# Research in and Development of School Mathematics: Delving Deeper into Concepts to Revitalize the Mathematics Taught and Learned in Schools

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## Contextualizing the Research

Mathematics teaching and learning happen every school day. Albeit sometimes celebrated and advocated for, most mathematics teaching and learning are often questioned, criticized or negatively judged by people involved in mathematics education. These criticisms have often led these same people to call for changes to how mathematics is taught and learned in classrooms. It has always seemed odd to me that these criticisms are addressed mainly to teachers and their teaching practice. For me, as a mathematics education researcher, the problem has always been something else. To contextualize this better, I offer you this quote from French didactician of mathematics, Guy Brousseau:

> I am never critical toward teaching as it is practiced. If you see 200 000 teachers doing the same thing and it looks stupid to you, it is not because there are 200 000 stupid people. It is because there is a *phenomenon* that orients this same type of reaction in these people. And it is this *phenomenon* that we need to understand. ... We won't improve it with an ideology, nor by moralizing to teachers. (Brousseau 1988; my translation)

It is this "phenomenon," this "something else," that the research project I report on in this paper attempts to address. In short, the underlying belief of this project is that it is not the people who are problematic in mathematics teaching and learning, but the mathematics itself: the problem resides in the mathematics being taught and learned and not how it is taught or learned. My entry into the problem is, therefore, mathematical.

This paper outlines the significance of this research for the continuing improvement of mathematics teaching and learning. I explain the various phases of the research to give the reader an idea of the approach. Then, through this, I report on the specific project conducted, focused on the notion of area of planar figures, and discuss the results and products of this research orientation and their outcomes for the teaching practice.

## The Research Approach: Delving Deeper into the Mathematics

The intention to address the mathematics being taught and learned in schools came as a result of diverse reform movements in mathematics that advocated for "less mathematics, but deeper," to promote more conceptual forms of understandings in students. But what does it mean to go deeper into the mathematics? It is this question that this research project explored, taking the notion of area of planar figures as a specific example.

In this project, the choice of area of planar figures as an example to work on was based on personal interest and on my teaching experience in secondary schools, where I have often been unsatisfied with the treatment of this topic. Area is often seen as an easy topic simply consisting of helping students memorize a number of diverse formulas. Thus, learning area is pinned down to knowing and memorizing formulas, recognizing which ones to use and applying them in a problem. Consider this typical quote taken from the Purplemath website (www.purplemath.com/modules/geoform.htm) addressed to learners:

> Some instructors like to give all needed geometric formulas, so your test will have a listing of anything you might need. But not all instructors are this way, and you can't just expect a new instructor ... to give you all this information. Ask your instructors for their policies, but remember

that there does come a point (high school? SAT? ACT? College? "real life"?) at which you will be expected to have learned at least some of these basic formulas. Start memorizing now!

Thus, the central question of this research project became, what would it mean to work deeper in area of planar figures? To address this question, a number of phases were designed. I report on them below.

## Phase 1: What Is Out There?

Interesting as it may seem as a research orientation, wanting to delve deeper in mathematics supposes that one has some awareness of the mathematics usually worked on in schools. This led to the consideration of what is out there in the area of planar figures. Thus, the first phase of the research reviewed various forms of textbooks and curricular materials to gather a sense of what is out there concerning the teaching and learning of area of planar figures.

This review of Grades 4 to 8 resources revealed a somewhat poor treatment of the notion of area of planar figures. From most of the resources reviewed, three important tendencies could be highlighted:

1. An explicit and important focus given to area formulas of planar figures, which were either given, constructed or explained
2. A large number of isolated formulas, one for each planar figure; for example, rectangle [$L \times l$], parallelogram [$B \times h$], square [$s^2$], rhombus [$\frac{(D \times d)}{2}$], trapezoid [$\frac{(B + b) \times h}{2}$]
3. A focus placed on numerical calculations of areas for the planar figures, often triggered by a command, such as "Calculate the area of the following rectangle."

It was therefore felt that area received a rather poor treatment through these various resources. Thus, the project's central question was again brought to the fore: what would it mean to work deeper in area of planar figures than what these resources already offer? The challenge was to delve into the topic and draw out more of it than this review outlined to enrich mathematically the concept of area of planar figures. This paved the way for the second phase of the research concerning the development of a deeper approach to the concept of area of planar figures.

## Phase 2: Digging into and Developing the Mathematics

As mentioned, the challenge was to probe into the concept of area to draw out an approach that enriched its treatment. Therefore, the work centred on exploring, making sense and delving into the mathematical concept of area of planar figures to unpack some of its underlying meanings, relations and subtleties often hidden within it.

However, it is important to note that this work was explicitly seen as being able to produce/develop only one of the many possible approaches to the concept of area of planar figures, as many other ideas could be explored. The intention was not to develop all possible treatments for area of planar figures but to offer one rich approach to illustrate what it could mean to delve deeper into a topic of study (here, with area of planar figures taken as an example).

