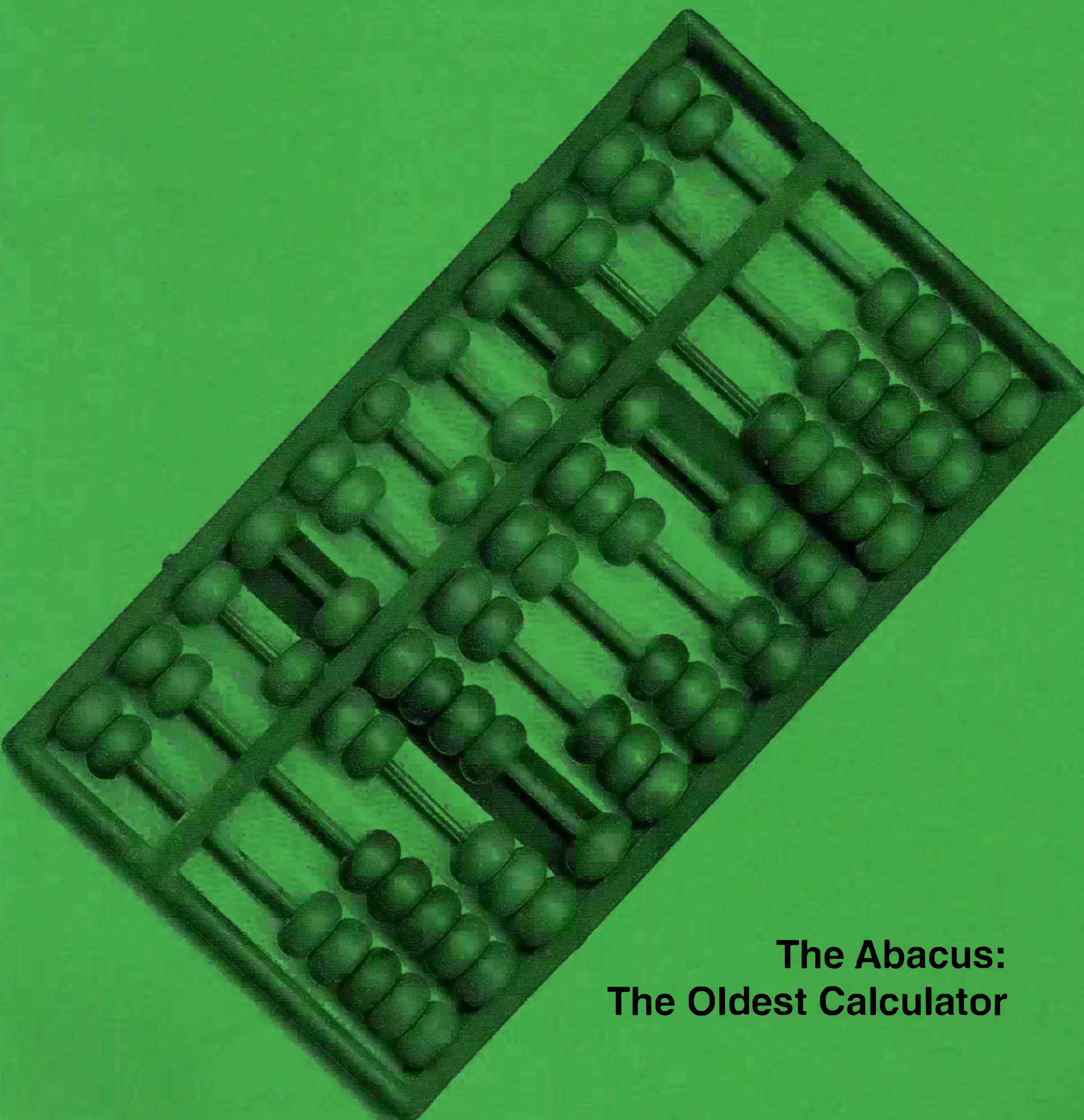

The Abacus: The Oldest Calculator

GUIDELINES FOR MANUSCRIPTS

delta-K is a professional journal for mathematics teachers in Alberta. It is published twice a year to

- promote the professional development of mathematics educators, and
- stimulate thinking, explore new ideas and offer various viewpoints.

Submissions are requested that have a classroom as well as a scholarly focus. They may include

- personal explorations of significant classroom experiences;
- descriptions of innovative classroom and school practices;
- reviews or evaluations of instructional and curricular methods, programs or materials;
- discussions of trends, issues or policies;
- a specific focus on technology in the classroom; or
- a focus on the curriculum, professional and assessment standards of the NCTM.

Suggestions for Writers

1. *delta-K* is a refereed journal. Manuscripts submitted to *delta-K* should be original material. Articles currently under consideration by other journals will not be reviewed.
2. All manuscripts should be typewritten, double-spaced and properly referenced. All pages should be numbered.
3. The author's name and full address should be provided on a separate page. If an article has more than one author, the contact author must be clearly identified. Authors should avoid all other references that may reveal their identities to the reviewers.
4. All manuscripts should be submitted electronically, using Microsoft Word format.
5. Pictures or illustrations should be clearly labelled and placed where you want them to appear in the article. Please also include all graphics as separate files (JPEG, GIF, TIF). A caption and photo credit should accompany each photograph.
6. References should be formatted using *The Chicago Manual of Style*'s author-date system.
7. If any student work is included, please provide a release letter from the student's parent/guardian allowing publication in the journal.
8. Limit your manuscript to no more than eight pages double-spaced.
9. A 250- to 350-word abstract should accompany your manuscript for inclusion on the Mathematics Council's website.
10. Letters to the editor and reviews of curriculum materials are welcome.
11. Send manuscripts and inquiries to the editor: Gladys Sterenberg, 131 Woodbend Way, Okotoks, AB T1S 1L7; e-mail gladys.sterenberg@uleth.ca.

MCATA Mission Statement

Providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics.

delta-K

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TEACHING IDEAS

EDITORIAL

With this issue of *delta-K* I give many thanks to the reviewers and the authors for their willingness to critically respond to new ideas and be challenged to elaborate on their thoughts. The process of review has been both engaging and inspiring as we work together to continue conversations about teaching and learning mathematics.

Launching the conversation in this issue is a reader's response to some past articles on the roles of understanding and memorization. Issues of good pedagogy remain a focus of curriculum development and continue to challenge teachers. Two of the feature articles remind us why we continue to strive for excellence in a context of curriculum change. The other, also related to context, presents a historical look at an ancient method for calculating arithmetic problems.

A special feature of this issue is the inclusion of student voices. The two articles in the Student Corner highlight the impressive results of mathematics teachers' efforts in our province and showcase mathematical thinking.

Our reviewers found the submissions for Teaching Ideas compelling, and I hope that you will also find them interesting and practical for your classroom. Although the ideas in this issue focus on secondary education, I encourage teachers of elementary mathematics to submit their ideas, too. It is through sharing these ideas that we build our community and enhance our teaching.

delta-K is a publication by teachers for teachers; it is possible only through teachers' participation as authors. I encourage you to contact me if you are thinking about writing for *delta-K* and need some guidance. I am more than willing to make suggestions and help write articles. As the school year comes to a close, please take time to read and reflect on the articles in this issue. Write down your responses as a reader, or jot down some of your own teaching ideas. Write an article, or gather student work that can be shared. In your preparations to finish the school year, remember to share your most vivid memories with the rest of the mathematics education community.

Have a wonderful summer!

Gladys Sterenberg

From the President's Pen

In the introduction to his book *Beyond Numeracy,* John Allen Paulos (1991) refers to a misconception held by many that mathematics is like a totem pole—"first arithmetic, then algebra, then calculus, then more abstraction, then whatever." It interests me to think about the different ways those who are (and those who are not) literate thinkers in their fields see the essentials of the various arts and sciences.

As the child of a cutting-edge research scientist, I remember hearing about the roles imagination, dead ends and noticing the unexpected played in the early development of gene identification in animals. In school, I was exposed to *the* scientific method, which reduced an extremely complex, multifaceted process to the categories of aim, materials, method, observations and conclusion. Exclusive focus on the scientific method—without speculation, imagination and the ability to notice—does not lead anyone to scientific literacy.

I have watched the competing claims in the literacy wars argued and demonstrated over years. It continues to be that some children learn to read without apparent effort and without instruction, most learn to read through whatever method of instruction is used, a small number learn through idiosyncratic processes, and an extremely small number never learn to read well. This has led me to an instant suspicion of any approach that seeks to simplify the complex and varied ways in which we learn.

In my reading, I have come across the suggestion that teaching for children in the early school years should focus on number to the virtual exclusion of other aspects of mathematics. Often this is accompanied by the suggestion that early remedial math (focused again on number) is needed to prevent later failure.

The math achievement levels in some Asian countries and the focus on number in the early grades in those countries are offered as evidence that this is a promising approach. Alberta, which has been following a well-rounded curriculum based on all the strands and processes of the standards of the US-based National Council of Teachers of Mathematics (NCTM 2000), is also achieving at these high levels on international assessments such as the Programme for International Student Assessment (see Bussière et al 2004).

Is mathematics more about number or more about patterns and relationships? Do the ways in which mathematics and culture interact affect how children learn as well as what they learn? If we want to adopt the math curricula of Asian countries, do we also need to adopt their cultures and languages?

Can we reduce the complexity and beauty of mathematics to a totem pole? Would we be wise to do so?

References

Bussière, P, W T Rogers, T Knighton and F Cartwright. 2004. *Measuring Up: Canadian Results of the OECD PISA Study: The Performance of Canada's Youth in Mathematics, Reading, Science and Problem Solving: 2003 First Findings for Canadians Aged 15*. Ottawa: Statistics Canada. Also available at www.cmec.ca/pisa/2003/Pisa2003.en.pdf (accessed 2006 03 09).

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Janis Kristjansson

Conference 2005: "Mathematics for Teaching"

Conference Report

Conference 2005 was held November 3–5 at the Fantasyland Hotel in Edmonton. We chose the theme "Mathematics for Teaching" in recognition of the specialized and complex knowledge mathematics teachers bring to their craft.

The conference was enthusiastically received by teachers from all over Alberta and even some from British Columbia, Saskatchewan, Manitoba and the territories. More than 60 sessions were offered to meet a wide range of needs—from those of the experienced elementary or secondary teacher to those of the first-year teacher teaching Grades 5–9 in a rural school. This year all the sessions were 90 minutes long, allowing for more in-depth investigation of topics. Keynote speakers Brent Davis and John Allen Paulos gave inspiring talks on mathematics for teaching and mathematics for living a well-informed life.

This year we didn't even try to compete with the shopping and entertainment available at the West Edmonton Mall. Participants and their families were spotted enjoying activities throughout the mall. I hear that some even pursued the mathematical possibilities offered by the roller coaster and the waterslide.

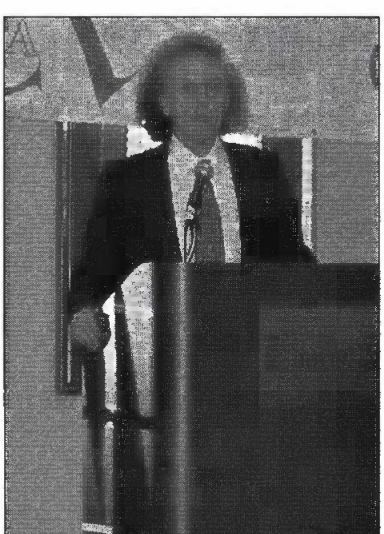
The Dr Arthur Jorgensen Chair Award is presented annually to an education student in Alberta to encourage continued commitment to mathematics education. The award includes a one-year term on the MCATA executive and an opportunity to attend the annual conference. This year's recipient was Rebecca Steel, a student at the University of Calgary.

Through Friends of MCATA awards, we recognize those who have given generously of their time and expertise to support the work of MCATA. This year's recipients were Chenoa Marcotte, Susan Ludwig, Scott Petronech and Jodee Brennan-Frois.

We hope you will be able to join us October 20–22 for Conference 2006 at Jasper Park Lodge in Jasper!

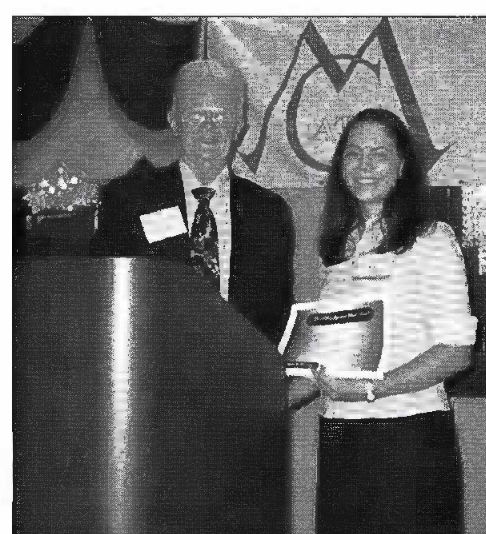
Janis Kristjansson

Photographic Memories

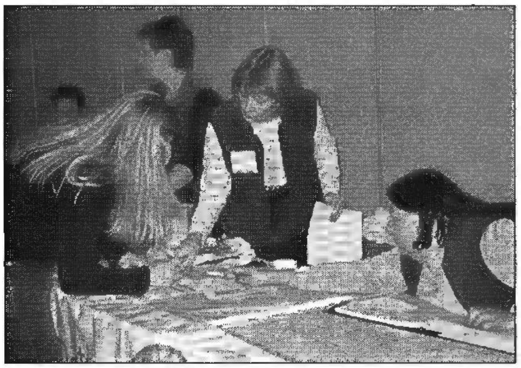


John Allen Paulos, Saturday's keynote speaker

Brent Davis, Friday's keynote speaker

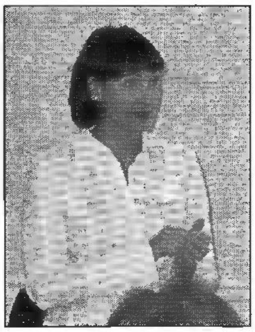


Art Jorgensen presenting the Dr Arthur Jorgensen Chair Award to Rebecca Steel



The registration desk

Participants at Florence Glanfield's workshop Multiple Representations of Fraction Knowing



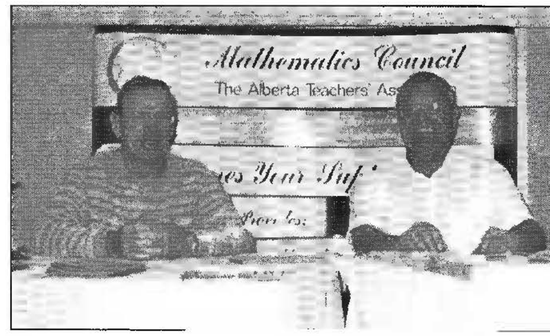
Katherine Willson

Lynda Burgess

Participants at Jennifer Krueckl and Rosalind Carson's workshop The Adventures of Rene D!

Participants at Joanne Currah's workshop Double Dare You

Florence Glanfield with workshop participants

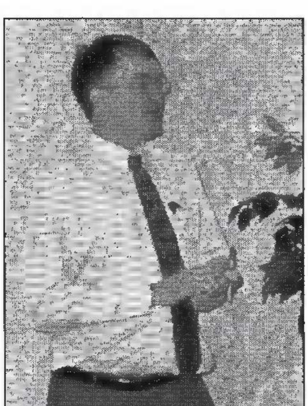


Displays by our MCATA friends

Jennifer Krueckl with workshop participants

Participants at Elaine Simmt and Florence Glanfield's workshop Teaching Mathematics for the First Time?

Lissa Howes



Ralph Mason

Participants at Ralph Mason's workshop Teaching for Understanding: Decimal-to-Fraction Conversions

The Right Angle: Report from Alberta Education

Jennifer Dolecki

Curriculum Revisions

Alberta Education is providing leadership for the Western and Northern Canadian Protocol (WNCP) to make revisions to the WNCP Common Curriculum Framework (CCF) for K–12 mathematics. A publishers draft for K–7, based on feedback received during consultations, is available on the WNCP website (www.wncp.ca) with the final consultation report. Consultations on outcomes for Grades 8 and 9 were held in December 2005 after initial work on the revisions to the Grades 10–12 CCF to ensure a smooth transition from junior to senior high school.

