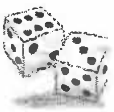
The Unusual Die: Exploring a ProblemThrough Technology

A Craig Loewen

I recently had a wonderful problem-solving experience that taught me a great deal about ways to explore problems, the role technology can play in exploring problems and the qualities of a truly intriguing problem. It started with this seemingly simple game that I introduced to a group of beginning teachers:



Kara is playing a dice game. In this game each player selects a die, rolls it five times and adds the values rolled. The player with the greatest sum wins. Kara may choose between two dice. One die is a regular six-sided die; the other die has three ones and three sixes on its six sides. Kara knows that the selected die must be used for the entire game, and she knows that her opponent must use the other die. Which die should Kara choose?

It is amazing how this problem tends to divide people into two groups: optimists and pessimists. The optimists will select the unusual die knowing it has three sixes and therefore the greatest chance of rolling a six. The pessimist will select the regular die given that the unusual die has three ones and the greatest chance of rolling a one.

This simple game is a wonderful problem to introduce to groups of people as it sparks a lot of good discussion. Of all the information given, what is the most important information, and what is it that makes the problem so difficult? To many solvers it is disconcerting that the player must stick with whatever die is chosen, and it is a bit confusing in that the player will make only five rolls but the die has six sides. In other words, no matter which die is selected only a maximum of five of the different faces on each die can possibly appear. But does this matter?

The number of rolls is really not a significant variable in this problem. After all, if the die is fair (unusual, but fair), each face has exactly the same probability of turning up on each roll (that is, 1/6). We know it is quite possible for the same face to turn up five times in a row, but we also know it is more likely that more than one face will appear within the five rolls.

How do we tie all this information together, or more to the point, how can we reasonably compare the dice?

A Solution

What we need to calculate is the average possible roll on each die, and then we can simply multiply that value by five to find the average score that die would likely produce.

The average roll is calculated as the sum of the values on the faces divided by the number of faces. For the regular die:

$$\frac{1+2+3+4+5+6}{6} = \frac{21}{6}$$

The average roll is 3.5 and the average score across many games with this die would be 17.5 (= 5 × 3.5). Likewise, the average score obtained with the unusual die is:

$$\frac{1+1+1+6+6+6}{6} = \frac{21}{6}$$

The unusual die also has an average roll of 3.5, so the average score over many games with this die will likewise be 17.5 (= 5 × 3.5).

In other words, it theoretically doesn't matter which die we pick. Ultimately, these two dice give the same results at least while playing this game.

Exploring the Problem

It was at this point that the solvers decided that they would like to try it for themselves; they were hesitant to give up their pessimistic or optimistic view of the unusual die. To conduct the experiment the teachers took two dice and rolled them, treating one die as the regular die and the other die as the unusual die (counting twos and threes as ones, and fours and fives as sixes). They kept track of which die seemed to win most often and after several games they began to share and compile their results. However, even with a large group there was still doubt whether enough examples had been generated to clearly prove which was the better die.

A Monte Carlo Simulation

I came prepared to show my students a Monte Carlo simulation, a way to help generate more examples quickly using a computer. In the case of this problem, I had simply programmed a computer to play this game repeatedly, keeping a tally of the winning die. As long as we can assume that the computer generates values in a manner comparable to that of rolling a die, the simulation should give similar results to playing the game many hundreds, even thousands of times.

The following program was entered into my TI-83+ calculator to simulate the game.

```
PROGRAM:DICE
:ClrHome
:Prompt M
:0→R:0→S:0→T
:Output(3,1,1234
56)
:Output(4,1,1116
66)
:Output(5,1,"TIE
S")
:For(A,1,M)
:0→D:0→U
:For(B,1,5)
:D+randInt(1,6)→
D
:randInt(1,2)→X
:If X=2:6→X
:U+X→U
:End
:If D>U:R+1→R
:If U>D:S+1→S
:If D=U:T+1→T
:Output(3,9,R)
:Output(4,9,S)
:Output(5,9,T)
:End
:■
```

To run the program all that needs to be entered is the number of times you wish the calculator to play the game, and the screen will display the results of the simulation, showing the number of times each die wins and the number of tie games. The following screen resulted from having the calculator play the game 10,000 times.

```
M=?10000

123456   4649
111666   4751
TIES     600
```

From these results it is easy to calculate that in this Monte Carlo simulation the regular die won approximately 46.5 per cent of the games, the unusual die won approximately 47.5 per cent of the games, and 6 per cent of the games resulted in ties. But, would we get the same results if we ran the program again, or if we had the computer play the game 100,000 times? To do that, we would need to use a much faster computer!