Two important issues emerge from the overabundance of formulas, numbers and calculations in the study of area: (1) the absence of geometry and (2) the enormous number of isolated and disconnected formulas to memorize. Richard Skemp discusses this second issue:

> There is a seeming paradox here, in that it is certainly harder to learn. It is certainly easier for pupils to learn that "area of a triangle = ½ base × height" than to learn why this is so. But they then have to learn separate rules for triangles, rectangles, parallelograms, trapeziums; whereas relational understanding consists partly in seeing all of these in relation to the area of a rectangle. It is still desirable to know the separate rules; one does not want to have to derive them afresh everytime [sic]. But knowing also how they are inter-related enables one to remember them as parts of a connected whole, which is easier. (Skemp 1978, 12–13)[1]

These issues triggered specific guidelines in the research inquiry for developing the approach that would delve deeper into area of planar figures: (1) attempting to go back to and work in geometry in area of planar figures and (2) finding a way to draw out the links existing between the usual area formulas. To illustrate where these guidelines led the work, I offer a glimpse at ideas that were brought together for delving into area of planar figures. (However, because of space constraints I cannot go into great length about these ideas, so I refer the reader to three other papers that were produced on these ideas: Proulx 2007; Proulx 2008; Proulx and Pimm 2008).

## A Glimpse at the Ideas Developed for Area of Planar Figures

Let's begin by introducing a mathematical principle that influenced the inquiry: the Cavalieri principle.

This principle, for planar figures and solids, asserts that:

> If between the same parallels any two plane figures are constructed, and if in them, any straight lines being drawn equidistant from the parallels, the included portions of any one of these lines are equal, the plane figures are also equal to one another; and if between the same parallel planes any solid figures are constructed, and if in them, any planes being drawn equidistant from the parallel planes, the included plane figures out of any one of the planes so drawn are equal, the solid figures are likewise equal to one another. ... The figures so compared let us call analogues, the solid as well as the plane, ... (Cavalieri 1653)

Gray (1987, 13) expresses the two-dimensional aspects of the Cavalieri principle as follows:

> The principle asserts that two plane figures have the same area if they are between the same parallels, and any line drawn parallel to the two given lines cuts off equal chords in each figure.

Figure 1 offers an illustration of the Cavalieri principle with two plane figures:

Informally, the principle asserts that if you cut each polygon horizontally at the same height and each chord obtained is of equal length, then the two polygons are of the same area. To make the comparison, both polygons need to be the same height; if not, the comparison appears not possible because one polygon would be cut where there would not be any of the other polygon left to cut.

When used for the study of area, the Cavalieri principle allows for insightful comparisons between figures, and in that sense appears helpful in establishing geometric links and relating planar figures. For example, one can establish links between rectangles and parallelograms (of same base and same height), where both figures lie between the same parallels (see Figure 2).

The 2-D version of the Cavalieri principle helps us also to see "families of planar figures;" for example, a family of rectangle and parallelograms of same area. To see this, let the parallelogram be as slanted as you want: any parallelogram with the same height and same base would have the same area, because each cross-section is always of the same length, creating the equivalent family of rectangles and parallelograms (Figure 3).

**Figure 1. An Illustration of the Cavalieri Principle in Two Dimensions**

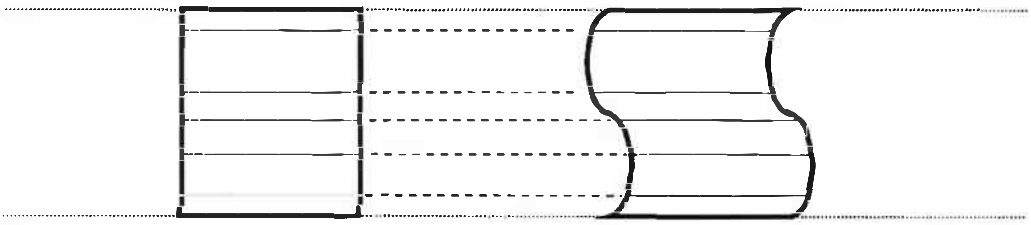


**Figure 2. Using the Cavalieri Principle with a Rectangle and a Parallelogram**

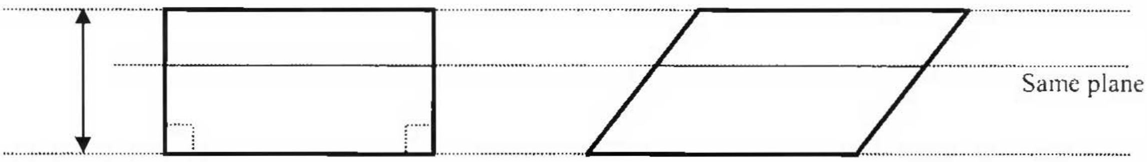
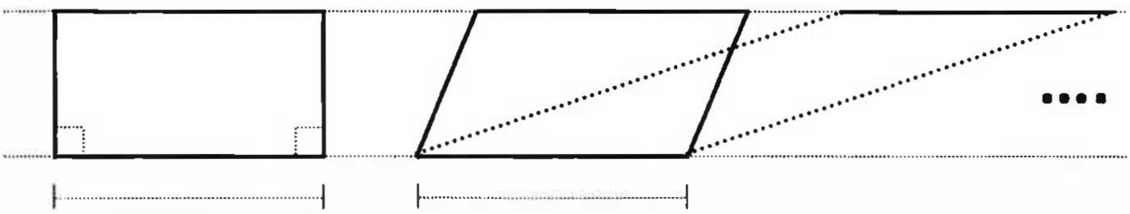