New Alberta programs of study will be developed based on the revised K–12 CCF. Implementation timelines are published in the *Implementation Schedule for Programs of Study and Related Activities* (www.education.gov.ab.ca/k_12/curriculum/impshed.pdf) on the Alberta Education website. The K–12 CCF will be reviewed to ensure that First Nations, Métis and Inuit perspectives are represented.

Alberta Education, in cooperation with WNCP partners, is engaged in ongoing communication with publishers to ensure that the best possible resources are available for Alberta schools in both English and French before provincial implementation.

Through the Alberta Regional Professional Development Consortia, Alberta Education continued a series of workshops for Grades 7, 8 and 9 mathematics teachers. The workshops, based on current research in mathematics education, provide teachers with strategies for teaching the concepts in the junior high mathematics program that students often find challenging. This year the following workshops were provided:

- Teaching Algebra Concepts in Junior High
- Measurement Activities to Develop Understanding
- Learning Strategies That Enhance Understanding in Mathematics

For information on the availability of these workshops, please contact your regional professional development consortium. Other questions can be addressed to Debbie Duvall, Alberta Education's K–12 mathematics resource manager, at Debbie.Duvall@gov.ab.ca.

Knowledge and Employability Courses

The Integrated Occupational Program (IOP) is making the transition to Knowledge and Employability courses. Provincial implementation of the new Knowledge and Employability policy, courses and provincial achievement tests is scheduled for September 2006. The current IOP policy will remain in effect until then.

The September 2005 field test drafts of the programs of study for the Knowledge and Employability courses have been posted at www.education.gov.ab.ca/k_12/curriculum/bySubject/iop/default.asp. The revised Certificate of Achievement for Knowledge and Employability will now require students to complete Mathematics 26 and Science 26.

The Knowledge and Employability Studio, an online guide to the implementation of Knowledge and Employability courses for Grades 8–12, provides information, strategies, sample activities and tools to help teachers meet curricular outcomes and enhance student learning. It can be found at www.learnalberta.ca in the Teachers section, or accessed directly at www.learnalberta.ca/content-teacher/kes/index.html?launch=true.

Maintaining Consistent Standards over Time

Alberta Education aims to make diploma examination results directly comparable from session to session, thereby enhancing fairness to students across exam administrations. To achieve this goal, a number of questions, called anchor items, remain the same from one exam to the next. Anchor items are used to find out if the student population writing in one administration is different in achievement from the student population writing in another administration. Anchor items are also used to find out if an exam's unique items—questions that are different on each exam—are different in difficulty from the unique items on the baseline exam. A statistical process called equating adjusts for differences in examination form difficulty. The resulting equated scores have the same meaning regardless of when and to whom the exams were administered.

Baseline exams were written for Chemistry 30, Physics 30 and Pure Mathematics 30 in January 2005. In January 2006, baseline exams were written for Applied Mathematics 30 and Biology 30. Students who write any of these diploma exams in a future administration will have their marks directly equated to the baseline exam. Exam marks may be adjusted slightly upward or downward depending on the difficulty of the exam relative to the baseline exam. These equated marks will be reported to students. As a result of equating, students' marks will accurately reflect their level of achievement regardless of the particular exam.

Because of the security required to enable fair and appropriate assessment of student achievement over time, both Part A and Part B of the January 2006 Applied Mathematics 30, Biology 30, Chemistry 30 and Pure Mathematics 30 diploma exams were secured and were not released at the time of writing. The June 2006 Applied Mathematics 30 and Biology 30 diploma exams will also be secured. Only one form of the exam will be written during an administration.

In the fall of 2006, Alberta Education will once again provide schools with print copies of *Released Items* from past diploma exams. Alberta Education will also provide *Assessment Highlights* on its website (www.education.gov.ab.ca) to educators, students and parents. Teachers will still be able to peruse copies of these exams during exam administrations.

For more information on this initiative, go to www.education.gov.ab.ca/k%5F12/testing/diploma/consistentstandards.asp.

New Applets and Interface for Applied Mathematics Multimedia Resources

Eleven new discovery applets that address outcomes in applied and pure senior high mathematics have been added to the LearnAlberta.ca website (www.learnalberta.ca). These applets allow students and teachers to manipulate parameters and easily visualize concepts being addressed. This resource has also adopted an improved user interface. In addition to grouping the applets according to strand and substrand, the new interface allows the user to select applets according to course (pure mathematics or applied mathematics) and level (10, 20 or 30).

Preview Zone Launched on LearnAlberta.ca

The Preview Zone, available on the LearnAlberta.ca website, offers learning resources in the final stages of technical development. These resources will be available through the Preview Zone to authorized users on a temporary basis during development and testing. New resources will be added as they become available.

To find the Preview Zone, go to the What's New? section or the Teachers page at www.learnalberta.ca.

The following resources are currently available:

- *The Learning Equation 11* (Pure Mathematics 20)
- *The Learning Equation 12* (Pure Mathematics 30)
- *Applied Mathematics 10* (Applied Mathematics 10)
- *Mathématiques appliquées 10* (Applied Mathematics 10, in French)

CD-ROMs with complete content are available through the Learning Resources Centre (www.lrc.education.gov.ab.ca).

The following resources will soon be available:

- *La Formule du savoir 11* (Pure Mathematics 20, in French)
- *La Formule du savoir 12* (Pure Mathematics 30, in French)

Your feedback on these resources is encouraged and valued.

READER'S RESPONSE

Context

Indy Lagu

> The definition of a good mathematical problem is the mathematics it generates rather than the problem itself.
>
> —Andrew Wiles

Growing up in the Calgary school system, I had a pretty dim view of mathematics. I never had any desire to become a mathematician, and I would not have believed you if you had told me that I someday would. It is now clear to me that I had no idea what a mathematician did. All right, I *thought* I knew what a mathematician did, but my speculations could not have been further from the truth. Recently, I began to wonder why that was, and I have come to some partial conclusions. I think that, given the rotten attitudes toward mathematics of the high school graduates I encounter, the reasons have not changed at all; the reasons the students in my class hate mathematics are the same reasons I hated it. One reason is that we have taken an essential element out of mathematics classes: context.

Context, in this context, does not mean merely putting a student's name into a question, *à la*

> Johnny has three quarters, two dimes and four pennies. How much money does he have?

Nor does it mean succumbing to an idiotic political correctness and changing the name Johnny to Jane, Isaac, Sunil, Ahmed or any of a host of wonderful names. What I mean by *context* is a situation that allows for exploration. Now, such situations do not have to be real-life situations, whatever that means. Johnny's financial woes may be a real-life situation, but they do not lead to a question that is interesting (mathematically or otherwise).

I do not understand how or why we have lowered mathematics to this level, but I do know that questions of this type are what most of my students believe mathematics is all about. They also have a misguided belief that the answer is the most important thing in mathematics and that communicating ideas and clear exposition are a nuisance.

We often encourage these destructive and insulting beliefs. Why destructive? Because they relegate mathematics from a useful way of thinking to a form of *Jeopardy*. Why insulting? What other word is there to describe such a denigration of the enormous accomplishments of Gauss, Euler, Newton and Hopper, to name a few?

One way we encourage these beliefs is by making it appear to our students that only the answers are important. This is most easily accomplished through giving our students multiple-choice tests.

Another way we encourage these beliefs is by presenting mathematics as an unrelated string of facts. I call this unit mentality—the belief that factoring, graphing and finding the roots of a polynomial are three distinct activities (units) that come with three different exams (unit tests). We must divorce ourselves from the notion that mathematics comes in discrete and disparate pieces. To that end, I offer the following problem.

In the old days of college basketball, before the shot clock and the three-point line, a team could score only one or two points at a time. For example, if a team had five points, those points must have been obtained in one of the following eight ways:

$$1 + 1 + 1 + 1 + 1,$$
$$2 + 2 + 1,$$
$$2 + 1 + 1 + 1,$$
$$1 + 2 + 1 + 1,$$
$$1 + 1 + 2 + 1,$$
$$1 + 1 + 1 + 2,$$
$$1 + 2 + 2 \text{ or}$$
$$2 + 1 + 2.$$

How many ways could the team score n points?

If your first instinct is to see the words *how many ways* and think, *This is a question about perms and combs,* forget that instinct. The expression *perms and combs* belongs in a hair salon, not in mathematics. Mathematicians do not use those terms, so why do we? And why do we expose our students to that false terminology?

Now, why is the basketball problem a good one? Well, basketball teams likely do not care about the answer, and therefore it is certainly not a real-life problem. However, it is a good problem because of where it leads—to exploration. We can play with the problem.

If we let w_n be the number of ways to score n points, it is not too difficult to see that $w_0 = 1$ and $w_1 = 1$, because there is only one way to have zero points (score no points) and only one way to have one point (score only one point). Since there are two ways to get two points (1 + 1 and 2), $w_2 = 2$. And so on. Play with this problem some more before reading on, trying to generate the sequence.

The sequence generated is as follows:

$$w_0 = 1, w_1 = 1, w_2 = 2, w_3 = 3, w_4 = 5, w_5 = 8, \ldots .$$

Do you see a pattern? After w_1, each number is the sum of the two that came before. Is that really the pattern? Why? (Astute readers will recognize this as the Fibonacci sequence.) Notice that the basketball problem has led us to sequences. In particular, it has led us to the important fact that a sequence is defined not by a few terms but, rather, by its context. If you are not convinced, try the following sequence[1]:

$$x_n = \lceil \sqrt{e^{n-1}} \rceil.$$

You will get $x_0 = 1$, $x_1 = 1$, $x_2 = 2$. As it turns out, x_n and w_n agree for $n = 0, 1, 2, \ldots, 9$. However, for $n = 10$, we get $w_{10} = 89$, whereas $x_{10} = 91$. Thus, it is not sufficient to look at a list of numbers.

For the basketball problem, the pattern wherein each term is the sum of the two previous terms does hold, and I will leave it to you to discover why. Algebraically, $w_0 = w_1 = 1$, and

$$w_{n+2} = w_{n+1} + w_n \qquad \text{for } n \geq 0.$$

Typically, people will ask for an explicit formula for w_n. Try this: assume that $w_n = x^n$ for some (non-zero) value of x. Then,

$$x^{n+2} = x^{n+1} + x^n.$$

Hence, we must have

$$x^2 - x - 1 = 0,$$

and we have reduced the original question to finding the roots of a quadratic. This leads to factoring, working with the quadratic formula and graphing.

So we see that a good problem—that is, a problem with a context—can lead us to many of the mathematical places our curriculum mandates: sequences, polynomials, factoring, graphing and combinatorics, to name a few. The basketball problem also leads to the theory of partitions, difference equations, the golden rectangle, phyllotaxis (the pattern of bumps on a pineapple) and the solution of certain differential equations.

What we need is not only a bank of such problems but also teachers who have the ability and desire to investigate good problems, and the belief that finding the answer is not the reason for trying to solve a problem.

Note

1. The symbol $\lceil x \rceil$ means "round up," so that $\lceil 1.1 \rceil = 2$, $\lceil \pi \rceil = 4$ and so on. It is called the ceiling function.

Indy Lagu completed his PhD in numerical analysis at the University of Calgary in 1996 and is currently the chair of the Department of Mathematics, Physics and Engineering at Mount Royal College in Calgary. He has served on the MCATA executive for five years and has been working with K–12 teachers and students for nearly 10 years. Outside of mathematics, his interests include playing guitar, cooking, golfing and wondering when the Montreal Canadiens will win another Stanley Cup.

Assessing What Matters

David Geelan

The following article has been adapted from a workshop for MCATA members held at Barnett House in Edmonton on April 22, 2005.

Educators, both those who work in the classroom and those who support them, are increasingly getting involved in assessing and evaluating programs and innovations. This kind of work used to be the province of university faculty and outside consultants. In many ways, the involvement of those who are implementing programs in the assessment of those programs is an exciting development. However, teachers may find it challenging to undertake assessment activities when they feel that they do not have the relevant training and expertise. This article provides an overview of some of the issues in program assessment and evaluation in educational contexts, and points to further reading for those who need support in conducting assessments. It also aims to make you a more informed reader of research and evaluation reports, so that you can critically evaluate the claims made by others.

What Matters?

In conducting an evaluation (I will define the term *evaluation* in the next section), it is important to know what matters. Why is it important to assess this program? What are the goals of the program? To what extent is the program meeting those goals? Are the goals appropriate? Deciding what matters is essentially a value decision: what matters to us is what we value highly. The decision should be made reflectively rather than reactively; that is, rather than letting someone else dictate what matters, we should make a principled choice.

It is also important to ask, Matters to whom? Whose interests are served by the program or innovation, and how well? Whose interests might suffer?

Deciding what is important gives the assessment a centre that it might otherwise lack. Many of the other decisions that must be made in conducting an assessment become simpler when the key values are clear. In a collaborative assessment project, deciding what matters becomes even more important: if what matters remains implicit and is not discussed, the members of the team might be working in different directions and toward different goals, and applying different standards.

Assessment, Evaluation, Research

Perhaps my working definitions of the key terms I will be using—*assessment, evaluation* and *research*—would be useful here. You would think that by now we would have these terms pretty well defined in education, but I have seen them used in different ways in different places. Your own definitions might vary, but I want to make it clear what I mean when I use these terms.

In its simplest terms, *assessment* means measuring something. Assessment does not include a value judgment: it notes that your speed is 80 km/h, but it is not concerned with whether that is good or bad. Assessment consists of finding ways to measure things. It need not always yield a number. A story that richly describes what happens in a classroom, without making judgments about what is described, is a form of assessment.

Evaluation includes the root word *value*, so I define *evaluation* as making a value judgment based on the evidence collected in an assessment. (When someone makes a value judgment independent of the evidence, we call that prejudice!) Value judgments depend on context: on the highway on a sunny day, 80 km/h might be too slow, but on a snowy city street in front of a school, 80 km/h is much too fast.

It is necessary to make value judgments in education. Essentially, we do what we do because we value it; if we do not value it, we stop doing it. For example, if our assessment of an innovative teaching strategy shows that its use has increased students' grades across the board and has particularly helped students who were failing the course, we are likely to use the results of that *assessment* to make a positive *evaluation* of the strategy. Of course, an evaluation can be more sophisticated than a single measure. We might also notice that students and teachers are more tired and that absenteeism increases when the new strategy is used. That makes the value judgment more difficult and brings us back to the question, What matters (most)?

Of course, in practice it is often difficult to make an assessment without also making an evaluation. Moreover, those who commission an inquiry into a program often *want* an evaluation; they want to know whether the program should be continued, expanded or scrapped. For that reason I will here tend to talk about *evaluation* rather than *assessment*.

I define *research* as seeking to understand something in a new way. That might include discovering genuinely new ideas or theories, but it might also include activities such as re-evaluating old theories or looking at old practices through new theories. Research usually includes assessment, but it usually should not be evaluative in itself; that is, research should aim to present a strong assessment of *what is* rather than focus on *what should be* and whether *what is* measures up. This rule (which might really be merely a preference of mine!) is not always followed, but it is usually better if researchers can assess a situation richly and leave evaluation to the reader.

I have done many kinds of research in many contexts, and I have even written a sort of textbook on qualitative research (Geelan 2003). I have also done a number of program evaluations. The rest of this discussion mingles the two fields, because many of the tools, methods and approaches used in research are also used in evaluation.

Purposes of Assessment and Evaluation

Assessment in education might be done for any of a number of reasons, including the following:

- *Measuring achievement.* Every teacher conducts simple assessments when grading tests and assignments and writing report cards. The data from these assessments can also be used in research and program evaluations.
- *Comparisons.* Though we might have ethical misgivings about the ways some comparisons (for example, league tables of school achievement) are used, some comparisons between students, between schools, between provinces and so on can help us improve learning.
- *Evidence in research.* Some kinds of assessment are done purely for the purposes of research or program evaluation.
- *Diagnosis, and support for teaching and learning.* Teachers also conduct many formal and informal assessments of student understanding, both formatively and summatively, to improve teaching and learning.