Going Deeper

After showing these teachers this simulation, one of them asked me this question: "Why is it that 6 per cent of the games are ties?" After a careful explanation of how it is possible for games to result in ties, the student re-asked his question: "Yes, I can see that it is possible for there to be ties, but what I don't understand is why it is 6 per cent of the games that would result in ties. Why not 5 per cent or 8 per cent or any other number?" This is a marvellous question! And frankly, I didn't know the answer! So, I began to play with the problem again, this time with the help of a spreadsheet.

To construct the spreadsheet I had to calculate all of the possible outcomes (final scores) for each die and the probabilities of obtaining those scores. For example, with the regular die the smallest possible game score is five, and that can be achieved only one way: rolling a one every time the die is tossed. It is also possible to get a score of six, seven, eight and so on all the way up to thirty. Just like there is only one way to roll a game score of five, there is only one way to roll a game score of thirty; there is a certain symmetry to the table of values. Similar computations were necessary for the unusual die. The table below shows all of the possible game scores and their probabilities for the regular die along the left edge, and all of the possible game scores and probabilities for the unusual die along the top.

		Score:	5	10	15	20	25	30	Sums:
		# Ways:	243	1215	2430	2430	1215	243	7776
		Probability:	0.031250	0.156250	0.312500	0.312500	0.156250	0.031250	1.000000
Score	# Ways	Probability							
5	1	0.000129	0.000004	0.000020	0.000040	0.000040	0.000020	0.000004	
6	5	0.000643	0.000020	0.000100	0.000201	0.000201	0.000100	0.000020	
7	15	0.001929	0.000060	0.000301	0.000603	0.000603	0.000301	0.000060	
8	35	0.004501	0.000141	0.000703	0.001407	0.001407	0.000703	0.000141	
9	70	0.009002	0.000281	0.001407	0.002813	0.002813	0.001407	0.000281	
10	126	0.016204	0.000506	0.002532	0.005064	0.005064	0.002532	0.000506	
11	205	0.026363	0.000824	0.004119	0.008238	0.008238	0.004119	0.000824	
12	305	0.039223	0.001226	0.006129	0.012257	0.012257	0.006129	0.001226	
13	420	0.054012	0.001688	0.008439	0.016879	0.016879	0.008439	0.001688	
14	540	0.069444	0.002170	0.010851	0.021701	0.021701	0.010851	0.002170	
15	651	0.083719	0.002616	0.013081	0.026162	0.026162	0.013081	0.002616	
16	735	0.094522	0.002954	0.014769	0.029538	0.029538	0.014769	0.002954	
17	780	0.100309	0.003135	0.015673	0.031346	0.031346	0.015673	0.003135	
18	780	0.100309	0.003135	0.015673	0.031346	0.031346	0.015673	0.003135	
19	735	0.094522	0.002954	0.014769	0.029538	0.029538	0.014769	0.002954	
20	651	0.083719	0.002616	0.013081	0.026162	0.026162	0.013081	0.002616	
21	540	0.069444	0.002170	0.010851	0.021701	0.021701	0.010851	0.002170	
22	420	0.054012	0.001688	0.008439	0.016879	0.016879	0.008439	0.001688	
23	305	0.039223	0.001226	0.006129	0.012257	0.012257	0.006129	0.001226	
24	205	0.026363	0.000824	0.004119	0.008238	0.008238	0.004119	0.000824	
25	126	0.016204	0.000506	0.002532	0.005064	0.005064	0.002532	0.000506	
26	70	0.009002	0.000281	0.001407	0.002813	0.002813	0.001407	0.000281	
27	35	0.004501	0.000141	0.000703	0.001407	0.001407	0.000703	0.000141	
28	15	0.001929	0.000060	0.000301	0.000603	0.000603	0.000301	0.000060	
29	5	0.000643	0.000020	0.000100	0.000201	0.000201	0.000100	0.000020	
30	1	0.000129	0.000004	0.000020	0.000040	0.000040	0.000020	0.000004	
Sums:	7776	1.000000							