**Figure 3. Family of Rectangle and Parallelograms**



A parallelogram thus possesses an associated rectangle, which has the same base and same height, implying that the area of a parallelogram can be obtained through the same formula as for rectangles:

**Area of the parallelogram** = **length of the base** × **height**

The Cavalieri principle, and families of parallelograms, offers an interesting entry into rhombuses and squares. Obviously the square, frequently defined as a rectangle with four equal sides, is directly related to the preceding formula of *Area = base × height*. Thus, this prevents the need for inventing a specific formula for the square (often given with $s^2$), a prevention that in fact strengthens the link between the square and the rectangle.

The rhombus can be defined as a parallelogram that possesses all identical sides and therefore can be seen as part of a family of parallelogram and directly associated with a specific rectangle (having the same base and height as the rhombus). However, defining a rhombus as a parallelogram raises some questions concerning the usual "data" given to calculate its area. Diagonals are usually provided (or to be found) for finding the area of the rhombus; its formula being in fact directly related to it $[\frac{(D \times d)}{2}]$. Hence, defining the rhombus as a parallelogram leads one to aim for the "base" and "height" of the rhombus, something quite unusual for rhombuses.[2] However, one has to acknowledge that this strengthens the link between rhombuses, parallelograms, squares and rectangles; a link that, in addition, simplifies the various formulas for these four planar figures by providing a single and general formula for them: *Area = base × height*.

Concerning families, the same thing can be said for triangles, where all triangles with the same base and height are part of the same family, be they as slanted as one wants (Figure 4 or Figure 5).

It is also possible to establish a family of trapezoids, where the small base slides on the same plane, producing a family of trapezoids with the same base and the same height (Figure 6).

**Figure 4. A Family of Triangles with Same Height and Same Base**

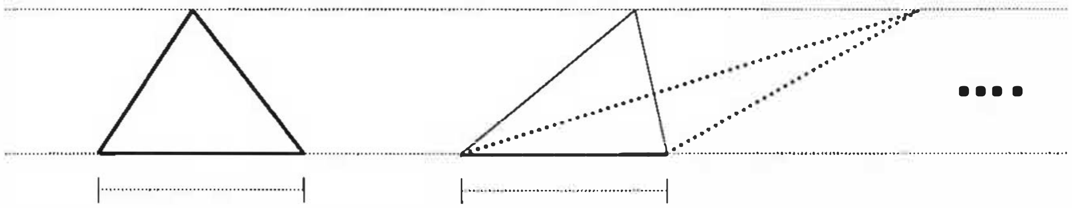


**Figure 5. Another Representation of a Family of Triangles**

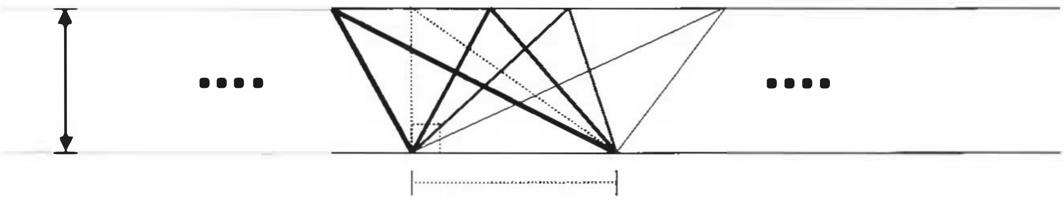
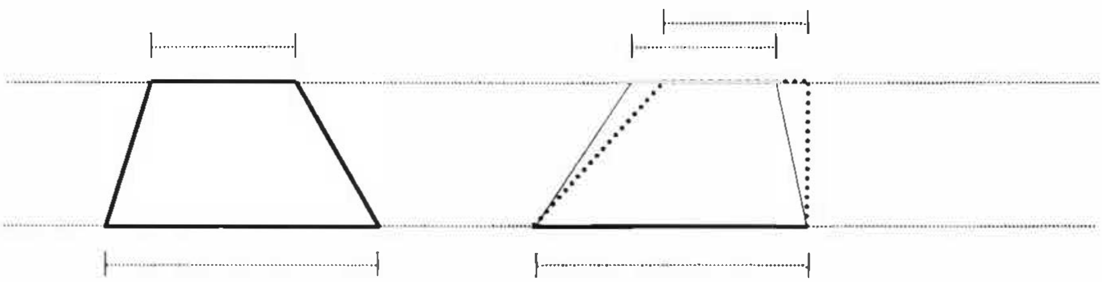


**Figure 6. A Family of Trapezoids with Same Height and Same Base**



Moreover, as the family of trapezoids gets established, it is interesting to note that a trapezoid is indeed defined as "a quadrilateral with two sides parallel" (see, for example, Wolfram MathWorld website http://mathworld.wolfram.com), which takes into account not only standard trapezoids that we are often used to seeing (Figure 7) but also any quadrilateral that has a pair of opposite sides that are parallel, which are indeed trapezoids, without being parallelograms (Figure 8)[3].