Similarly, evaluation might be done for any of a number of reasons, including the following:

- *Decision making about programs, strategies and technologies.* Should we put energy, money and other resources into a new program, or redirect them elsewhere? Do we need the latest and greatest technology, or will what we already have serve our students' real learning needs? These questions are better answered using carefully designed evaluations rather than ideology, bias or gut instincts.
- *Ranking.* Given that not everyone who wants to go to university can have a place, how do we decide who gets to go? And given that not all schools are the same, what do we do about funding? (Hint: Taking it away from schools that are already struggling is the wrong answer!)
- *Decisions about how to apply scarce resources.* Given that we do not have unlimited resources in education, where can existing resources best be applied? Where will they be the most effective and efficient in supporting better teaching and learning for all?

Research is generally done because the researchers imagine that it will contribute new understandings, but—let's face it!—it is also done because academics must publish or perish.

You can probably think of more purposes to add to these lists.

Types of Research and Evaluation

There is a huge range of types of research and evaluation. We often think first of what Shulman (1986) has described as "process–product research"; that is, we try something new in the classroom and find out what happens. We might do this with all the trappings of experimental models like those in the sciences: a single independent variable (the thing we change) and dependent variable (the thing we measure), controlling as many of the other variables as possible, including using a randomly chosen experimental group and control group. Or, realizing that humans are not as simple as atoms and that they do not quite fit into a true experimental framework, we might use a quasi-experimental design in which students' earlier behaviour acts as the control for their later behaviour. Process–product research and evaluation is often (though not always) quantitative; that is, what is measured is expressed in numbers and subjected to various kinds of statistical analysis.

At the opposite extreme in many ways but important in education is action research, in which the aim is to change what is happening in a particular context by understanding it better rather than to merely capture a snapshot of the situation. The evidence used in action research can include numbers, but it may also include teachers' observations of their students, reflective journal entries, interviews, qualitative and open-ended surveys, and a variety of other information. In between these two extremes—experimental, process–product research and action research—remains a range of research commitments, approaches and methods.

A program evaluation can use a wide variety of evidence, from the quantitative to the qualitative (defined below), in seeking to make judgments about the program's value.

Paradigms and Research Methods

A paradigm is a set of related beliefs, theories and assumptions (Kuhn 1970). The two main paradigms in educational research are often called *quantitative* and *qualitative,* although maybe *positivist* and *post-positivist* would be better labels. The basic characteristics of each paradigm are listed below.

The Quantitative Paradigm

- Is modelled on the methods used in the physical sciences
- Measures quantities of things; yields numbers as data
- Uses simple models of cause and effect
- Uses the scientific method—dependent, independent and controlled variables
- Uses validity and reliability as the standards for judging quality

Quantitative methods are usually fairly linear: formulate the question(s), create an instrument (survey or test), gather the data, analyze the data and write a report.

The Qualitative Paradigm

- Is modelled on methods from the social sciences and humanities
- Measures qualities of things; yields sophisticated descriptions
- Recognizes the complexity of educational contexts
- Uses trustworthiness and authenticity as standards

Qualitative methods tend to be iterative: formulate the question(s), gather some data, analyze the data, revise the questions, gather more data, analyze again, gather more data, analyze, report, revisit . . . .

The basic assumptions of quantitative research and qualitative research are different. Nevertheless, combining aspects of both paradigms can often be much more powerful than limiting yourself to one paradigm and set of methods.

Context

To be useful, research and evaluation reports in education must explain in great detail the context in which the research was conducted. That allows readers to make sense of the findings and to think about how their own contexts are similar and different in order to determine how useful the results might be to them. Relevant variables include the students' age, grade and socio-economic status; whether the context is urban or rural; the subject areas taught; the characteristics of the teacher(s); whether the students have any special needs; the history surrounding the program or innovation; and a host of other factors. Of course, at some point you have to stop reporting the factors (unless you want the report to be the size of a phone book), so you must think carefully about which contextual factors are the most important in allowing readers to understand and apply the findings. As a reader of research and evaluation reports, be aware of the presence or absence of these contextual cues.

Research Question

Your research question should be rooted in your values—in your own notion of what matters. That does not mean that you should go into research looking for ammunition to shoot down those who disagree with you; rather, it means that the research should be meaningful and important to you. That is what helps keep you interested and committed when the work gets hard (and occasionally boring).

How the research question is phrased will depend on the kind of project you are doing—research, action research or evaluation. Trying to answer too many questions can lead to an unfocused research process, so try to have only one research question or a small set of related questions. Phrase the question early in the process, but realize that it will likely evolve as the project goes on, particularly if you are working in a qualitative, iterative mode.

A key consideration is what evidence you will need to gather to answer the question. I have been known to say, "If you can't get the data you love, love the data you can get"—but that is not really good advice. If we do research that draws on only the information that is easy to get (the low-hanging fruit), whole aspects of education will be ignored or misrepresented. Find the question you *really* want to answer, and then think seriously about what kinds of evidence you will need in order to credibly answer the question.

The final important issue to think about is the scope of the question. Is it big enough to be worth pursuing but small enough to be manageable? Taking on a question that is too big might mean that you never finish, but taking on a trivial question fails the test of catalytic authenticity (discussed later).

Standards

Twenty-five years ago, the education community largely agreed on the appropriate standards that were to be applied to educational research and evaluation—quantitative (positivist) standards of validity, reliability, objectivity and generalizability, defined in terms of how the research accurately represented reality. That meant that, in writing up evaluation reports, no one had to be explicit about what standards they were using; it was assumed that the quantitative standards applied.

That is no longer the case. Below, I outline the quantitative standards in a little more detail and then outline an important set of qualitative standards for contrast.

Quantitative (Positivist) Standards

- *Validity.* Does the study measure what it claims to measure?
  - *Construct validity.* Is this variable related to other variables in the way the theory requires?
  - *Face validity (or content validity).* Is the measure actually measuring variables in the right domain?
  - *Criterion validity.* Does this measure of the variable correlate with a known correct measure of the variable?
- *Reliability.* If we measure the same thing again, will we get the same results?
- *Objectivity.* Are the results independent of the biases of the researcher(s)? Would everyone get the same results? Are the results researcher-independent?
- *Generalizability.* Can these results be applied everywhere? Can they be generalized to other contexts? Are the results context-independent?

In short, validity is concerned with whether a test measures what we think it measures, and reliability is concerned with whether the test measures it the same way repeatedly. It is possible for an instrument to be reliable without being valid. For example, a reliable rifle will have a tight cluster of bullets on the target, whereas a valid rifle will have the bullets clustered around the centre of the target. It is possible to imagine validity without reliability and vice versa. A good quantitative study or evaluation will aspire to both. To extend the rifle metaphor, objectivity measures the extent to which a different shooter would get the same results, and generalizability measures whether changing to a different rifle range would change the results.

Qualitative (Postpositivist) Standards

There are many sets of quality standards for qualitative research, but Guba and Lincoln's (1989) parallel criteria, or trustworthiness criteria, have been very influential. These criteria attend to the same issues addressed by validity and reliability in the quantitative paradigm, using the assumptions of the qualitative paradigm.

Trustworthiness Criteria

- *Credibility (parallels validity).* To what extent can the research credibly claim to be measuring what it has set out to measure?
- *Transferability (parallels generalizability).* To what extent are the research results useful in contexts other than those in which they were obtained?
- *Dependability (parallels reliability).* To what extent will the results be similar if the research is done again?
- *Confirmability (parallels objectivity).* To what extent will the results be similar if the study is done by another researcher or team?

The concerns of qualitative inquiry, however, are broader than these technical standards for the quality of the research. Qualitative research also applies moral standards in relation to protecting the research participants (the term *participants* is preferred over *subjects*) and to the vexed question of how we as researchers can be so audacious as to claim that our work is a fair representation of the views and needs of others. Guba and Lincoln (1989) add the following authenticity criteria to remind researchers to pay attention to these ethical and political dimensions of their work.

Authenticity Criteria

- *Fairness.* Are the representations of others fair? Will the participants recognize themselves in the accounts of them?
- *Educative authenticity.* Does everyone involved (the researchers and the participants) learn something?
- *Ontological authenticity.* Does the research enhance understanding of its social context (the constructed realities in which the research is occurring)?
- *Catalytic authenticity.* Does the research make something happen? Does something change because the research was done?
- *Tactical authenticity.* Are the methods used in the research consistent with the values implicit in the work and in the educational context of the work?

A variety of standards corresponding to the two paradigms are available now. Thus, it is crucial to carefully choose the standards you will apply, and to clearly state in the evaluation report which standards you chose and perhaps why.

Writing and Publishing

Academics are driven by publication, but classroom and district educators are far less so (unless they are doing graduate studies). So why write a report of your results? Where is the payoff?

The payoff may not be great in financial or career terms (although a publication listed on a CV is increasingly coming to mean something for teachers). However, you will be reporting your results as a service to the profession. Writing a high-quality report that outlines the context of the study in detail and in clear, accessible language allows you to share your ideas, experiences and results with colleagues in similar and different contexts. We all value learning, and we are obliged to share what we learn with others who are trying to enact similar values in their teaching.

For teachers, publishing in peer-reviewed academic journals is much less important than it is for academics. It is more valuable for teachers to publish their work in publications that make it accessible to other teachers—such as *delta-K.* Web publishing is another way to share your knowledge and experiences as widely as possible.

Conclusion

An evaluation is about assessing what matters. Do not do an evaluation with only the evidence that is easy to find; rather, use the evidence that allows you to do a high-quality, well-supported evaluation of a program or innovation that is important to you, using clearly defined standards. Then, share your findings as broadly as you can.

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Secondary Mathematics Education Curriculum Developments: Reflecting on Canadian Trends

Florence Glanfield

I recently attended a professional meeting where a conversation focused on the potential changes in the Western and Northern Canadian Protocol (WNCP) secondary mathematics curriculum. The suggestion was made that Alberta drop its applied mathematics courses in favour of returning to the Mathematics 10–20–30, Mathematics 13–23–33 and Mathematics 14–24 programs of the early 1990s. I was saddened to hear this conversation.

I am a mathematics teacher educator in Saskatchewan, a province that does not have an applied or consumer mathematics program. There is one mathematics program for all secondary students. Most of the teacher candidates in the teacher-education program at the University of Saskatchewan are graduates of the provincial system. I am saddened when I hear about their experiences in the secondary mathematics program—experiences that have led many of them to believe that they cannot "do" mathematics. Yet, the teacher candidates who tell these stories are the elementary mathematics teachers of the future, the teachers who will be expected to excite children and help them develop a passion for mathematics.

A 1981 report by the Council of Ministers of Education, Canada (CMEC) provided the impetus for provinces and territories to consider how they might humanize mathematics (Wheeler 1982) through their curricula. O'Shea (2003) describes the recommendations of the CMEC report as follows:

> The use of applications as a motivation for learning mathematics and as an experience in using mathematical models received increased emphasis. Respondents asserted that this reflected an effort to link the mathematics of the classroom to the real world in the interest of helping students see the practical uses of mathematics, understand the relationship of mathematics to other disciplines, and understand more clearly the mathematical ideas themselves.

O'Shea continues,

> In general, the trends at this level reflected a reduction of emphasis on rigor and formal structure. The reason usually given for this was that students might not be ready for formalization until they had developed concepts through informal approaches, including finding patterns, working with physical models of mathematical concepts, and using other manipulable aids. The result was an evolving mathematics curriculum that included emphasis on topics such as numerical skills, applications, and problem solving, and an accompanying de-emphasis on abstract topics such as set theory and algebraic manipulations.

Across Canada we see mathematics curricula that include a focus on problem solving, applications and the development of mathematical concepts from a concrete approach; the design of programs intended for students who are postsecondary-bound but are not going into science and mathematics; and the inclusion of computer technology (O'Shea 2003). Below are examples of how the provinces and territories have responded to the CMEC report and taken a more humanistic approach to mathematics.

Applied Mathematics

To address the needs of postsecondary-bound (but not necessarily science- or mathematics-bound) students, the Western Canadian Protocol (WCP) (now the WNCP) developed a series of three courses called applied mathematics. The curriculum document states that the "applied clusters . . . emphasize applications of mathematics rather than precise mathematical theory. The approaches used are primarily numerical and geometrical" (WCP 1996, 19). Throughout the applied mathematics courses, students are engaged in projects and activities that explore the mathematics in a given context. The textual resources that support the courses are filled with projects and laboratory activities. Students are encouraged to work collaboratively throughout the courses, and the mathematical ideas are expected to be developed from the numerical and geometrical approaches.

An example of an applied mathematics expectation is "Use properties of circles and polygons to solve design and layout problems" (WCP 1996, 107). Students might work on a problem such as the following:

> The pattern on a piece of vinyl flooring consists of a square and four equilateral triangles. Each equilateral triangle has as its base one side of the square. Circles are inscribed in each triangle and in the square.
>
> a) Start with a square of side length 6 cm. Draw the design, full size.
>
> b) Determine the ratio of the area of the small circle to the area of the large circle. (p 107)

In the Atlantic Canada collaboration, the applied mathematics courses start in Grade 10, similar to the WCP collaboration. In Ontario, an applied mathematics course has been developed for Grade 9.

Consumer Mathematics

A second area in curriculum development for students who may or may not be university- or college-bound is the development of consumer or basic mathematics courses for Grades 10–12. Currently, three provinces (British Columbia, Manitoba and Ontario) and the three territories have adopted such courses. An example of a Grade 12 consumer mathematics curriculum outcome is "Develop, use, and justify mathematical strategies by analyzing puzzles and games" (Manitoba Education 2004). (As a side note, Manitoba Education has negotiated with its postsecondary institutions to accept consumer mathematics as a general admission requirement into postsecondary programs that are not mathematics intensive.) In Ontario, the courses are called Mathematics for Everyday Life and are referred to as workplace-preparation courses (Craven 2003).

Role of Computer Technology

The use of graphing calculators has been evident in the curricula of most provinces and territories since the early 1990s. More recently, however, computer technology has played a more prominent role in secondary mathematics. Across Canada, secondary mathematics students are expected to learn to use spreadsheet programs and geometry software. You will see curriculum expectations such as "Students will solve problems using spreadsheet functions and templates" and "Students will use geometry software to develop the geometric properties of circles."

Reflection

The mathematics education community and curriculum developers must not give up on the idea of investigating mathematics from a variety of perspectives. I believe that when we offer secondary students a variety of ways to see mathematics, they will begin to see themselves as mathematical beings and recognize that they, too, are part of the mathematical community. The development of programs such as applied mathematics and consumer mathematics and the use of computer technology in mathematics classes invite students to be engaged in mathematics from multiple perspectives.

Kissane (2002, 191) notes Wheeler's (1982) prophetic suggestion of more than 20 years ago that "mathematics teachers were in the midst of three major educational upheavals: mass secondary education, the rise of new and available technologies, and the revolution of humanizing mathematics." In Canada, we could say that in the past 20 years secondary mathematics curriculum development has also been in the midst of these three upheavals. I believe that secondary mathematics curriculum development has been addressing "the rise of new and available technologies" and "humanizing mathematics" in order to address "mass secondary education." Generally, we now see secondary mathematics curricula that include the use of technology (not just computer and calculator technology) to study mathematics and the introduction of courses, such as applied and consumer mathematics, that humanize mathematics.

We must continue to look for ways, through secondary mathematics curriculum development, to humanize mathematics. We must work hard as a profession and engage in problem solving to overcome issues around implementation and postsecondary acceptance of mathematics courses other than those labelled as pure mathematics in order to begin to make a difference in how our students view themselves within the mathematics community.

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The Abacus: The Oldest Calculator

Sandra M Pulver

The abacus—the world's oldest calculating machine—can be used for performing all basic arithmetic operations. It has taken many forms: the dust abacus, the line abacus, the grooved abacus and the bead abacus, which is still used today.

The first known abacus was the dust abacus, which was probably invented by the Babylonians. Lines were drawn as place holders with a finger or a sharp instrument, either on the ground or on a table covered with sand or dust. Pebbles were put on the lines as counters to indicate the numbers. The dust abacus was used until the end of the ninth century AD.

Before the fourth century AD, the Romans developed a grooved abacus from the line abacus. The Romans based their numbers on the decimal system, and their abacus had 19 grooves and 45 stones that slid in the grooves to count numbers. It was the first abacus in which the counters were fixed in their columns.

From the grooved abacus, the Chinese developed a bead abacus, which is still used in many forms in many Oriental countries. The bead abacus consists of a series of rods holding five or six beads each. It can be used for counting both very large and very small numbers, and it is a fast and efficient tool for daily arithmetic calculations. Of all the ancient calculating devices that have come down to us, the bead abacus is the only one on which all four arithmetical operations can be performed rapidly and relatively simply. For those trained in its use, it is an efficient means of adding, subtracting, multiplying and dividing—and even for extracting square and cube roots.

