An Exploration of Per Cents and Fractions Through a Study of Fractals

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In the winter of 2011, four classes of Grade 8 students at the Calgary Girls' School, a public charter school for girls in Calgary, Alberta, explored the fascinating world of fractals. Alberta's *Mathematics Program of Studies, Grades K–9*, states that students are to "gain an understanding and appreciation of the contributions of mathematics as a science, philosophy and art" (Alberta Learning 2007, 4). At the Calgary Girls' School, significant efforts are made to connect students' mathematics learning to other subject areas to help students see the applications of concepts addressed in class. What follows is a discussion of an integrated project involving fractals that sought to develop students' understanding of several mathematics concepts, while simultaneously requiring them to call on their emerging understanding to create fractal art.

Two specific outcomes listed in the number strand of the *Mathematics Program of Studies, Grades K–9* (2007) at the Grade 8 level are as follows:

- Demonstrate an understanding of percents greater than or equal to 0%, including greater than 100%.
- Demonstrate an understanding of multiplying and dividing positive fractions and mixed numbers, concretely, pictorially and symbolically.

When faced with the task of developing an engaging learning activity intended to assist students in meeting these outcomes, the Grade 8 mathematics teachers at the Calgary Girls' School saw an opportunity in the study of fractals. Fractals are repetitive geometric patterns that go on infinitely; they consist of increasingly smaller and smaller shapes that are similar to preceding shapes in the fractal pattern. Objects that are similar have the same shape, but different size, and the same interior angles. The Sierpinski triangle (see Figure 1), for example, consists of increasingly smaller, similar equilateral triangles.

Preceding Work

Prior to our work re-creating fractals, students developed their ability to solve authentic problems involving percentages with the assistance of percentage benchmarks (see Figure 2). For example, students used their emerging understanding of benchmarks to calculate 30 per cent of $80.

Figure 2. Percentage-benchmark example

Please find 30% of $80.

÷5

	50%	10%	30%
$80	$40	$8	$24

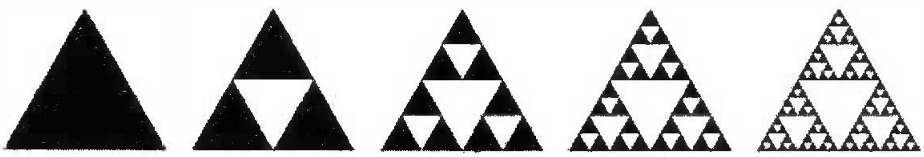
÷5 ×3

10% of $80 is $8. 10% × 3 = 30%

so, $8 × 3 = $24

30% of $80 is $24.