In that sense, the family is composed of any quadrilateral that has the same height and a pair of opposite and parallel sides, making the family of trapezoids look like the following (Figure 9).

One notices, however, that no family of equivalent trapezoids contains a rectangle. However, there is a fixed relationship between a trapezoid and its associated rectangle. By rotating the rectangular trapezoid (that is part of any trapezoid family) about the midpoint of the remaining slant side, a rectangle that is the double of the trapezoid is produced (Figure 10), establishing a significant relationship between trapezoids and rectangles. Trapezoids become perceived as the half of a related rectangle that has the same

**Figure 7. Some Examples of Standard Trapezoids**

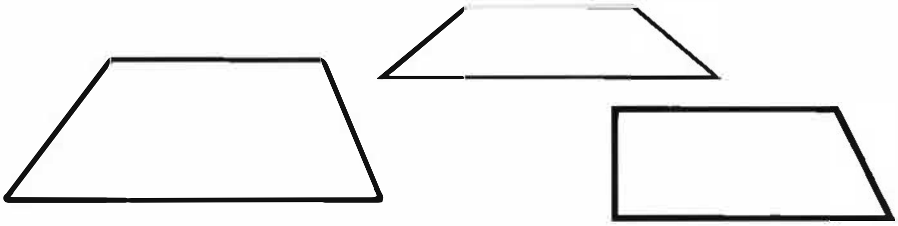


**Figure 8. Another Type of Trapezoid**

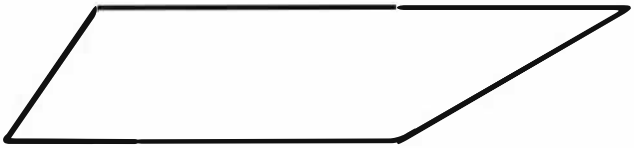


**Figure 9. A Family of Trapezoids with Nonstandard Ones**

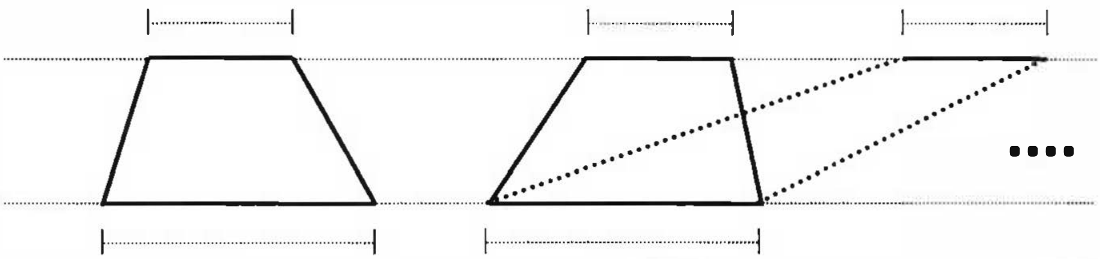
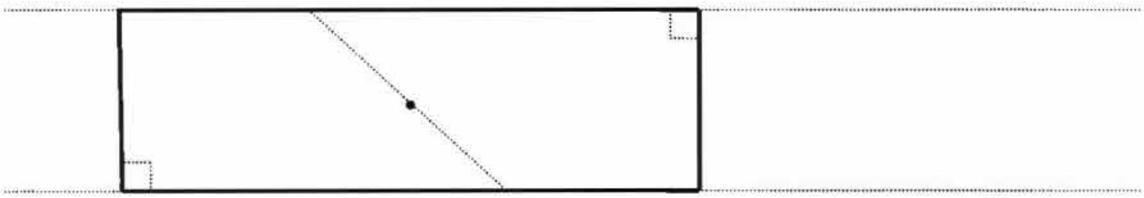


**Figure 10. Two Copies of a Right Trapezoid Creating a Rectangle**



height and a base equal to the sum of both the trapezoid's bases.

This helps establish a relationship between trapezoids and rectangles, where the trapezoid is seen as a half of its associated rectangle (with same height and a base being the sum of both trapezoid's bases).

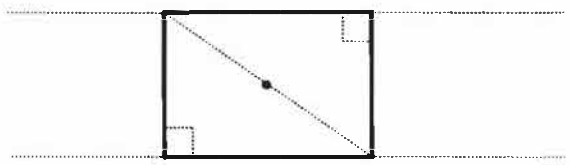
**Area of trapezoid** = ½ × **area of the *associated* rectangle (same height and base is the sum of the two bases of the trapezoid)**

↓

**Area of trapezoid** = ½ × (**Length of both bases** × **height**)

Like trapezoids, there are no rectangles in the triangle's family either (see Figures 4 and 5). But, again, it is possible to establish a relationship between the right-angled triangle and its associated rectangle, which is once more the double of it obtained as well by rotation about the midpoint of the hypotenuse (see Figure 11). As well, this rotation or double of the triangle can lead triangles to be called "half rectangles," a connotation that emphasizes the relationship between a triangle and its associated rectangle (see for example, Jamski 1978).