In China, the bead abacus (the *suan p'an*) is still used by peddlers, merchants, accountants, bankers and hotelkeepers—and even by astronomers and mathematicians. It is so deeply rooted in the Far Eastern tradition that the Westernized Chinese and Vietnamese in Bangkok, Singapore, Taiwan, Europe and North America often continue to do calculations on an abacus, even though they have easy access to modern calculators. Even more striking is the fact that in Japan, one of the world's foremost producers of pocket calculators, the abacus (the *soroban*) is still the main calculating instrument in everyday life, and every child learns how to use it in school. Even in the former Soviet Union, the abacus (the *s'choty*) appears alongside modern calculators and is often used for calculating what customers must pay in shops, supermarkets, hotels and department stores.

Today, the best-known form of the bead abacus is the Chinese suan p'an. The suan p'an has 10–12 rods, with two beads on the upper part of each rod and five beads on the lower part. Each of the five beads on the lower part represents 1 unit, and each of the two beads on the upper part represents 5 units. Numbers are set by moving beads toward the crossbar that separates the upper and lower parts of the rods. To set the number 3, for example, raise three beads on the lower part of the first rod on the right. For the number 9, lower one upper bead (worth 5) and raise four lower beads (worth 1 each).

The value represented by a bead depends on which rod the bead lies. A lower bead on the ones rod represents a value of 1; on the tens rod, it represents a value of 10; on the hundreds rod, it represents a value of 100; and so on. Similarly, an upper bead on the ones rod has a value of 5, an upper bead on the tens rod has a value of 50 and so on. Thus, relatively few beads are needed for representing large numbers.

The Japanese imported the abacus from China in the 16th century and used it in that form until the end of the 19th century, when they modified it by having one upper bead instead of two. In 1938, they further modified it, this time by having four lower beads instead of five. This is what became known as the soroban. The soroban is usually one foot long and two inches wide, and it usually has 21 rods (*keta* in Japanese).

On the soroban, as on the Chinese abacus, each bead on the lower part of a rod represents 1 unit, and each bead on the upper part represents 5 units. Again, each bead is pushed toward the horizontal crossbar in a calculation. The value represented depends on which rod the bead is on. Three lower beads on the ones rod represent the number 3, while three lower beads on the hundreds rod represent the number 300.

The horizontal bar on the soroban has a unit point marked on every third rod (the first rod, the fourth rod, the seventh rod and so on). These unit points are used either to represent a decimal point or to indicate where the ones place is. On the soroban, the first rod represents the thousandths place, the second rod represents the hundredths place and the third rod represents the tenths place. Whole numbers begin to be represented at the fourth rod.

Adding on the Soroban

In performing addition, push beads toward the crossbar.

For 2 + 6, first set 2 on the ones rod, using two lower beads. To add 6, push one 5-unit (upper) bead and a 1-unit (lower) bead to the crossbar. A total value of 8 appears.

When there are not enough lower beads, use complements with respect to 5. The complements of 5 are 1 and 4, and 2 and 3. To add 1, 2, 3 or 4 to 4, add 5 and then subtract its complement, as follows:

$$4 + 1 \text{ becomes } 4 + (5 - 4),$$
$$4 + 2 \text{ becomes } 4 + (5 - 3),$$
$$4 + 3 \text{ becomes } 4 + (5 - 2).$$

The complements of 10 are 1 and 9, 2 and 8, 3 and 7, 4 and 6, and 5 and 5. To add the numbers 1 through 9 to 9, add 10 and then subtract its complement, as follows:

$$9 + 1 \text{ becomes } 9 + (10 - 9),$$
$$9 + 2 \text{ becomes } 9 + (10 - 8),$$
$$\vdots$$
$$9 + 9 \text{ becomes } 9 + (10 - 1).$$

In adding multidigit numbers on the soroban, work from left to right, not right to left as you would when adding with pencil and paper. For 22 + 12, first set 22. Then add 10 (using a lower bead on the tens rod) and 2 (using two lower beads on the ones rod). The result forms mechanically. In the last step, you do not need to use the complement of 5 to add 2, because there are enough beads on the ones rod.

For 374 + 918, which involves carrying, first set 374. There are not enough beads remaining on the hundreds rod to represent 900. Therefore, to add 900, use its complement (1,000 – 100). Add 10 (using a lower bead from the tens rod) and then add 8 using its complement (10 – 2). You will then have the correct result.

Subtracting on the Soroban

In subtraction, beads are pushed away from the crossbar.

For 3 – 2, first set 3. To subtract 2, push down two 1-unit beads. One 1-unit bead remains against the crossbar, so you have a result of 1.

The soroban has only four 1-unit beads, so to subtract 1, 2, 3 or 4 from 5, you must subtract and add its complement. Think of the numbers –1 through –4 as follows:

$$-1 = -5 + 4,$$
$$-2 = -5 + 3,$$
$$-3 = -5 + 2,$$
$$-4 = -5 + 1.$$

To perform 5 – 4, first set 5 by pushing the 5-unit bead on the ones rod toward the crossbar. Then add –5 + 1 by pushing the 5-unit bead away from the crossbar and pushing a 1-unit bead toward the crossbar. Only the 1-unit bead remains at the crossbar, so you have a result of 1.

To subtract the numbers 1 through 9 from 10, subtract 10 and add its complement. Think of the numbers –1 through –9 as follows:

$$-1 = -10 + 9,$$
$$-2 = -10 + 8,$$
$$-3 = -10 + 7,$$
$$\vdots$$
$$-9 = -10 + 1.$$

For 781 – 377, first set 781. Then use the complement of –300 (–500 + 200) to subtract the 300. Then subtract the 70. To subtract 7, use the complement of –7 (–10 + 3).

Conclusion

You need only know the addition and multiplication tables for the numbers 1 through 9 in order to do arithmetic operations on the bead abacus.

This ingenious calculating device has its drawbacks, however. It requires intensive training and a precise touch (though this becomes instinctive with practice). Also, the slightest error makes it necessary to do the whole calculation over, and intermediate results (such as partial products in multiplication and remainders in division) disappear in the course of operations.

Nevertheless, students who learn how to manipulate the abacus will get better at mental calculations and will obtain a deeper appreciation of mathematics while learning about other cultures.

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Two Great Escapes

Jerry Lo

The Great Amoeba Escape

The world of the amoeba consists of the first quadrant of the plane divided into unit squares. Initially, a solitary amoeba is imprisoned in the square in the bottom left corner. The prison consists of six shaded squares, as shown in Figure 1. It is unguarded, and the Great Escape will have succeeded when the entire prison is unoccupied.

Figure 1

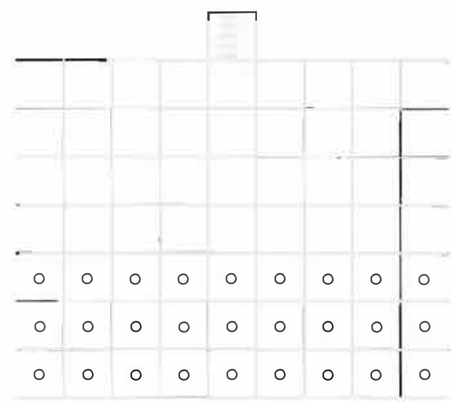
In each move, an amoeba splits into two, with one going to the square directly north and one going to the square directly east. However, the move is not permitted if either of those two squares is already occupied.

Can the Great Escape be achieved?

The Great Beetle Escape

The world of the beetle consists of the entire plane divided into unit squares. Initially, all the squares south of an inner wall constitute the prison, and every square is occupied by a beetle. Freedom lies beyond an outer wall four rows north of the inner wall. If any beetle reaches any square outside the unguarded prison, such as the shaded one in Figure 2, it will trigger the release of all the surviving beetles. Then the Great Escape will have succeeded.

Figure 2



In each move, a beetle can jump over another beetle in an adjacent square and land on the square immediately beyond. However, the move is not permitted if that square is already occupied. The beetle that is jumped over is removed, making a sacrifice for the common good. The jump may be northward, eastward or westward.

Can the Great Escape be achieved?

Remark 1. *The reader may wish to attempt to solve these two problems before reading on. At the least, the reader should delay reading beyond Strategies.*

Strategies

In both problems, the configuration keeps changing, with more and more amoebas in one case and fewer and fewer beetles in the other. The changes must be carefully monitored before things get out of hand. What we seek is a quantity that remains unchanged throughout. Such a quantity is called an invariant.

In the amoeba problem, the situation is simpler at the start, with only one amoeba. After one move, we have two amoebas. However, each is less than one full amoeba. Suppose we assign the value 1 to the initial amoeba, x to the one going north and y to the one going east. After the move, the initial amoeba is replaced by the other two. If we want the total value of amoebas to remain 1, we must have $x + y = 1$. By symmetry, then, $y = x$.

In the beetle problem, the situation is simpler at the end, with one beetle beyond the outer wall. Let's assign the value 1 to that beetle. It has reached its current position by jumping over another beetle. Let's assign x to the jumped-over beetle and y to the beetle before making the jump. After the move, the final beetle replaces the other two. For the total value of the beetles to be invariant, we must have $x + y = 1$, as in the amoeba problem.

A beetle with value z could jump over the beetle with value y to become the beetle with value x. If we choose $y = x$, as in the amoeba problem, then we must make $z = 0$ to maintain $z + y = x$. This is undesirable. A better choice is $y = x^2$. Then we can make $z = x^3$. Since $x^2 + x = 1$, we indeed have $z + y = x^3 + x^2 = x(x^2 + x) = x$.

The idea of an invariant is an important problem-solving technique. For further discussion and practice, see Fomin, Genkin and Itenberg (1996, 123–33, 254–57) and Tabov and Taylor (1996, 93–109).

Solution to the Amoeba Problem

We now put into practice the strategy discussed earlier. Clearly, the value of an amoeba is determined by its location. So we may assign values to the squares themselves, as shown in Figure 3.

Figure 3

1/16	1/32	1/64	1/128	1/256
1/8	1/16	1/32	1/64	1/128
1/4	1/8	1/16	1/32	1/64
1/2	1/4	1/8	1/16	1/32
1	1/2	1/4	1/8	1/16

The total value of the squares in the first row is

$$S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots.$$

Then,

$$2S = 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots.$$

Subtracting the first equation from the second, we have $S = 2$. Since each square in the second row is half the value of the corresponding square in the first row, the total value of the squares in the second row is 1. Similarly, the total values of the squares in the remaining rows are ½, ¼, ⅛, ... Hence, the total value of the squares in the entire quadrant is 4.

Note that the total value of the six prison squares is 2¾. Remember that the total value of the amoebas is the invariant 1. If the Great Escape is to be successful, the amoebas must fit into the non-prison squares with total value 1¼. Though there is no immediate contradiction, we do not have much room to play about.

Each of the first row and the first column holds exactly one amoeba at any time. If the amoeba on the first row is outside the prison, its value is at most ⅛. The remaining space with total value

$$\frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots = \frac{1}{8}$$

must be wasted. Similarly, we must leave vacant squares in the first column with a total value of at least ⅛. Since

$$1\frac{1}{4} - 2 \times \frac{1}{8} = 1,$$

we have no room to play at all.

For the Great Escape to be successful, all squares outside the prison and not in the first row or first column must be occupied. However, this requires that the number of moves be infinite. Hence, the Great Escape cannot be achieved in a finite number of moves.

Solution to the Beetle Problem

As in the amoeba problem, the value of a beetle is determined by its location. So we may assign values to the squares themselves, as shown in Figure 4.

Figure 4

			1			
x^4	x^3	x^2	x	x^2	x^3	x^4
x^5	x^4	x^3	x^2	x^3	x^4	x^5
x^6	x^5	x^4	x^3	x^4	x^5	x^6
x^7	x^6	x^5	x^4	x^5	x^6	x^7
x^8	x^7	x^6	x^5	x^6	x^7	x^8
x^9	x^8	x^7	x^6	x^7	x^8	x^9

The total value of the squares in the central column in the prison is

$$S = x^5 + x^6 + x^7 + x^8 + \cdots.$$

Then,

$$xS = x^6 + x^7 + x^8 + x^9 + \cdots.$$

Subtracting the second equation from the first, we have

$$S = \frac{x^5}{1-x}.$$

Since each square in the adjacent column on either side is x times the value of the corresponding square in the central column, the total value of the squares in either column is

$$\frac{x^6}{1-x}.$$

Similarly, the total values of the squares in the remaining columns on either side are

$$\frac{x^7}{1-x}, \frac{x^8}{1-x}, \frac{x^9}{1-x}, \ldots.$$

The total value of the squares in the prison east of the central column and including this column is

$$\frac{1}{1-x}(x^5 + x^6 + x^7 + x^8 + x^9 + \cdots) = \frac{x^5}{(1-x)^2}.$$

Similarly, the total value of the squares in the prison west of the central column but excluding this column is

$$\frac{x^6}{(1-x)^2}.$$

Hence, the total value of the squares in the entire prison is

$$\frac{x^5 + x^6}{(1-x)^2}.$$

Recall that $x^2 + x = 1$, so $1 - x = x^2$. Hence, the denominator of the total value is $(1 - x)^2 = (x^2)^2 = x^4$. The numerator of the total value is $x^6 + x^5 = x^4(x^2 + x) = x^4$ also, so the total value is exactly 1. Thus the Great Escape can succeed only by sacrificing all but one beetle, and it cannot be achieved in a finite number of moves.

Remark 2. *Everything up to this point has been adapted from material in existing literature. The Great Amoeba Escape is from Kontsevich (see Taylor 1993, 31, 37–39), and the Great Beetle Escape is from Conway (see Honsberger 1976, 23–28). What follows is largely my own contributions.*

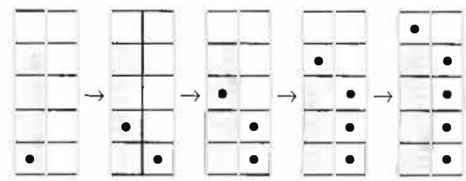
Further Amoeba Problems

We define a prison in the amoeba world as a set of squares consisting of the southernmost a_i squares in the i-th column for $1 \leq i \leq n$ such that $a_1 \geq a_2 \geq \cdots \geq a_n$. Such a prison is denoted by $(a_1, a_2, \ldots, a_n)$. We wish to determine all prisons from which the Great Escape is achievable. We consider the following cases.

Case 0. $a_2 = 0$.

The Great Escape from all such I-shaped prisons is easily achieved. Figure 5 illustrates the Great Escape from the prison (4) in $a_1 = 4$ moves.

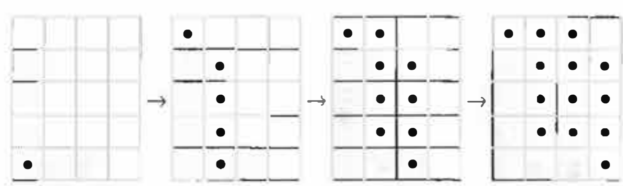
Figure 5



Case 1. $a_2 = 1$.

The Great Escape from all such L-shaped prisons is achievable in two stages. Figure 6 illustrates the Great Escape from the prison (4,1,1) in 12 moves. The first stage is the northward breakout in $a_1 = 4$ moves, exactly as in Case 0. The second stage is the eastward breakout in $n - 1 = 2$ phases, each involving $a_1 = 4$ moves.

Figure 6

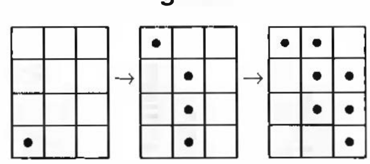


Case 2. $a_2 = 2$.

By symmetry, we may assume that $a_1 \geq n$. Since the Great Escape from the original (3,2,1) prison is not achievable, we may assume that $a_3 = 0$. The principal result is that the Great Escape from the prison (3,2) is not achievable. It then follows that it is not achievable from any P-shaped prisons $(a_1,2)$ where $a_1 \geq 3$.

Suppose the Great Escape from (3,2) is achievable. The order of the moves is irrelevant, as long as we allow temporary multiple occupancy of squares. Thus, there is essentially one escape plan, if any exists. So we may begin an attempt by making a three-move northward breakout followed by a three-move eastward breakout, as shown in Figure 7.