Figure 1. The Sierpinski triangle



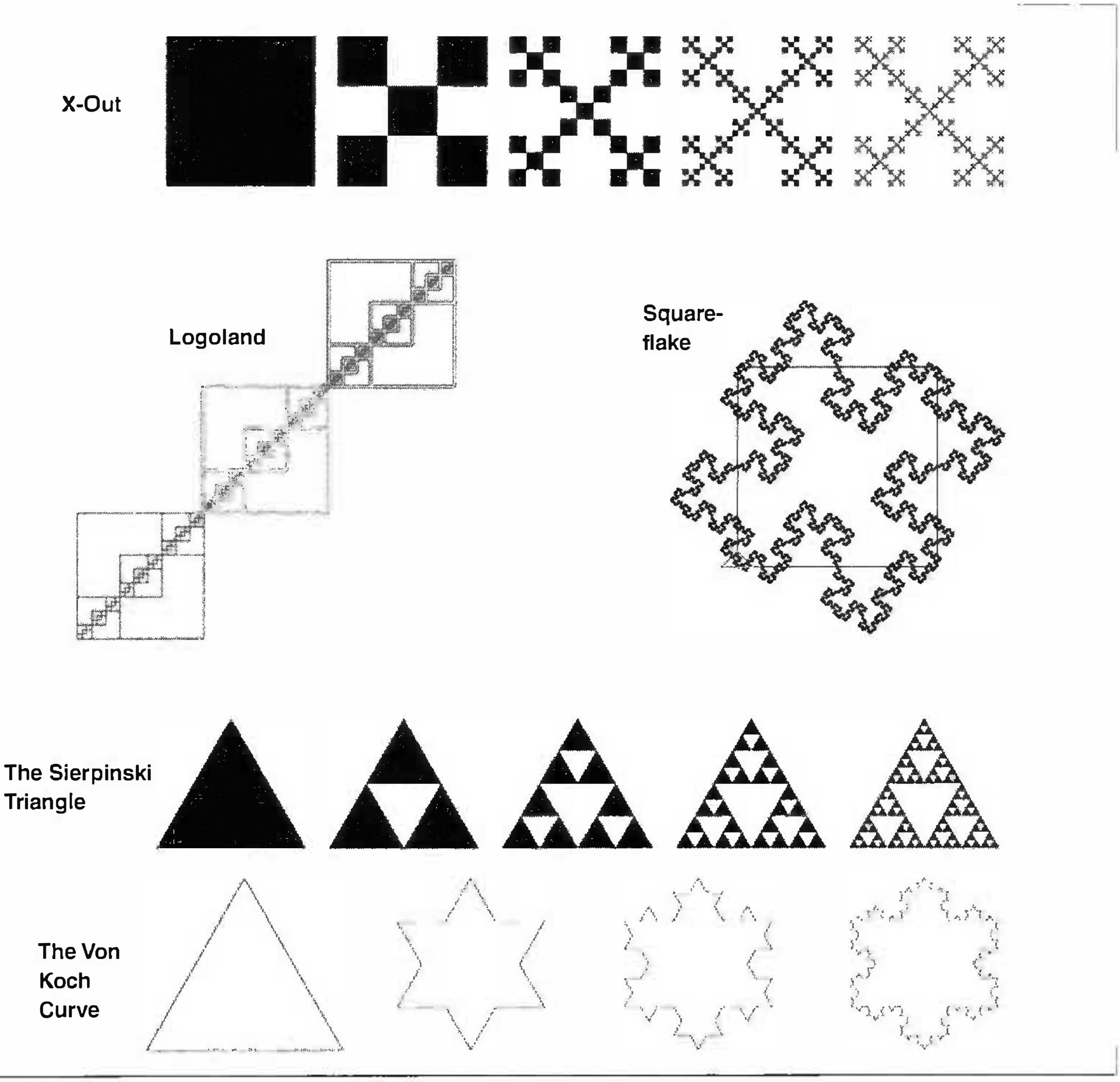
The Fractal Patterns

For our study of fractals, we selected five fractal patterns for students to analyze, investigate and ultimately re-create. We referred to them as X-Out, Logoland, the Sierpinski triangle, the Von Koch curve and Squareflake (see Figure 3).

These fractal patterns exhibited differing levels of complexity. The Sierpinski triangle and Von Koch curve consisted of similar equilateral triangles; each of the remaining three fractals consisted of similar squares.

After selecting these fractal patterns, the Grade 8 teachers placed students in each class of 25 into groups of two to three students, then assigned each group a specific fractal pattern whose complexity aligned with the collective attributes of those in the group (collaborative skill, fine-motor skill, perseverance, interest level and so on). Student groups were then given graph paper, pencils, rulers, right angles and protractors, and were required to work collaboratively in an effort to re-create a rough design of their assigned fractal. It should be noted that at the Calgary Girls' School, great emphasis is placed on collaborative group work, and students are regularly expected to engage in relevant dialogue with one another as they seek to construct their own meaning of the various mathematics concepts addressed in class.