**Figure 11. Two Copies of a Right-Angled Triangle Forming a Rectangle**



At the level of formulas, this offers a specific conceptualization because the triangle is not defined anymore in regard to its own formula [$\frac{(B \times h)}{2}$] but mainly in its relationship with its associated rectangle [½ of rectangle]. Subtle as it may seem, it provides the occasion to draw out a strong link between the rectangle and all other planar figures mentioned above. The triangle is therefore defined in relation to its relationship with its associated rectangle; that is, as being half of the area of a rectangle that has the same height and the same base.

**Area of triangle** = ½ × **area of *associated* rectangle**

↓

**Area of triangle** = ½ × (**Length of base** × **height**)

The glimpse at some of the ideas presented above illustrates how the approach developed for conceptualizing more deeply the area of planar figures offers a different view. It can transform one's view of the area of planar figures far away from a calculational view (Thompson et al 1994) and from a collection of disconnected formulas. In addition to attempting to strengthen the existing link between figures and their formulas, this approach grounded these links in geometrical aspects, an approach different from a view focused on numbers and calculations for area.

However, this conceptual work was still incomplete. The project intended to work on a more concrete phase, structured around the construction of physical devices that would support the approach aimed at delving into the concept of area of planar figures. Thus, a third phase was designed and consisted of building devices and materials that would embody and support the mathematical ideas and issues developed in phase 2 of the project. I present a number of the designed devices below.

## Phase 3: Building the Supporting Devices[4]

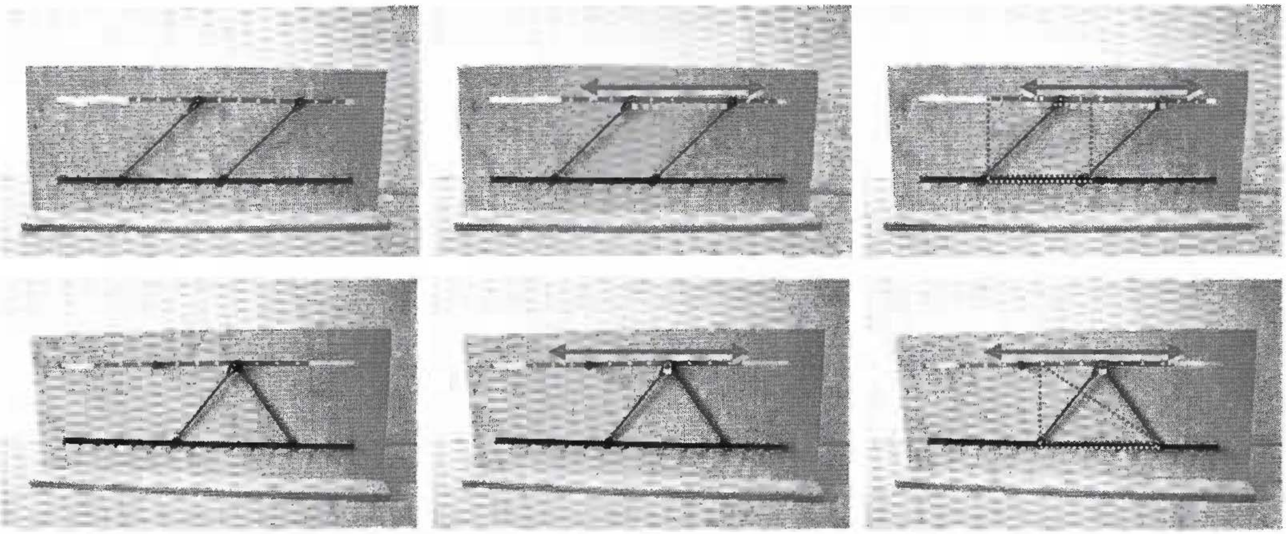
### Device 1: Variations Keeping the Area Constant

This device was designed to show, dynamically, that the area stays constant as one moves from left to right or right to left the upper base of the figure (for a quadrilateral) or the vertex (for a triangle). One important mathematical property that can be drawn out of this device is that for quadrilaterals, for example, any parallelogram with equal bases and equal height is of the same area independently of how slanted it is (in relation to Figure 3). It can also help to show that this family of equivalent-area parallelograms can be related to the rectangle with same base and same height. For the triangle, it also can be related to a right-angled triangle of the same base and same height (in relation to Figures 4 and 5). As well, the same could be represented for trapezoids in relation to Figures 6 and 9.