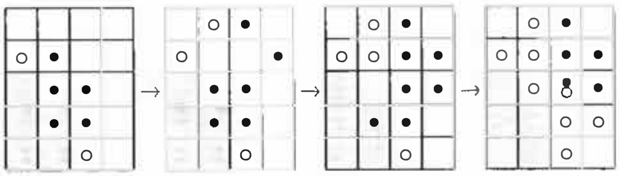
Figure 7



At this point, note that the amoeba on the first column and the one on the first row should not be moved any further, since they are outside the prison and not blocking the escape paths of any other amoebas.

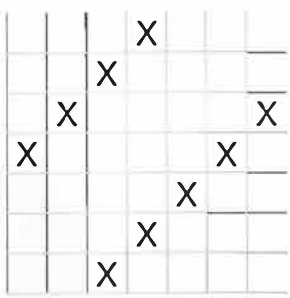
We mark them with white circles. We now move the other five amoebas one row at a time, as shown in Figure 8.

Figure 8



We have five more amoebas to move, and they form the same configuration as before except shifted one square diagonally in the northeast direction. It follows that in the Great Escape from (3,2) the amoebas do not venture outside the two diagonals of squares, as indicated in Figure 9.

Figure 9



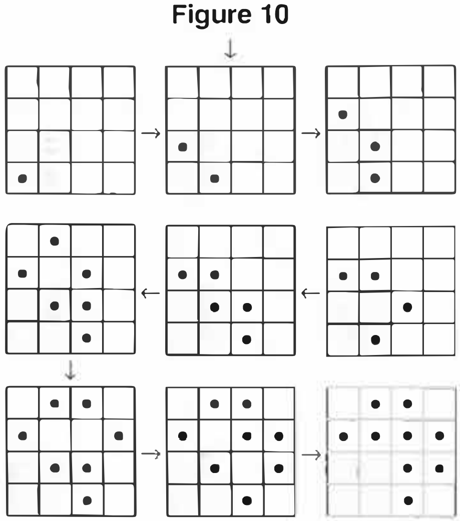
The total value of the squares between and including these two diagonals but outside the prison is

$$\frac{1}{4}+3\left(\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots\right)=\frac{1}{4}+\frac{3}{4}=1.$$

Hence, the Great Escape cannot be achieved in a finite number of moves.

Finally, the only prison for which $a_2 = 2$ and from which the Great Escape is achievable is (2,2), in eight moves, as shown in Figure 10.

Figure 10



CASE 3. $a_2 \geq 3$.

Such a prison contains the prison (3,2) as a subset. By Case 2, the Great Escape from (3,2) is not achievable. Hence, it is also not achievable for any prison with $a_2 \geq 3$.

Further Beetle Problems

We have already shown that the Great Escape from the original prison in the beetle world is not achievable. We modify the prison by reducing the distance d between the outer wall and the inner wall. It turns out that for $t \leq 3$ the Great Escape can be achieved in a finite number of moves. Thus, it involves a team of beetles, all but one of which will be sacrificed. What we want is to minimize the size of the team. We consider the following scenarios.

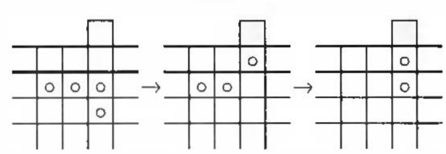
SCENARIO 0. $d = 0$.

Clearly, two beetles lined up directly in front of the target square can serve as the escape team. A team of size one is insufficient, because the maximum value of the lone beetle is x, and $x < x + x^2 = 1$.

SCENARIO 1. $d = 1$.

Four beetles positioned as shown in Figure 11 can serve as the escape team. After the first two moves, we can continue as in Scenario 0. A team of size three is insufficient, because the maximum total value of the beetles is $x^2 + 2x^3 < 2x^2 + x^3 = x + x^2 = 1$.

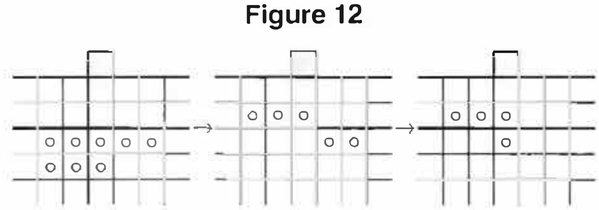
Figure 11



SCENARIO 2. $d = 2$.

Eight beetles positioned as shown in Figure 12 can serve as the escape team. After the first four moves, we can continue as in Scenario 1. A team of size seven is insufficient, because the maximum total value of the beetles is $x^3 + 3x^4 + 3x^5 < x^3 + 4x^4 + 2x^5 = 3x^3 + 2x^4 = 2x^2 + x^3 = 1$.

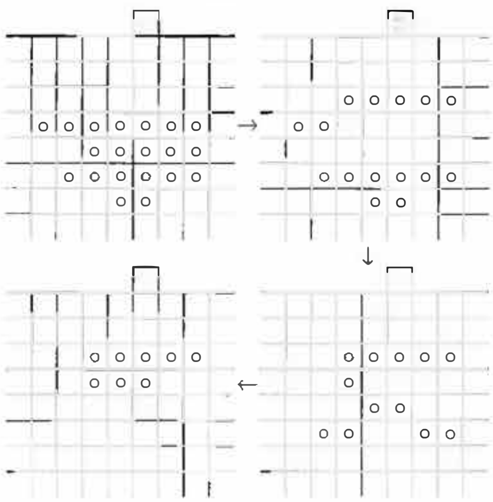
Figure 12



SCENARIO 3. $d = 3$.

Twenty beetles positioned as shown in Figure 13 can serve as the escape team. After the first 12 moves, we can continue as in Scenario 2.

Figure 13

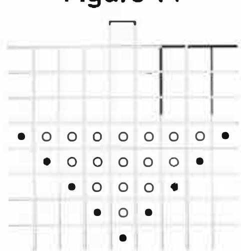


An escape team of size 19 may just be sufficient, because the maximum total value of the beetles is

$$\begin{aligned} &x^4 + 3x^5 + 5x^6 + 7x^7 + 3x^8 \\ &= x^4 + 3x^5 + 8x^6 + 4x^7 \\ &= x^4 + 7x^5 + 4x^6 \\ &= 5x^4 + 3x^5 \\ &= 3x^3 + 2x^4 \\ &= 1. \end{aligned}$$

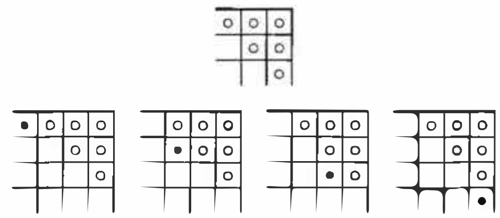
If this is the case, the escape team must consist of the 16 beetles in Figure 14, plus three more on the squares marked with black circles.

Figure 14



By symmetry, we may assume that at most one of the three additional beetles appears to the left of the central column. In each of the five cases shown in Figure 15, it is easy to verify that at least one beetle will remain to the left of the central column. This means that an escape team of size 19 is insufficient.

Figure 15



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2005 Kaohsiung Invitational World Youth Mathematics Intercity Competition

Robert Wong

Last year, Andy Liu, a professor at the University of Alberta, asked me to lead a team from Edmonton in competing in the 2005 World Youth Mathematics Intercity Competition (WYMIC) in Kaohsiung, Taiwan. Without much hesitation, I accepted the task.

WYMIC aims to provide an opportunity for young people from around the world to meet, to promote friendship and understanding between the world's young people, and to identify and nurture mathematical talent.

After meetings with parents and e-mail and phone conversations with organizers in Taipei and Kaohsiung, the Edmonton team was set for two weeks of competition and sightseeing in Taiwan. This was the first year Canada sent a team to WYMIC.

The Team

Each WYMIC team is made up of a leader (a math teacher, coach, professor, representative from the education ministry or parent); a deputy leader, who works closely with the leader and the local organizer (I was fortunate to have Alan Tsay, who speaks Mandarin fluently); and four students in Grade 9 or under. The members of the Edmonton team were deputy leader Alan Tsay (Grade 12 student, Harry Ainlay High School), Chenxi Qiu (student, Vernon Barford), Ray Yang (student, Vernon Barford), Sean Jia (student, Grandview Heights School) and Sven Zhang (student, Avalon Junior High School).

Coaching

The four contestants went through extracurricular training from Andy Liu and tried sample problems from the 1999 and 2000 competitions (available at www.nknu.edu.tw/~math/kiwymic/index.htm).

Three of the four students were also invited to the week-long Calgary Math Summer Camp as a result of good standing in the Edmonton Junior High Mathematics Invitational, Part II.

Cost

The University of Alberta sponsored each of the six team members for $500, but we bore the balance of the cost of flights and prearranged local tours. We each had to contribute about $1,500 Canadian, not including local expenses such as food, transportation and accommodations. WYMIC paid for all expenses for the duration of the event (August 2–5). We kept our accommodation costs to a minimum by staying at Alan's home in Taipei before the competition and with other host families after the competition. We were also fortunate that Mr Wen-Hsien Sun, executive director of the Chiu Chang Mathematics Education Foundation in Taipei, paid for part of our postcompetition tour. Without all this generous support, our costs would have been much higher.

Participants

The 2005 WYMIC was the largest ever, involving 34 teams from 11 regions (Macau, Philippines, Fuzhou, Hong Kong, South Africa, Thailand, Indonesia, Singapore, Canada, India and Taiwan). Including contestants, leaders, parents and companions, participants numbered around 250.

WYMIC Involvement

In addition to the WYMIC core committee made up of math educators (including professors), organizations involved included the Taiwan Ministry of Education's National Science Council, the Education Bureau of Kaohsiung City Government, the National Kaohsiung Normal University, the Mayoral Office of Kaohsiung, the Chiu Chang Mathematics Education Foundation, Kaohsiung Municipal San Min Junior High, and Shu-Te Home Economics and Commercial High School. Many volunteers, student teachers and students helped run the event.

The cost to put on the 2005 WYMIC was approximately 2 million Taiwanese dollars (C$80,000). When WYMIC was seeking a host country for the seventh WYMIC in 2006, only three countries—India, Thailand and South Africa—expressed interest. No other representatives had the authority or financial funding to be contenders.

The Competition

WYMIC has two parts: an individual competition and a team competition.

The individual competition is two hours long. Section I contains 12 numerical-response questions worth five points each. Section II contains three full-solution questions worth 20 points each.

The team competition is one hour long and consists of 10 full-solution questions worth 40 points each. During the hour, the team members work together to solve as many questions as possible, with partial marks given for incomplete solutions.

Awards

Approximately half of the contestants receive individual awards in the form of gold, silver and bronze medals. The total score of the top three contestants in each team is considered the team score. The top three teams in each grouping receive a group award. Other group awards are given to teams for outstanding efforts in the categories of spirit of cooperation, originality, modelling and popularity.

The Edmonton team won the group award for modelling. The top three countries that won the academic awards were Taiwan, Hong Kong and Thailand.

Events

Day 1

Competitors checked in at the Military Hostel. A team leaders meeting was held. Everyone was provided with lunch and supper.

Day 2

The opening ceremony with dignitaries was held at the university and received local news coverage. Competitions ran from 10 AM to about 2:30 PM. While the contest was under way, leaders and companions were treated to a guided tour of the National Science and Technology Museum, the Kaohsiung Museum of Fine Arts and the old British Consulate. During dinner, students took turns performing on stage in a cross between a talent show and a cultural show. Members of the Edmonton team had brought along plenty of small gifts to exchange with others; giving and receiving souvenirs is customary at these events.

Day 3

A tour of the Taiwan Aboriginal Culture Park was cancelled due to an approaching typhoon. Instead, everyone went on a boat tour along the Kaohsiung Harbor followed by a visit to the National Science and Technology Museum. A formal banquet was held at the prestigious Grand Hotel in Kaohsiung. We had brought along some more-expensive gifts for the formal banquet, because we knew that we would be receiving the same from the dignitaries.

Day 4

We attended the closing ceremony, which was exceptional. We were all pleasantly surprised by the greeting, the orchestra, the performance and the prizes and couldn't help but feel important.

Tips and Warnings

Here is some advice for travelling to Taiwan and for participating in WYMIC.

Cash your Canadian dollars at the airport. Many banks in Taiwan either do not buy Canadian cash or accept only Thomas Cook traveller's cheques (you must bring along your passport to cash traveller's cheques). Instead of desperately trying to find a bank that will cash your Canadian dollars, pay the equivalent of a dollar transaction fee at the airport. It is well worth the money!

The streets of Kaohsiung contain hundreds of thousands of mopeds driven by men and women of all ages. Be careful when you walk down the street, because the mopeds can sneak up on you in an instant. The same goes for driving. Constantly be aware of your surroundings. Oh, yeah—don't try to hitch a ride on one of the mopeds. It could be a hair-raising experience (at least that's what my Hong Kong counterpart told me)!

In Taiwan, there is no tipping when eating out, but saying thank you is always appreciated.

Exchanging souvenirs with other participants is customary at events such as WYMIC, so bring along various gifts. We brought Canadian flags, as well as pins from Edmonton, Alberta and Canada. We also brought more-expensive gifts for the dignitaries, the members of the organizing committee (from the National Kaohsiung Normal University) and our host families. Be prepared to receive gifts that are difficult to bring home on the plane. For example, our gift from the vice-minister of education was a rather large porcelain dragon figurine in a glass encasement.

Make sure to have a team uniform of some sort. There will be plenty of photo opportunities, and wearing street clothes won't do on many occasions.

More Information

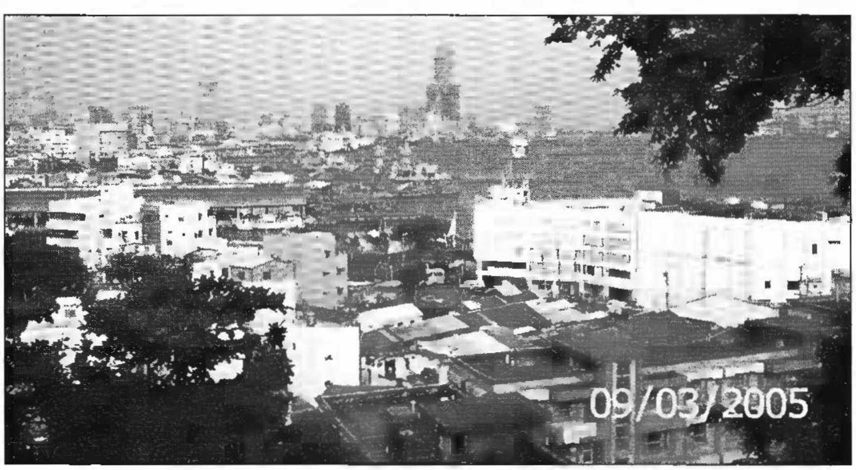
Visit www.nknu.edu.tw/~math/kiwymic/index.htm for results and photos from the 2005 WYMIC, as well as sample problems from past competitions. This year's WYMIC will be held this summer in the city of Fuzhou in the Fujian province of China.

Robert Wong is a teacher at Vernon Barford Junior High School in Edmonton.

The Edmonton team at the opening ceremony

Dignitaries and competitors at the opening ceremony

The Edmonton team with the Hong Kong team



Kaohsiung, Taiwan

National Science and Technology Museum

The Edmonton team presenting at the cultural evening

Left to right: Alan Tsay, Chenxi Qiu, Sven Zhang, Sean Jia, Ray Yang, Mr Wen-Hsien Sun, Robert Wong

Dinner at the Military Hostel

The closing ceremony

Root Multiples and Polynomial Coefficients

Bonnie H Litwiller and David R Duncan

Teachers are always looking for situations in which their algebra students can investigate numerical/symbolic patterns. Polynomial equations provide a setting for such processes.

Consider the following quadratic equation: $x^2 + 3x - 4 = 0$. Either by factoring or by using the quadratic formula, most algebra students would readily conclude that the roots are $x = 1$ and $x = -4$.

Could we write a quadratic equation whose roots are twice those of the original equation? Since this new equation would have roots of $x = 2$ and $x = -8$, the new equation would be

$$(x-2)(x+8)=0,$$
$$x^2+6x-16=0.$$

Comparing this new equation to the original equation ($x^2 + 3x - 4$), we note that the x-coefficient has been multiplied by 2 and the constant term has been multiplied by 4.