Probability of Normal Die Winning (Sum of shaded boxes):	0.47130
Probability of Unusual Die Winning (Sum of non-shaded boxes):	0.47130
Probability of Ties (Sum of blackened boxes):	0.05740

The values in the middle of the table show all of the probabilities of the various game outcomes; that is, the product of the probabilities of the given final scores. For example, there is a probability of 0.3125 that the final score with the unusual die will be 15, and there is a probability of 0.100309 that the final score with the regular die will be 17. Therefore, the probability of that particular result (a score of 15 and 17 on the respective dice) is 0.031346.

The values in the boxes that are shaded show all of the probabilities related to game outcomes where the normal die wins. Likewise the values in the nonshaded boxes show all of the probabilities of game outcomes where the unusual die wins. The values in the blackened boxes show the probabilities of tied game outcomes.

There are lots of patterns of symmetry in this table, but a quick inspection shows that there exists the same number of ways for the unusual die and the regular die to win, and that there are six ways for the game to result in a tie. Further, by separately summing the values in the shaded boxes and nonshaded boxes (I had the spreadsheet do this), we see that the probability of each die winning is about 47 per cent of the time. More importantly, by adding the values in the blackened boxes, I could show that the probability of tie games is about 5.7 per cent!

Creating a table like this would be almost impossible without the aid of a spreadsheet. There are just too many values to tabulate and compute, and it is likely that too many errors would be made. The spreadsheet however does this accurately and quickly, and provides a credible answer to the question that started the exploration.

Looking Back

Solving this problem with these teachers helped to remind me of some of the most important qualities of a good problem-solving experience.

First, a good problem-solving experience should be well rounded; it should involve exploration (preferably a hands-on component where possible), the asking of why, and the opportunity to explore the problem on many levels. This problem provided each of these when the teachers first picked their favourite die, conducted an experiment, compiled and compared their results, and then began to challenge the conclusions.

This particular problem also provided an opportunity to integrate technology into the search for a solution, including both a Monte Carlo simulation and the creation of a complex spreadsheet. The simulation provided a way to test our hypotheses and the spreadsheet aided in making very complicated calculations that would otherwise be too tedious or difficult.

The most important outcome though was this: it reminded me yet again that in most problem-solving experiences the answer is much less important than the process of solving the problem. The process we engaged in brought enjoyment, debate and a realization of the power of technology, and above all else, stimulated a question arising from genuine curiosity. These are important hallmarks of a successful problem-solving experience.

Extension Problems

1. If you need to roll a value of 20 or greater in five rolls, with which die will you have the best chance?
2. How do you calculate the number of ways each game outcome can be reached? In other words, how do we know there are only 15 ways of getting a score of 7 with a regular die in this game?
3. Design a third die and compare it to the two dice used in this game. Does your die improve your odds of winning? How do you know? Design a different unusual die that has the same chance of winning as a regular die.
4. Playing the same game, assume you may pick between a regular four-sided die, and an unusual four-sided die that has two ones and two fours on it. Which die would you pick? Why?

A Craig Loewen is a professor of mathematics education at the University of Lethbridge, where he has been teaching for 23 years. He is particularly interested in acts of problem solving, building problem-solving competence and the role that technology can play in the solution of many different kinds of problems. His other areas of interest include the use of manipulatives and manipulative-based games in enhancing both the understanding and enjoyment of mathematics. In his spare time, Loewen enjoys music and woodworking in his small home shop.