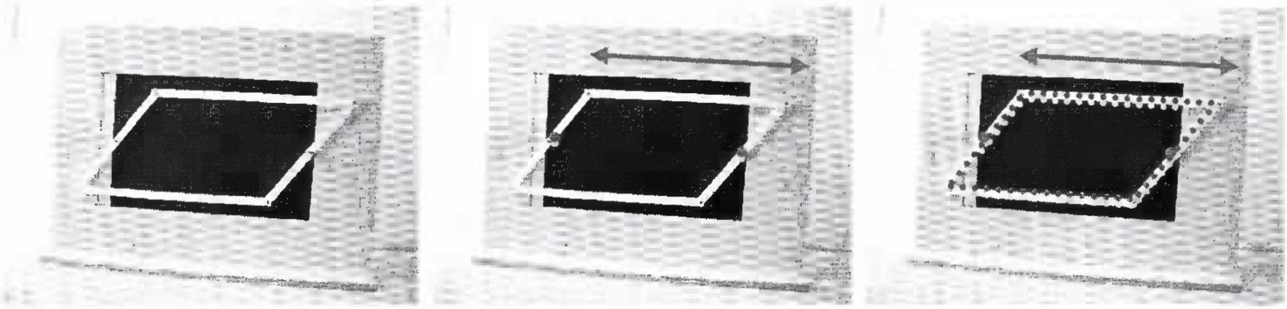
### Device 2: Variations Keeping the Perimeter, But Not the Area, Constant

This device was built to contrast with Device 1, where the variation changes the area while keeping the perimeter constant. Albeit similar work can be done with geostrips, the decrease in area can be shown in relation to the black rectangle drawn on the back of the device offering something to compare the new area with. Thus, as one varies the angles, the area of the rectangle decreases in comparison with the black-initial rectangle that had the same area as the white rectangle when all angles were at 90 degrees.

**Device 1: Variations Keeping the Area Constant**



**Device 2: Variations Keeping the Perimeter, But Not the Area, Constant**



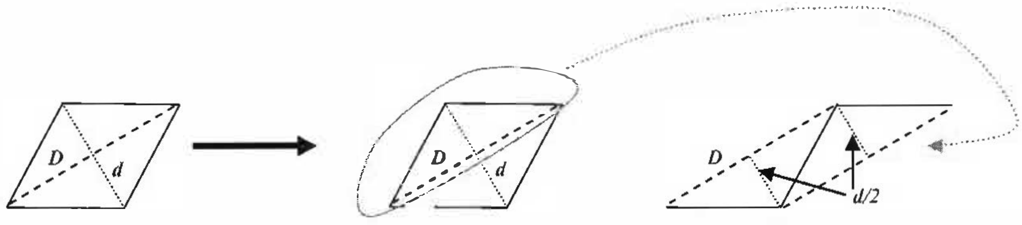
## Device 3: An Accumulation of Cross-Sections

Device 3 shows aspects of the Cavalieri principle in regard to cross-sections of equal length that can be taken from a figure (between a rectangle and a parallelogram) in relation to Figures 1 and 2. The stripes or cross-sections can be tossed aside to create a parallelogram (or another figure) that keeps the same area.

## Device 4: The Area of the Rhombus Calculated as the Area of a Parallelogram

Device 4 illustrates how to overcome the difficulty of considering the rhombus as a parallelogram in regard to the data given to calculate its area (normally its diagonals). It shows that a rhombus, for which one does not know the length of its side and thus cannot compute its height using the parallelogram/rectangle formula of base × height, can be reorganized to create a parallelogram for which the long diagonal is the base and half of the small diagonal is the height. This device intends to help link the rhombus with parallelograms and rectangles to continue contrasting with the need to opt for having a different and disconnected formula for the rhombus. See Figure 12.

**Figure 12. Reorganizing a Rhombus into Another Parallelogram, Using the Device Developed**



## Device 5: The Relation Between Trapezoids and Rectangles

This device illustrates the idea of Figure 10 in order to see, through doubling the figure, that twice a trapezoid constitutes a rectangle (or a parallelogram if the trapezoid is not a right-angled one). Hence, this uses the ratio of 1:2 between the trapezoid and the rectangle that has the same height and a base constituted of the sum of the trapezoid's bases, supporting the idea that the area of a trapezoid can be calculated in relation to the rectangle (again setting aside the need for another specific formula for the trapezoid).

## Device 6: The Relation Between Triangles and Rectangles

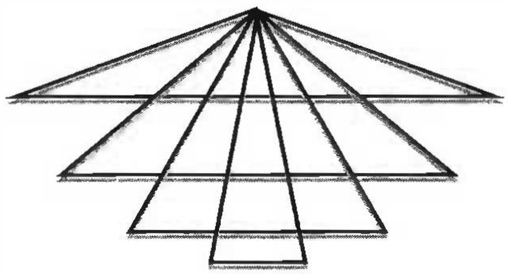
In the same vein as for the trapezoid, this device supports the idea of the triangle as half of the rectangle with the same base and same height (illustrated in Figure 11). This device, however, embodies a very specific case, the one with a right-angled isosceles triangle leading to a square (one could think of a variation in the triangle used, leading to various rectangles and parallelograms that would be the double of area).