Does this pattern hold for any quadratic equation? Consider the equation $(x - a)(x - b) = 0$, which has roots of $x = a$ and $x = b$. In expanded form, this equation would be

$$x^2-(a+b)x+ab=0.$$

To find a quadratic equation with roots $x = 2a$ and $x = 2b$, we proceed as follows:

$$(x-2a)(x-2b)=0,$$
$$x^2-(2a+2b)x+4ab=0,$$
$$x^2-2(a+b)x+4ab=0.$$

The x-coefficient of this new equation is 2 times the x-coefficient of the original equation, and the original constant term has been multiplied by 4. The pattern does indeed hold generally.

This pattern generalizes to any higher-degree polynomial equation and to any multiple of the original roots. To see its extension to the cubic case, suppose that the roots of the original cubic equation are r_1, r_2 and r_3. The cubic equation is then $(x - r_1)(x - r_2)(x - r_3) = 0$. (If the leading coefficient is not 1, divide both sides by that coefficient to simplify the problem.) When the left side of the equation is expanded using the distributive property, we obtain the following equation:

$$x^3-(r_1+r_2+r_3)x^2+(r_1r_2+r_1r_3+r_2r_3)x-(r_2r_2r_3)=0.$$

Suppose that we want a cubic equation whose roots are k times those of the original cubic—namely kr_1, kr_2 and kr_3. Such a cubic can be written in factored form as $(x - kr_1)(x - kr_2)(x - kr_3) = 0$. Expanding the left side of the equation (with repeated use of the distributive property), we obtain

$$x^3-k(r_1+r_2+r_3)x^2+k^2(r_1r_2+r_1r_3+r_2r_3)\,x-k^3(r_1r_2r_3)=0.$$

Note the presence of k in increasing powers in the coefficients of the new equation.

Suppose that we want to find a quartic (fourth-degree) equation whose roots are 3 times the roots of an original quartic equation. We verify symbolically that the coefficients of the original quartic equation should be multiplied by 3^0, 3^1, 3^2, 3^3 and 3^4, respectively. Large numbers result!

Challenge for the Reader

Construct a general proof for the nth-degree equation. You will need to determine a way of writing coefficients in terms of the roots.

Bonnie H Litwiller and David R Duncan are professors of mathematics at the University of Northern Iowa in Cedar Falls. They teach a number of content and methods courses for prospective and inservice teachers. Both are past presidents of the Iowa Council of Teachers of Mathematics. Litwiller is also a past board member of both the National Council of Teachers of Mathematics and the School Science and Mathematics Association. They have published more than 900 articles for mathematics teachers at all levels.

Where Is the Directrix of a Circle?

David E Dobbs

The Case of the Missing Directrix

The conic sections (parabola, ellipse and hyperbola) get their name from the fact that each can be obtained as the intersection of a plane with a double-napped right circular cone. By changing the position of the plane relative to the cone, one finds that certain positions produce an intersection that is typically referred to as a degenerate or limiting case of a conic, such as a point, a line, two intersecting lines or a circle. In particular, a circle can thus be viewed as a limiting case of an ellipse (see, for instance, Dobbs and Peterson 1993, Figure 8.1, 441).

Other unifying approaches to introducing the conic sections have a similar feature. Consider, for instance, the approach that involves the shadow cast on a wall by the nonfluted lampshade of a reading lamp. If the shade is pointed almost directly at the wall, the shadow is an ellipse that is nearly circular. As the lamp is gradually tilted more and more, the elliptic shadow becomes less circular until it becomes a parabola. Increased tilting eventually produces a shadow that consists of both branches of a hyperbola. Reversing the tilting process, one finds that the hyperbola changes back into a parabola, then an ellipse and finally, when the shade is pointed directly at the wall, a circle. In this way, a circle can once again be physically obtained as a limiting case of an ellipse.

A third approach to the three basic conic sections is the one usually used in high schools today—namely, as graphs of quadratic polynomials in two variables (with real number coefficients). However, the graph of such a polynomial can also be a point, a line, two intersecting lines, two parallel lines, a circle or (most degenerate of all) an empty set. Elsewhere, I (Dobbs 1992, 803) examined the effect of subjecting a familiar equation of an ellipse, $x^2/a^2 + y^2/b^2 = 1$, with parameters $a > b > 0$, to three limiting processes of the kind studied in precalculus and calculus. The result was to produce equations whose graphs were two parallel lines, a line segment or a circle. For example, one can obtain the upper half of the above ellipse as the graph of the function $f(x) = b(1 - x^2/a^2)^{1/2}$ over the domain $-a \le x \le a$. Now, if a is fixed and we let b approach a from the left, we have the one-side limit $\lim_{b \to a^-} f(x) = b(1 - x^2/a^2)^{1/2} = (a^2 - x^2)^{1/2}$, a function whose graph (over the above domain) is the upper half of the circle whose equation is $x^2 + y^2 = a^2$. When the same limiting process is applied to an equation of the lower half of the ellipse, the limit is a function whose graph is the lower half of the circle. Thus, in a sense that could be very effective in a precalculus classroom, we have seen an algebraic way to view a circle as a limiting case of an ellipse.

The last approach mentioned above is part and parcel of studying conic sections through analytic geometry. In this approach, a conic section with focus F, directrix L and eccentricity e is the set of points P such that e is the ratio of the distance from P to F and the distance from P to L. The familiar conics are obtained as follows: parabolas have eccentricity $e = 1$, ellipses have eccentricity satisfying $0 < e < 1$ and hyperbolas satisfy $e > 1$. It is customary to say that circles are ellipses with eccentricity 0. This makes some sense if one views the circle $x^2 + y^2 = a^2$ as having been obtained through the limiting process considered above. Indeed, the foci of the above ellipse are the points $(c,0)$ and $(-c,0)$, where $c^2 + b^2 = a^2$ and $c = ae > 0$, and the limit process $\lim_{b \to a^-}$ sends c to 0. (More precisely, $\lim_{b \to a^-} c = \lim_{b \to a^-} (a^2 - b^2)^{1/2} = 0$.) The effect of the limiting process is to identify the foci with the centre of the limiting circle. Moreover, an ellipse whose eccentricity is a small positive number is only slightly oval and is often indistinguishable from a circle to the naked eye. The mathematical use of the term *eccentricity* comes from the fact that one can view an ellipse as having evolved from a circle whose centre has split into two foci, with the distance c from the centre of the ellipse to either focus measuring the amount that each focus has moved away from the centre. (The Latin origins of the terminology reveal this interpretation, with *ex* meaning "away from" or "out of" and *centrum* meaning "centre.")

The above point of view leads to the basic question we will study in this article. We have seen how algebra (together with functions and limits) allows us to view a circle as the limit of an ellipse, and we have also seen how that limiting process converts the foci of the ellipse to the centre of the circle. Our basic question is, What happens to the directrices of the ellipse under that limiting process? Since the circle can be viewed as a degenerate conic, it should have at least one focus and a corresponding directrix. If we view the centre of a circle as its focus, where is the corresponding directrix of the circle?

Naysayers may point out that the above ellipse has directrices $x = \pm a/e$ and that there would be no sense in considering these equations for a circle—which we have seen should have eccentricity $e = 0$—since it is said that you can't divide by 0. We will show that much mathematics has been based on the refusal to let the circle-as-conic analogy die at the hands of that tired bromide. Parts of this article could be useful in high school courses in algebra, geometry, precalculus/functions, and calculus, especially as enrichment material for the unit on one-sided or infinite limits. It is also hoped that geometry teachers will find this information effective in introducing students to the line at infinity and, more generally, to projective geometry and modern algebraic geometry.

Can Algebra Explain the Nature of $x = \infty$?

We are trying to avoid having to say that the directrices of the circle $x^2 + y^2 = a^2$ are given by $x = \pm a/0$. How can we do this? For inspiration, let's recall that the founders of calculus (especially Leibniz, with his *dy/dx* notation) managed to view the derivative of a function as a ratio whose denominator was infinitesimally small. Nowadays, we view the derivative as the limit of a certain ratio whose denominator is approaching 0. Let's take a similar approach in addressing our current difficulty. Rewrite the above equation of an ellipse as $x^2/(b + \varepsilon)^2 + y^2/b^2 = 1$, where $a = b + \varepsilon$ and $\varepsilon > 0$. The limit process $\lim_{b \to a}$ is equivalent to $\lim_{\varepsilon \to 0^+}$ (where b is fixed and a is varying). What is the effect of applying this limit process to the right-hand directrix $x = a/e$ of the above ellipse? It is

$$\begin{aligned} x &= \lim_{\varepsilon \to 0^+} \frac{b+\varepsilon}{e} \\ &= \lim_{\varepsilon \to 0^+} \frac{b+\varepsilon}{\dfrac{((b+\varepsilon)^2 - b^2)^{1/2}}{b+\varepsilon}} \\ &= \lim_{\varepsilon \to 0^+} \frac{b+\varepsilon}{\left(1-\left(\dfrac{b}{b+\varepsilon}\right)^2\right)^{1/2}}. \end{aligned}$$

In this limit problem, the numerator has limit b, which is a fixed positive number, and the denominator has limit 0, which is taken through positive values. Calculus teaches us how to understand such a limit. It is $x = \infty$.

We have managed to reinterpret $x = a/0$ as $x = \infty$. Are we any better off now? Yes, indeed. In this section, we will use an engineer's numerical and algebraic intuition to try to interpret the equation $x = \infty$ graphically. That effort will be instructive but ultimately unsuccessful. In the next section, we will succeed in interpreting the graph of $x = \infty$ by turning to the machinery of the real projective plane.

Let's focus here on the following question: Can $x = \infty$ be equivalent to (that is, have the same solution set as) some homogeneous linear equation $ax + by + cz = 0$ (where a, b and c are real numbers, not all of which are 0)? This leads to the question, Which ordered pairs (y,z) of real numbers can satisfy an equation $a\infty + by + cz = 0$? Let's assume that there are such points (whatever the directrix of a circle ends up being, it should not be empty!). Then $a\infty = -by - cz$ must be a real number. Our experience with products involving the ∞ symbol in calculus—along with common sense—tells us that a must equal 0. The only restriction on (y,z) is then that $by + cz = 0$. We proceed to understand the graph of this equation for different possible values of b and c.

One situation can be ruled out at once. Indeed, it cannot be the case that both b and c are 0, for the graph of $by + cz = 0$ would then consist of all ordered pairs (y,z) of real numbers. That is intuitively unacceptable because a directrix should be a line, not a plane. However, the three remaining cases each lead to a plausible interpretation. For instance, if $b = 0$ and $c \neq 0$, the graph of $by + cz = 0$ consists of the ordered pairs $(y,0)$, where y varies over the set of all real numbers. With one degree of freedom, this set could possibly be viewed as a line in some new geometry. By similar reasoning, the same type of conclusion holds for the case in which $c = 0$ and $b \neq 0$. In the final case, neither b nor c is 0. The graph of $by + cz = 0$ in this case is more complicated: it consists of all the points of the form $(y, -by/c)$, where, once again, y can be any real number. With one degree of freedom, this set could also plausibly be viewed as a kind of line.

Has the above intuitive analysis involving algebraic operations with the ∞ symbol been of any help? Not really! We have gone from having had no ready interpretation for the graph of $x = \infty$ to having three equally plausible interpretations. The feast is not preferable to the famine. We wanted one answer, not several. Since algebra (mixed with intuition from calculus) has not provided a satisfactory answer, we turn next to geometry in our quest to understand the graph of an equation such as $x = \infty$.

Projective Geometry Explains the Nature of $x = \infty$

In an intuitive sense, one might think that the graphs of equations such as $x = \infty$ and $y = \infty$ should be lines at infinity. In fact, there is an extension of the ordinary analytic geometry of the Euclidean plane where notions similar to these can be given rigorous mathematical meaning. That larger mathematical system is known as the real projective plane. Analytically, a point in the real projective plane is an ordered triple (x,y,z) of real numbers, not all of which are 0, with two such triples viewed as being the same if their corresponding components are proportional. (More precisely, this notion of *same* means that the points in the real projective plane are actually the equivalence classes arising from a certain equivalence relation on the set of certain ordered triples of real numbers.) Analytically, a line in the real projective plane is the graph of a homogeneous linear equation in the variables x, y and z. (The proportionality that defined sameness of points ensures that any two identified non-zero triples of real numbers satisfy the same homogeneous linear equations.)

The ordinary point (x,y) of the familiar analytic geometry of the real Euclidean plane can be regarded as the point $(x,y,1)$ of the real projective plane. The only other points of the real projective plane are of two types: the infinitely many points $(1,y,0)$, which are different for different values of y, and the point $(0,1,0)$. It is customary to say that these two types of points are points at infinity. Notice that the points at infinity are exactly the graph of the equation $z = 0$, which is then naturally called the line at infinity.

With the projective machinery now in hand, graphs of some familiar equations of lines in the real Euclidean plane become subsumed as subsets of projective lines in the real projective plane, as follows. The nonvertical, nonhorizontal line $y = mx + b$ (where $m \neq 0$) becomes part of the projective line given by $y = mx + bz$. Apart from the familiar points on the real Euclidean line $y = mx + b$, the only new point on this projective line is the point $(1,m,0)$. The x-axis, $y = 0$, becomes part of the projective line given by the same equation. Apart from the familiar points on the real Euclidean x-axis, the only new point on this projective line is the point $(1,0,0)$. Similarly, the y-axis, $x = 0$, is subsumed as part of the projective line $x = 0$, whose only new point is $(0,1,0)$.

Something similar happens when we try to embed the other horizontal or vertical lines of the real Euclidean plane into the projective environment. As c varies over the set of non-zero real numbers, the familiar horizontal line $y = c$ becomes part of the projective line $y = cz$, whose only new point is $(1,0,0)$. Notice that if c_1 and c_2 are unequal non-zero real numbers, then the projective lines $y = c_1z$ and $y = c_2z$ intersect at the point $(1,0,0)$, which is the same point at infinity that lies on the projective line $y = 0$. In fact, parallelism is not a useful concept in projective geometry, because you can check that *any* two distinct projective lines meet at exactly one point (which may be on the line at infinity).

The situation is similar when we extend the familiar vertical line $x = c$, with $c \neq 0$, to the projective line $x = cz$. Indeed, distinct projective lines $x = c_1z$ and $x = c_2z$ intersect at the point $(0,1,0)$, which is the same point at infinity that lies on the projective line $x = 0$.

Are we now ready to make any sense out of graphs of expressions such as $x = \infty$ and $y = \infty$? Yes! The process by which we embedded each Euclidean line as a subset of some projective line involved what algebraic geometers call homogenization: the variables x and y appearing in a Cartesian equation of a given Euclidean line L are replaced by x/z and y/z, respectively, so that cross-multiplying produces an equation of the projective line in which L is embedded. Since we arrived at the equation $x = \infty$ by using the analytic geometry of the Euclidean plane, it follows that an interpretation of $x = \infty$ in terms of the projective plane should be (after homogenization) as the graph of $x = z\infty$. What on earth is this?

Once again, our experience with calculus (or ordinary common sense) tells us that if z is a non-zero real number, then $z\infty = \pm\infty$, which is certainly not a real number. Thus, in the real projective plane, each point (x,y,z) on the graph of $x = z\infty$ must satisfy $z = 0$. In other words, the graphical interpretation of $x = \infty$—which we wanted to be a directrix and, hence, some sort of line—is that it is a subset of the line at infinity. But surely a line cannot be a proper subset of another line. The conclusion is inescapable: the graph of $x = \infty$ is the line at infinity.

By reasoning with homogenization as above, you can check that the graph of $y = \infty$ is also the line at infinity. Thus, to find a geometric answer to our basic question, we have come upon a geometry in which parallelism no longer matters and we can no longer tell horizontal from vertical. Moreover, now that we have argued that the line at infinity should be the directrix of the "circle" $x^2 + y^2 = z^2a^2$ (obtained from the Euclidean equation $x^2 + y^2 = a^2$ by homogenization), *focus* and *directrix* in projective geometry cannot continue to play their former roles. After all, *any* point P of the Euclidean plane is at a finite distance from the focus of this circle and at an infinite distance from the directrix. The ratio of these distances should surely be understood as 0 (since algebra, calculus and common sense agree that if a/∞ is to have a meaning for some real number a, that meaning must be 0). Since the circle has eccentricity 0, our earlier understanding of the terms *focus* and *directrix* would seem to imply that *each* point of the Euclidean plane lies on the circle $x^2 + y^2 = a^2$ that we started with. That conclusion is unacceptable, since the real projective plane is supposed to be a reasonable extension of the ordinary real Euclidean plane, where the only new phenomena involve the points (and line) at infinity. For this reason, we must abandon our earlier understanding of *focus* and *directrix* when working in projective geometry. In fact, the very definition of *conics* must be formulated anew in this geometry.