## Device 7: Variations on the Triangle

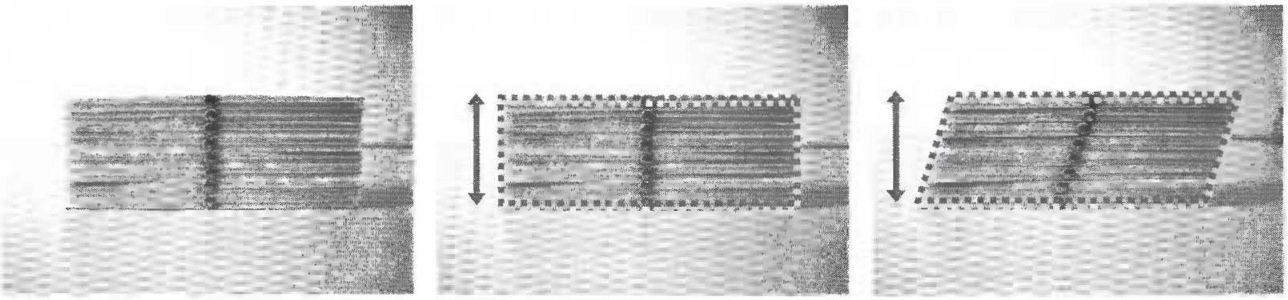
This device was built in the same spirit as Device 2, but for triangles to contrast with Device 1 that keeps the area constant in the case of the triangle. Here, it shows a different sort of family of triangles: isosceles triangles with the same equal sides and a third one that varies. This leads us to ask probing questions; for example, which triangle in the family has the bigger area? (see Figure 13 inspired by Avital and Barbeau 1991). What happens to a triangle in terms of area when only one side varies?

Thus, those devices were built to support the ideas put forth in Phase 2 of the project. Specifically, they were designed to draw out the geometric properties in regard to area through a dynamic interplay with the devices (changes, variations, constants, relationships

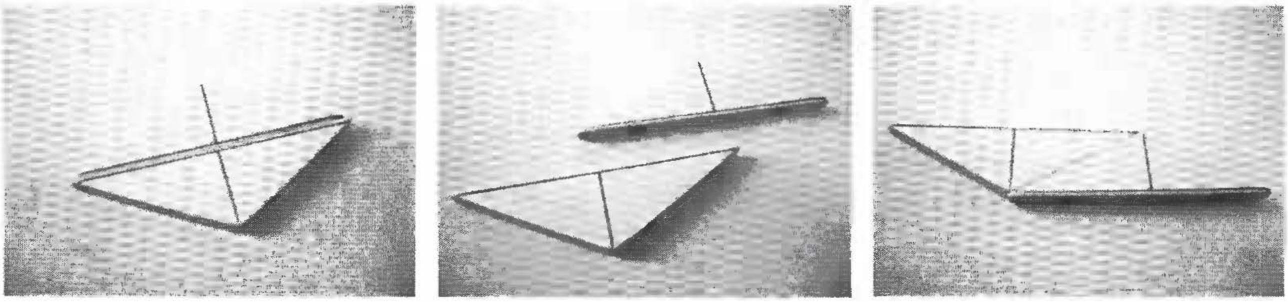
**Figure 13. Another Sort of Family of Triangles: The Isosceles Triangles Obtained Through a Variation of the Angle Joining Both Equal Sides**



**Device 3: An Accumulation of Cross-Sections**



**Device 4: The Area of the Rhombus Calculated as the Area of a Parallelogram**



between figures, families and so on), and supporting the "delving deeper approach" to area of planar figures developed.

## Final Remarks

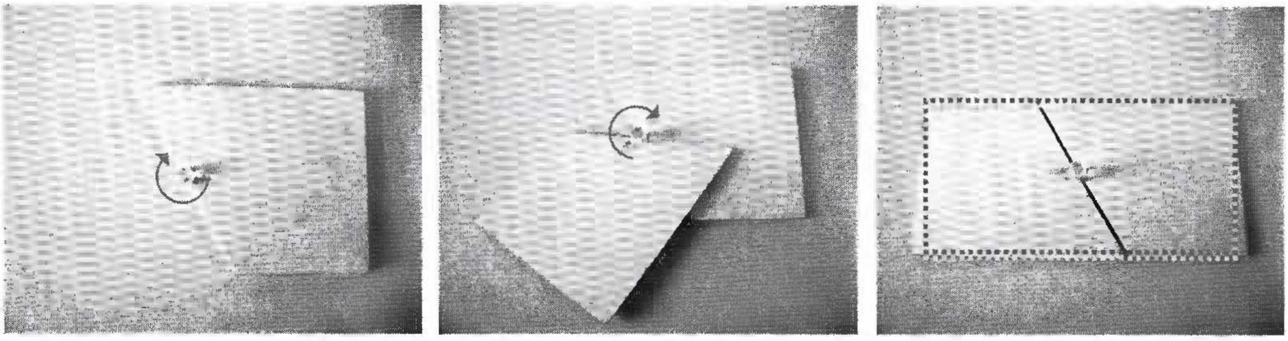
The development reported on about area of planar figures, specifically using the Cavalieri principle, brings forth the interrelations between the usually studied planar figures at a geometrical level. It also links these figures through their formulas, instead of having numerous area formulas to make sense of or memorize. Whereas the study of the many different formulas tends to isolate figures from each other, this geometrical approach helps to reunite them and stimulate significant reasoning for the concept of area.