H S M Coxeter, probably the most distinguished geometer in Canada's history, wrote often on this subject, including an accessible introduction to projective geometry (1964). A synthetic (in other words, non-analytic) approach to the real projective plane can be found in his book *The Real Projective Plane* (1993). Coxeter writes that "in the projective plane, there is only one type of conic; the familiar distinction between the ellipse, parabola, and hyperbola can only be made by assigning a special role to the line at infinity" (p 72). Thus, one consequence of enlarging the Euclidean plane to the projective plane is that we lose part of what we had thought we knew about conics. A venerable maxim in education is that to increase our understanding of a subject, it is often necessary to take one step backward before taking two steps forward. This is exactly what has happened as we have allowed considerations of infinity to affect our view of what *point* and *line* could mean in an extension of the Euclidean plane.

A survey of textbooks reveals that the notion of *conic*, when broadened beyond the Euclidean context as indicated above, plays a central role in most treatments of projective geometry. It is heartening to note that Coxeter's *Projective Geometry* (1964, 102–03) contains a short section called "Is the Circle a Conic?" and that he provides an elegant proof that answers this question with a resounding yes. One might say that in the real projective plane, the analogies with which we began have come full circle (pun intended).

Closing Comments

A nugget of literary wisdom seems appropriate here. In "Tintern Abbey," poet William Wordsworth writes, "Other gifts have followed." He is referring to the familiar process of mellowing whereby one discovers compensations while aging, but I believe that his words also have relevance for us here. In pursuing the possible meaning of $x = \infty$, we seem to have lost a role that directrices played in some more familiar situations. However, in losing a role, we have gained the entire subject of projective conics. An accessible book by Kendig (2005), which comes packaged with a CD containing 36 applets, includes eight ways of looking at a conic. Kendig also views conics within the broader framework of algebraic curves in projective spaces with complex number coordinates. Thus, Kendig's text could be used as an introduction to modern algebraic geometry that builds on the discussion of the real projective plane in the preceding section of this article.

Modern algebraic geometry has grown to encompass much more than projective geometry. By refusing to be stopped by the bromide that you can't divide by 0, mathematicians have opened up an entire area of geometry, where new applications to science and other areas of mathematics are still being discovered.

Two other recommended books serve as introductions to the computer-facilitated methods of modern algebraic geometry: *Algebraic Geometry for Scientists and Engineers* (Abhyankar 1990) is written for scientists and engineers, and *Using Algebraic Geometry* (Cox, Little and O'Shea 2005) is written in the spirit of modern commutative algebra.

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Euclid's Algorithm: Revisiting an Ancient Process

Duncan McDougall

Believed to be one of the oldest algorithms, Euclid's algorithm (also called the Euclidean algorithm) was presented in Proposition 2, Book VII of Euclid's *Elements* as a method for finding the greatest common factor (GCF) of two integers. To conventionally determine the GCF of the integers a and b, let $a = k \cdot a_1 \cdot a_2 \cdot a_3 \cdot \ldots \cdot a_n$ be a prime factorization of a, and let $b = k \cdot b_1 \cdot b_2 \cdot b_3 \cdot \ldots \cdot b_n$ be a prime factorization of b. Then, if none of the factors $a_1, a_2, a_3, \ldots, a_n$ are equal to any of the factors $b_1, b_2, b_3, \ldots, b_n$, we know that GCF$(a,b) = k$. However, if we also have $a_1 = b_1$, then GCF$(a,b) = k \cdot a_1$, and so on. That is, the GCF of two numbers will contain all factors common to both numbers, as the name suggests. In this article, I present two ways to apply this algorithm in the secondary mathematics classroom.

GCF: The Silent Partner

Why can't we reduce 3/7? Why do we multiply 2×3 to determine the lowest common multiple (LCM) of 1/2 and 1/3, but the product of 2×6 does not give us the LCM of 1/2 and 1/6? Why can't we combine $\sqrt{3} + \sqrt{7}$? Why does rationalizing

$$\frac{1}{\sqrt{6}} = \frac{\sqrt{6}}{6}$$

immediately give us a radical expression in lowest terms, but

$$\frac{2}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} = \frac{2\sqrt{6}}{6}$$

is not completely reduced? Why is the LCM of $\sin x$ and $\cos x$ their product? Why can we use a cross-multiplication rule to obtain an equivalent expression for

$$\frac{1}{x+h} - \frac{1}{x} \text{ as } \frac{x-(x+h)}{x(x+h)} = \frac{-h}{x(x+h)}?$$

The answer to all of these questions involves a GCF of 1. Like a silent partner, the GCF is always there when we need it, but it is never in the forefront. A GCF of 1 is almost always taken for granted, for without it we could not justify the steps of our procedure.

A GCF of 1 is the reason we can or cannot proceed with some of the most fundamental procedures in mathematics. It is our rationale for the answers we give to the questions raised above. Quite simply, we cannot reduce 3/7, since GCF(3,7) = 1. We perform $2 \times 3 = 6$ to obtain the lowest common denominator of 1/2 and 1/3 because GCF(2,3) = 1, but performing $2 \times 6 = 12$ does not give us the lowest common denominator of 1/2 and 1/6 because GCF(2,6) $\neq$ 1. We cannot combine $\sqrt{3} + \sqrt{7}$, since GCF(3,7) = 1, and we rationalize $1/\sqrt{6}$ immediately, since GCF(1,6) = 1. Rationalizing the denominator of $2/\sqrt{6}$ does not immediately give us a radical in simplest form, because GCF(2,6) $\neq$ 1. The same holds true for common denominators in the mathematics of trigonometry and calculus. For example, GCF($\sin x$,$\cos x$) = 1, and therefore the LCM is their product, $\sin x \cos x$. Finally, a cross-multiplication rule can be used to determine an equivalent expression (reduced to lowest terms) for

$$\frac{1}{x+h} - \frac{1}{x},$$

since GCF$(x + h, x)$ = 1.

The nature and calculation of the GCF should take a more prominent role in the curriculum and, hence, in the classroom because it is so fundamental. It is time to reinforce basic skills so that students can understand the logic behind the mathematical procedures they discover and are taught. Understanding and basic skills, such as the division algorithm, will greatly enhance students' mathematical literacy.

Now, let's pursue our discussion of the GCF by examining its role in reducing fractions and rational expressions. Consider the task of reducing 57/95 to lowest terms without using a calculator. Many students would first wonder if it was possible and then how to proceed. The quickest method is to calculate GCF(57,95) using Euclid's algorithm.

Step 1. Divide the smaller number into the larger, keeping track of the remainder.

$$\begin{array}{r} 1 \\ 57\overline{)95} \\ \underline{57} \\ 38 \end{array}$$

Step 2. Divide the remainder into the previous divisor, again keeping track of the remainder.

$$\begin{array}{r} 1 \\ 38\overline{)57} \\ \underline{38} \\ 19 \end{array}$$

Step 3. Repeat Step 2 until the remainder is 0.

$$\begin{array}{r} 2 \\ 19\overline{)38} \\ \underline{38} \\ 0 \end{array}$$

Step 4. The divisor that yields a remainder of 0 is our GCF.

Here, GCF(57,95) = 19, so to reduce 57/95, we simply divide both top and bottom by 19 to obtain

$$\frac{57 \div 19}{95 \div 19} = \frac{3}{5}.$$

This skill may also be applied to a task such as reducing the rational expression

$$\frac{x^2 - x - 6}{x^2 + x - 12}$$

to lowest terms, as is required in Grades 10, 11 and 12. Students are usually instructed to factor both top and bottom and then reduce. Logically, it would be better to first answer the key question, Will this rational expression reduce at all? If GCF ($x^2 - x - 6$, $x^2 + x - 12$) = 1, then the answer is no. However, if the GCF is not 1, then we must proceed.

Applying Euclid's algorithm, we get

$$\begin{array}{r} 1 \\ x^2 + x - 12\overline{)x^2 - x - 6} \\ \underline{x^2 + x - 12} \\ -2x + 6 \end{array}$$

$$\begin{array}{r} -\frac{1}{2}x - 2 \\ -2x + 6\overline{)x^2 + x - 12} \\ \underline{x^2 - 3x} \\ 4x - 12 \\ \underline{4x - 12} \\ 0 \end{array}$$

Here, $-2x + 6 \neq 1$ necessarily, and since $-2x + 6 = -2(x - 3)$, the binomial factor $(x - 3)$ is common to both $x^2 - x - 6$ and $x^2 + x - 12$. For students who have difficulty with factoring, half the work of reducing is now already done. That is, we know that the expression can be reduced, and we know the factor needed to begin the process.

The above procedure is another way to approach a common algebra problem, but it is not helpful for every student. Some students may have forgotten the division algorithm altogether, and others would rather try their luck at factoring. On the other hand, through this approach, some students will add to their understanding of mathematical process and become more independent learners. They may even recognize the power of division and Euclid's algorithm. A high-energy honours class may appreciate the power of reducing fractions without a calculator and how the approach serves as a natural lead-in to calculating LCMs using

$$\text{LCM}(a,b) = \frac{ab}{\text{GCF}(a,b)},$$

which is the method used in Asia. A teacher who is going back to the basics may also appreciate this application of the division algorithm.

In short, the calculation of the GCF is simple and direct, and an understanding of the significance of 1 as the GCF of two numbers or expressions may help students understand the logic behind many mathematical processes. This silent partner need not be silent anymore!

Radical Radicals

Today's math student does not like radicals any more than yesterday's math student did. Although the modern student is usually armed with a calculator, the process of guessing how to break down radicals or when to rationalize the denominator still dominates the thinking and strategy processes. Also, students are not always sure that their final answer is in lowest terms, especially when dividing. There has to be a set approach to all operations involving radicals that students can use to resolve these issues.

Consider $\sqrt{75} + \sqrt{27}$. We suggest to our students that they reduce the radicands before combining terms. The logical question, of course, is whether these terms can be combined and, if they can, how to proceed. In this case, how are numerically challenged students supposed to know that they should start with 3, especially if we have taught them to extract perfect squares from each of the given radicals? How is the student to proceed with confidence and certainty from the outset?

We may tell our students that we cannot add or subtract the terms if the GCF of the radicands is 1. In simplifying $\sqrt{75} + \sqrt{27}$, the trained eye observes that GCF(27,75) is not 1 but 3; thus, it may be possible to simplify the expression. Since 3 is a common factor of both 75 and 27, we can now express $\sqrt{75}$ and $\sqrt{27}$ in terms of $\sqrt{3}$. Instead of having to guess how to break down both 75 and 27, we already have one of the key factors. The student then has the simple task of dividing 3 into both 27 and 75 before simplifying the terms to obtain

$$\begin{aligned}\sqrt{75}+\sqrt{27} &= \sqrt{3\cdot 25}+\sqrt{3\cdot 9}\\ &= 5\sqrt{3}+3\sqrt{3}\\ &= (5+3)\sqrt{3}\\ &= 8\sqrt{3}.\end{aligned}$$

The expression $\sqrt{75}+\sqrt{27}$ can be simplified, since GCF(27,75) is 3 and removing a GCF of 3 from each radicand (that is, a common factor of $\sqrt{3}$ from each radical) reveals a factor that is a perfect square in each radicand.

We might also suggest that students not multiply or divide until they have determined the GCF of the radicands. Consider the conventional way of multiplying two radicals, such as $\sqrt{28}$ and $\sqrt{63}$. Many calculator-oriented students would perform $28 \times 63 = 1{,}764$ and then try to simplify ($\sqrt{1{,}764} = 42$). If we suggest that students simplify the radicals before multiplying, a guessing or guess/estimating process begins in an attempt to determine what numbers go into both 28 and 63. It is discovered that $\sqrt{28} = 2\sqrt{7}$ and that $\sqrt{63} = 3\sqrt{7}$, but only after dealing with the radicands one at a time.

The process of Radical Radicals involves considering both radicands at the same time by finding their GCF using Euclid's algorithm (without using a calculator).

For example, find GCF(28,63).

Step 1. Divide the larger number by the smaller number, keeping track of the remainder.

$$\begin{array}{r} 2 \\ 28\overline{)63} \\ \underline{56} \\ 7 \end{array}$$

Step 2. Divide the remainder into the previous divisor.

$$\begin{array}{r} 4 \\ 7\overline{)28} \\ \underline{28} \\ 0 \end{array}$$

Step 3. Continue Step 2 until the remainder is 0.

Step 4. The divisor that yields a remainder of 0 is our GCF.

Since the divisor of 7 gives us a remainder of 0, we know that 28 and 63 have 7 as a GCF, so $\sqrt{28}$ and $\sqrt{63}$ have $\sqrt{7}$ as a common factor. This brings about a different way of doing radicals, because we work with two radicands at a time, not one. Also, when it comes time to break down the radicals, we will already know one of the factors and, therefore, half the work will have already been done. Thus, guessing or guess/estimating is reduced dramatically.

Now consider $\sqrt{26}\cdot\sqrt{65}$. Instead of looking at $\sqrt{1{,}690}$, we determine that GCF(26,65) = 13 and express each factor in terms of $\sqrt{13}$. This gives us

$$\begin{aligned}\sqrt{26}\cdot\sqrt{65} &= \sqrt{2}\cdot\sqrt{13}\cdot\sqrt{5}\cdot\sqrt{13}\\ &= \sqrt{13}\cdot\sqrt{13}\cdot\sqrt{2}\cdot\sqrt{5}\\ &= 13\sqrt{10}\end{aligned}$$

The numbers never become large, so simplifying is easier.

The disadvantage of this method is that students must find the GCF of two numbers with little help from a calculator. Its advantages are that once the GCF has been found, half the work is already done; guessing is virtually eliminated; and the numbers never increase in size, which reduces the frustration and errors that come with working with large numbers.

In summary, the steps to this new approach are as follows:

Step 1. Find the GCF of the radicands.

Step 2. Express each radicand in terms of the square root of the radicand.

Step 3. Pair off like radicals.

Step 4. Simplify remaining terms.

If you thought multiplication done in this way was efficient, let's now look at division. This is where this method really shines! We do not rationalize the denominator unless the GCF of both the numerator and the denominator is 1.

Consider $\sqrt{175}/\sqrt{112}$. If students do not recognize that 7 is a common factor, they might multiply top and bottom by $\sqrt{112}$, giving them horrendous numbers to work with. Instead, we use Euclid's algorithm to find that GCF(112,175) = 7. Thus, we have

$$\frac{\sqrt{175}}{\sqrt{112}} = \frac{\sqrt{7\cdot 25}}{\sqrt{7\cdot 16}} = \frac{\sqrt{7}}{\sqrt{7}}\cdot\frac{\sqrt{25}}{\sqrt{16}}.$$

Since $\sqrt{25} = 5$, $\sqrt{16} = 4$ and the radical factors $\sqrt{7}$ divide to give us 1, we are left with the answer 5/4.

What happens if the GCF of the numerator and the denominator is 1? We simply multiply top and bottom by the denominator, knowing that we will not have to reduce the fraction after multiplying the terms. For example,

$$\frac{\sqrt{15}}{\sqrt{7}} = \frac{\sqrt{15}}{\sqrt{7}}\cdot\frac{\sqrt{7}}{\sqrt{7}} = \frac{\sqrt{105}}{7}.$$

We realize that $\sqrt{105}$ cannot be simplified, since GCF(15,7) = 1.

Now, what happens if the numerator is not a radical? Consider $2/\sqrt{10}$. Here, we do not multiply top and bottom by $\sqrt{10}$, since GCF(2,10) = 2 (not 1). Instead, we recall that $\sqrt{a} \cdot \sqrt{a} = a$, and in this case $\sqrt{2} \cdot \sqrt{2} = 2$. Since GCF(1,100) = 1, we express both numerator and denominator in terms of $\sqrt{2}$. This gives us

$$\frac{2}{\sqrt{10}} = \frac{\sqrt{2}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{5}}.$$

The $\sqrt{2} / \sqrt{2}$ divides to give us 1, and we are now left with $\sqrt{2} / \sqrt{5}$. Since GCF(2,5) = 1, we can now multiply top and bottom by $\sqrt{5}$ and not have to worry about reducing the final form of the quotient. Finally, we have

$$\frac{\sqrt{2}}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{10}}{5}.$$

We do not have to go backward or look over our shoulder to see if the quotient can be reduced, since the GCF is 1. Our rule, then, is to multiply top and bottom by the denominator factor only when the GCF of the radicands is 1.

In general, the steps for division are as follows:

Step 1. Find the GCF of both numerator and denominator.

Step 2. Express each radicand in terms of the square root of the GCF.

Step 3. Pair off like radicals and reduce.

Step 4. Simplify remaining terms.

Step 5. Multiply top and bottom by the denominator term only when the GCF of numerator and denominator is 1.

For division, then, we merely add Step 5 to the method used for multiplication.

These new processes for multiplication and division now simplify the processes for addition and subtraction, because we can operate only if the GCF is not 1. Hence, $\sqrt{3} + \sqrt{7}$ cannot be simplified any further, since GCF(3,7) = 1. For $\sqrt{24} + \sqrt{54}$, we find that GCF(24,54) = 6. We then have

$$\begin{aligned}\sqrt{24} + \sqrt{54} &= \sqrt{4} \cdot \sqrt{6} + \sqrt{9} \cdot \sqrt{6} \\ &= \sqrt{6}\,(2+3) \\ &= 5\sqrt{6}.\end{aligned}$$

Again, we look at two radicands at a time. We ascertain that it may indeed be possible to combine terms if their GCF is not 1. If we have more than two terms, we can look for two or more with the same GCF.

For combined operations, we again look at two GCFs as opposed to one. Consider $\sqrt{3}\,(\sqrt{15}+\sqrt{21})$. Since GCF(3,15) = 3 and GCF(3,21) = 3, we have $\sqrt{15} = \sqrt{3} \cdot \sqrt{5}$ and $\sqrt{21} = \sqrt{3} \cdot \sqrt{7}$. Thus, we can write

$$\begin{aligned}\sqrt{3}\,(\sqrt{15}+\sqrt{21}) &= \sqrt{3}\,(\sqrt{3} \cdot \sqrt{5} + \sqrt{3} \cdot \sqrt{7}) \\ &= 3\,(\sqrt{5}+\sqrt{7}).\end{aligned}$$

For division, consider

$$\frac{5}{\sqrt{7}-\sqrt{2}}.$$

Since GCF(5,7,2) = 1, we multiply top and bottom by the conjugate $\sqrt{7}+\sqrt{2}$. This gives us

$$\begin{aligned}\frac{5}{\sqrt{7}-\sqrt{2}} \cdot \frac{\sqrt{7}+\sqrt{2}}{\sqrt{7}+\sqrt{2}} &= \frac{5(\sqrt{7}+\sqrt{2})}{7-2} \\ &= \frac{5(\sqrt{7}+\sqrt{2})}{5} \\ &= \sqrt{7}+\sqrt{2}.\end{aligned}$$

In summary, by finding the GCF of the radicands, we introduce the idea of working with two or more radicands at a time. Once we have the GCF, we cut the work by at least half because we already have one of the factors of the radicand. We eliminate large numbers, the errors caused by large numbers and the frustration that results from guessing. We streamline the process by keeping the numbers simple and neat.

In my experience, student feedback on this process is always the same: "This is easy compared to what I used to do!"

Duncan McDougall has been teaching for 27 years and is a graduate of Bishop's University and McGill University. After 13 years of teaching in the public school systems of Quebec, Alberta and British Columbia, he left the classroom for private teaching. Over the last 15 years, he has taught all levels of high school mathematics, college and university calculus, and mathematics for elementary school teachers. This rich exposure to diverse programs and curricula has provided him with some of the ideas he is now publishing. He has given several workshops on rapid calculation and nonconventional approaches to operations on radicals and fractions. His latest project is developing a special family of polynomials for curve sketching called Diophantine polynomials. He owns and operates TutorFind Learning Centre in Victoria, British Columbia.

Alberta High School Mathematics Competition 2005: Part I

Andy Liu

The Alberta High School Mathematics Competition (AHSMC) is open to students in Alberta and the Northwest Territories. It is designed for high school students, but occasionally students in the earlier grades also take part. The prizes are books and scholarships ranging in value from \$50 for some of the Part I prizes to \$1,500 for the major prize of Part II.

For more information, visit the AHSMC website at www.math.ualberta.ca/~ahsmc/index.html.

Part I of the 2005 competition (with solutions) follows.

1. What is the value of $2,005 \times 20,042,004 - 2,004 \times 20,052,005$?

   (a) 0 (b) 10,000 (c) 2,003,000 (d) 2,005,000 (e) none of these

2. An ice cream store has 20 kinds of ice cream. A customer may get one or two scoops of ice cream. If she gets two scoops, the scoops can be the same or different, and the order of the scoops does not matter. How many different cones are possible?

   (a) 90 (b) 99 (c) 100 (d) 230 (e) none of these

3. In triangle ABC, let D be the midpoint of BC, and let E be on AD such that $ED = 2AE$. If the area of triangle ABC is 150, what is the area of triangle ABE?

   (a) 25 (b) 32.5 (c) 50 (d) 75 (e) none of these

4. Let a, b and c be real numbers and $x = 11c - a - b$. If $b - a - 3c \leq -2$ and $b - 2a + c \geq 3$, then

   (a) $x \in [0,1]$ (b) $x \in [2,5]$ (c) $x \in [6,9]$ (d) $x \in [10,11]$ (e) $x \in [12,\infty]$

5. Penelope has three red socks, three yellow socks and three blue socks. If she picks socks without looking, what is the smallest number of socks she must pick to guarantee that she has four socks that form two pairs of socks of matching colours?

   (a) 4 (b) 5 (c) 6 (d) 7 (e) 8

6. A building consists of a square-based pyramid on top of a square-based prism. All vertical edges of the prisms have lengths of 2 m. All other edges have lengths of 3 m. What is the height, in metres, of the top of this building from the ground?

   (a) $2 + \frac{\sqrt{2}}{2}$ (b) $2 + \frac{3\sqrt{2}}{2}$ (c) $\sqrt{20}$ (d) $\sqrt{21}$ (e) none of these

7. If f is a function defined on the positive real axis and

   $$f(x) + 3f\left(\frac{1}{x}\right) = x - \frac{5}{x} + 4 \quad \text{for all } x > 0,$$

   then $f(\frac{1}{2})$ is

   (a) $\frac{1}{2}$ (b) 1 (c) $\frac{3}{2}$ (d) 2 (e) 4

8. In the sequence obtained by omitting the squares and the cubes from the sequence of positive integers, 2,005 sits on which position?

   (a) 1,950 (b) 1,951 (c) 1,952 (d) 1,953 (e) none of these

9. The positive integer a is such that the inequality $2a + 3x \leq 101$ has exactly six solutions in positive integers x. How many possible values of a are there?

   (a) 1 (b) 2 (c) 3 (d) 4 (e) 5

10. P is a point inside a parallelogram $ABCD$. If the area of triangle PAD is one-third that of $ABCD$, and the area of triangle PCB is 6 cm^2, then what is the area, in square centimetres, of the parallelogram?

    (a) 24 (b) 36 (c) 48 (d) 60 (e) 72

11. There are three problems in a contest. Students win bronze, silver or gold medals if they solve one, two or three problems, respectively. Each problem is solved by 60 students, and there are

100 medallists. What is the difference between the number of bronze medallists and the number of gold medallists?

(a) 0 (b) 10 (c) 20 (d) 30
(e) not uniquely determined

12. The positive numbers a, b and c are such that $a + b + c = 7$ and

$$\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}=\frac{10}{7}.$$

What is the value, then, of

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}?$$

(a) 1 (b) 3 (c) 7 (d) 10 (e) dependent on a, b and c

13. A tile is obtained from a 3×3 square by first removing two squares in opposite corners and then removing one square adjacent to each of those squares, so that the remaining five squares form a connected piece. The tile is placed on an infinite piece of graph paper so that it covers exactly five squares. It may be turned over or rotated. If we wish to paint the squares of the graph paper in such a way that no matter where the tile is placed, it never covers two squares of the same colour, what is the smallest number of colours needed?

(a) 5 (b) 6 (c) 7 (d) 8 (e) 9

14. The number of ordered pairs of integers (x,y) such that $xy = x^2 + y^2 + x + y$ is

(a) 3 (b) 4 (c) 6 (d) 8 (e) none of these

15. The positive integer n that satisfies the equation $\log_2 3 \log_3 4 \cdots \log_n(n + 1) = 2{,}005$ should be a multiple of

(a) 2 (b) 3 (c) 5 (d) 31 (e) none of these

16. Three spheres of unit radius sit on a plane and are tangent to one another. A large sphere with its centre in the plane contains all three unit spheres and is tangent to them. The radius of the large sphere is

(a) $\sqrt{3}-1$ (b) $\sqrt{\frac{5}{3}}+1$ (c) $\sqrt{\frac{7}{3}}+1$
(d) $\sqrt{3}+1$ (e) none of these

Solutions

1. The answer is (a). We have $2{,}005 \times 20{,}042{,}004 - 2{,}004 \times 20{,}052{,}005 = 2{,}004 \times 2{,}005(10{,}001 - 10{,}001) = 0$.

2. The answer is (d). The number of different single cones possible is 20, as is the number of different double cones with identical scoops. The number of different double cones with different scoops is $(20 \times 19)/2 = 190$. Hence, the total is $20 + 20 + 190 = 230$.

3. The answer is (a). Since $BD = CD$, the area of triangle BAD is 75. Since $ED = 2AE$, the area of triangle ABE is 25.

4. The answer is (e). The first given inequality may be rewritten as $a - b + 3c \geq 2$. Adding three times this to two times $b - 2a + c \geq 3$, we have $x \geq 12$.

5. The answer is (c). If Penelope picks only five socks, she may get three red socks, one yellow sock and one blue sock, forming only one matching pair. If she leaves out three socks, she can leave out at least two socks of at most one colour. Hence, if she picks six socks, she can be guaranteed to have two matching pairs.

6. The answer is (b). Half the diagonal of the square base has length

$$\frac{3\sqrt{2}}{2}.$$

Since the pyramid is half of a regular octahedron, its height has the same length. Hence, the height of the top of the building from the ground is

$$2+\frac{3\sqrt{2}}{2}.$$

7. The answer is (d). We have

$$f(2)+3f\left(\frac{1}{2}\right)=\frac{7}{2} \text{ and } f\left(\frac{1}{2}\right)+3f(2)=-\frac{11}{2}.$$

Subtracting the second equation from three times the first, we have

$$8f\left(\frac{1}{2}\right)=16, \text{ so that } f\left(\frac{1}{2}\right)=2.$$

8. The answer is (c). Since $44^2 < 2{,}005 < 45^2$, $12^3 < 2{,}005 < 13^3$ and $3^6 < 2{,}005 < 4^6$, the number of squares, cubes and sixth powers less than 2,005 are 44, 12 and 3, respectively. Hence, 2,005 sits in position $2{,}005 - 44 - 12 + 3 = 1{,}952$.

9. The answer is (a). The six solutions for x must be 1, 2, 3, 4, 5 and 6. If $a = 42$, then $3x \leq 17$ and $x = 6$ is not a solution. Larger values of a eliminate further solutions. If $a = 40$, then $3x \leq 21$ and $x = 7$ is also a solution. Smaller values of a allow for further solutions. Hence, we must have $a = 41$. Then $3x \leq 19$, and we have the six solutions above.

10. The answer is (b). The total area of triangles *PAD* and *PCB* is half that of the parallelogram. Hence, 1/2 – 1/3 = 1/6 of the area of the parallelogram is 36, and its area is 36.

11. The answer is (c). If we have a gold medallist and a bronze medallist, we can convert them to two silver medallists. This way, we will end up with either no gold medallists or no bronze medallists. Altogether, 3 × 60 = 180 correct solutions are received. If all students were silver medallists, they would have turned in 2 × 100 = 200 correct solutions. Since we are 200 – 180 = 20 short, we have 20 bronze medallists and 0 gold medallists. This difference is not affected by our conversions.

12. The answer is (c). We have

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}$$
$$=\frac{7-(b+c)}{b+c}+\frac{7-(c+a)}{c+a}+\frac{7-(a+b)}{a+b}$$
$$=7\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)-3$$
$$=7.$$

13. The answer is (e). Divide the infinite piece of graph paper into 3 × 3 regions. If we paint the nine squares in each region in nine different colours in exactly the same way, the tile cannot cover two squares of the same colour. If we use fewer than nine colours, two squares in the same region will have the same colour. Both can be covered by a suitably placed tile.

14. The answer is (c). Adding $x + y + 2$ to both sides of the given equation, and letting $u = x + 1$ and $v = y + 1$, the equation becomes $u^2 + v^2 = uv + 1$. If $uv = 0$, the solutions are $(u,v) = (0,\pm 1)$ or $(\pm 1,0)$. If $uv > 0$, then $(u - v)^2 = 1 - uv$, and we must have $uv = 1$. The solutions are $u = v = \pm 1$. If $uv < 0$, let one of them be a and the other be $-b$, where a and b are positive. Then the equation becomes $a^2 + ab + b^2 = 1$, which has no solutions. Hence, there are six solutions for (u,v), and six corresponding solutions for (x,y).

15. The answer is (d). Through changing bases, the given equation becomes $\log_2(n + 1) = 2{,}005$. Hence $n = 2^{2{,}005} - 1$. It leaves a remainder of 1 when divided by 2, 3 or 5 but is divisible by $2^5 - 1 = 31$.

16. The answer is (c). Since the three equal spheres are pairwise tangent, their centres form an equilateral triangle of side 2. The distance of the centre of this triangle from any vertex is

$$\frac{2\sqrt{3}}{3}.$$

The projection of this centre onto the plane is the centre of the large sphere. Its distance from the centre of one of the spheres is

$$\sqrt{1+\frac{2\sqrt{3}}{2}}=\sqrt{\frac{7}{3}}.$$

Since the large sphere is also tangent to the others, its radius is

$$\sqrt{\frac{7}{3}}+1.$$

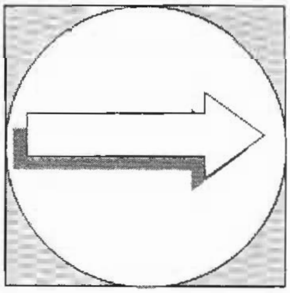
A Page of Problems
It's Do or Dice!

A Craig Loewen

Middle School

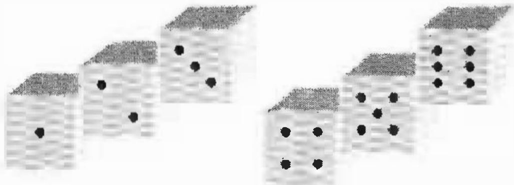
Create a spinner that would give you the same results as

- rolling a six-sided die, and
- rolling two four-sided dice together.



High School

On average, how many times will you have to roll six regular six-sided dice before getting a 1, 2, 3, 4, 5 and 6 all in one roll?



Hint: What is the probability of rolling a 1, 2, 3, 4, 5 and 6 all at once with six dice?

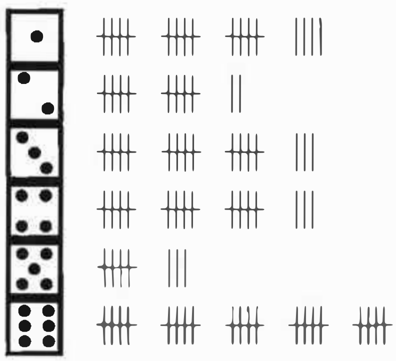
Junior High

Do you have a better chance of rolling an even sum with two regular four-sided dice, or rolling an odd sum with two regular six-sided dice?

Conduct an experiment to test your hypothesis.

Elementary

Jo rolled her six-sided die 100 times and recorded the results in the tally chart below. Do you think her die was "fair"? How do you know?



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