This sort of work on delving deeper in mathematical concepts, where only the example of area of planar figures was offered, appears important to stimulate the development of rich mathematical reasoning that enables a deeper understanding of mathematical concepts. This said, continuous work must be conducted along those lines as area of planar figures appear only as one of many topics in school mathematics that can be delved into and from which richer mathematics can be brought forth. For example, similar work has been conducted in a project led by Elaine Simmt around systems of equations (for details on the work done and its outcomes, see Proulx et al 2008) as well as a project I have led on the study of trigonometry (reported in Proulx 2003).

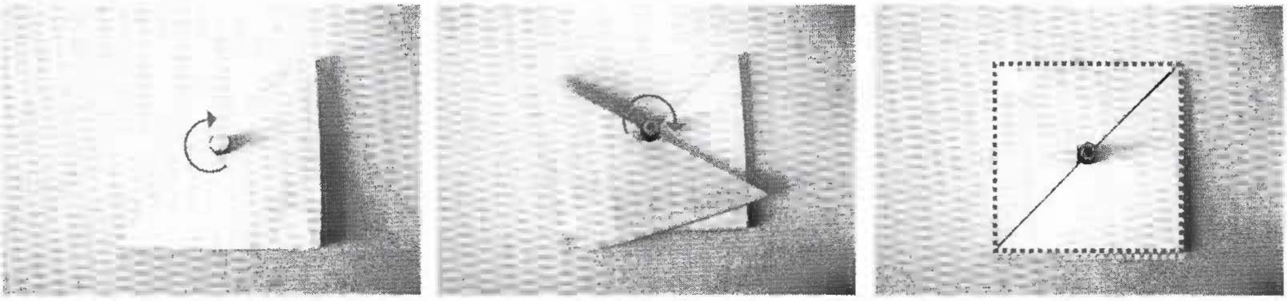
In each of these projects, the outcomes led to questioning the study of mathematical topics and the orientation they received in the curriculum. As mentioned, it is not the people who teach or learn mathematics who are problematic, it is mathematics itself. Thus, how can school mathematics topics be enriched and dug into deeper to revitalize the teaching and learning of mathematics in schools? Paying attention to the development of school mathematical concepts appears to be a fruitful line of inquiry in mathematics education in relation to the goal for continuing enhancement and revitalization of the teaching and learning of mathematics in classrooms.

Mathematics needs mathematicians to continue to evolve as a field of study. Perhaps what schools need are not mathematicians but school mathematicians: people who probe into the mathematics of the curriculum. Simply put, there is a lot of mathematics to delve into and develop within school mathematics. We need people to work intensely on these issues to enhance and revitalize the teaching and learning of mathematics in schools. This project has attempted to move toward that goal.

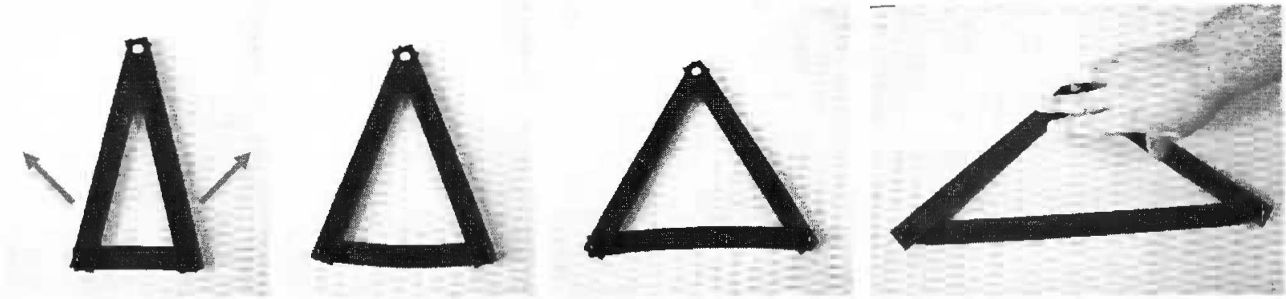
**Device 5: The Relation Between Trapezoids and Rectangles**



**Device 6: The Relation Between Triangles and Rectangles**



### Device 7: Variations on the Triangle



## Notes

1. A rapid Internet search concerning area formulas yields a horrifying list of numerous and different formulas for the area of planar figures, giving the impression that planar figures like rectangles, parallelograms, rhombuses and squares are very different figures requiring very different treatments concerning their area.

2. In this case, it is possible to transform the rhombus with long diagonal D and short diagonal d in a parallelogram of base D and of height d/2; a parallelogram then associated with a rectangle of base D and of height d/2.

3. I say "used to seeing" mostly because from looking at various textbooks and websites, rarely does one of them offer a picture different from the ones offered in Figure 7. It seems indeed that this type of trapezoid is not a current form that is often studied.

4. I need to acknowledge here the tremendous work done by one of my research assistants, Tom Hillman, in physically constructing these devices.

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