Figure 3. Fractal patterns



Mathematical Concepts Addressed

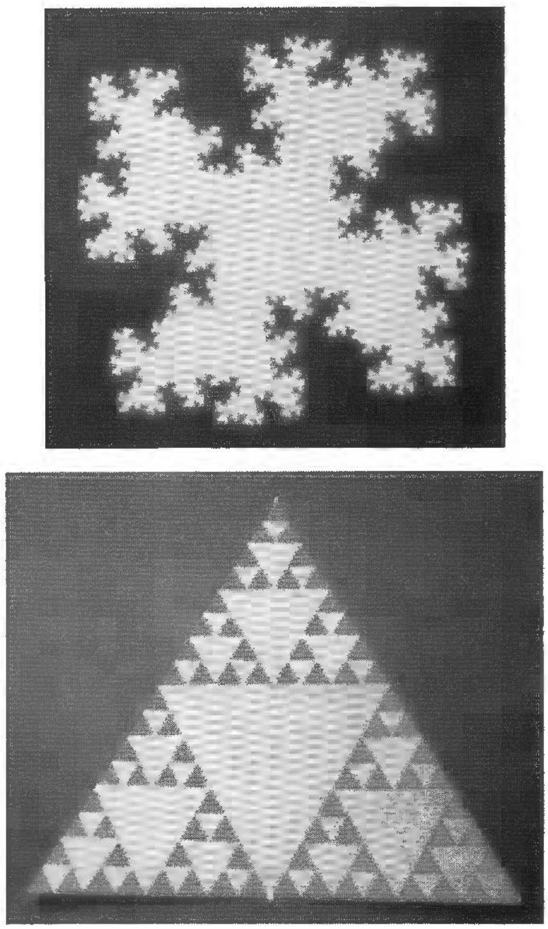
Once student groups were confident in their ability to re-create their assigned fractal pattern, they met with their mathematics teacher to share sketches of and dimensions for necessary shapes. For example, in meeting with their teacher, the majority of students who re-created the X-Out fractal-pattern first shared their observation that the largest square in their pattern required dimensions that were a multiple of three, as each subsequent square in the pattern had dimensions that were one-third that of the preceding square's dimensions. Most students in the X-Out groups went on to share their observation that the dimensions of the square in the first stage of their pattern needed to be a fairly large multiple of three or else they would quickly end up with decimal-number dimensions after several reductions in size, which many students wanted to avoid.

During these meetings, students in each group also worked with their teacher in developing their understanding of percentages, scaling, converting fractions into equivalent percentages and vice versa, as well as the multiplication of whole numbers by fractions. Students were asked to identify the key fraction at play in the re-creation of their assigned fractal (for example, 1/3 for X-Out) and to convert this fraction into an equivalent fraction with a denominator of 100 (the fraction's equivalent per cent). Students also pictorially modelled whole number by fraction multiplication pictorially by first identifying the dimensions of their given shapes on graph paper then calculating the dimensions of subsequent shapes in their fractal patterns after scaling down by some fraction. As an example, one X-Out group started their fractal pattern with a 27 × 27 cm square, then figured out the dimensions of the next squares in their pattern by calculating 1/3 of 27 cm. Last, students in these meetings were required to use their understanding of percentage benchmarks to calculate the length and width of a shape whose dimensions were either 50 per cent or 200 per cent that of some given shape. As an example, students re-creating the Sierpinski Triangle were asked to identify the side length of an equilateral triangle whose dimensions were 50 per cent that of an equilateral triangle whose side length was 32 cm; they were then asked to use benchmarks to figure out the dimensions of this shape if a scale factor of 200 per cent were used instead.

Building the Fractals

After meeting with their teachers, students in each group used rulers, right-angle tools, protractors, pencils, construction paper of varying colours, scissors and glue to assemble their assigned fractal pattern. While this phase of the work was quite time-consuming and could certainly be sped up with the use of technology, requiring students to actually measure, draw and cut out the shapes needed for their fractal re-creation enhanced their measurement skills and developed their appreciation for the exponential growth in the number of shapes at each successive stage of their fractal. One of the groups re-creating the Squareflake found the process too tiresome and pursued the re-creation of a simpler fractal pattern, while another Squareflake group was presented with the same opportunity, but chose to soldier on, later finding themselves highly satisfied with their finished product (see Figure 4). A few students became visibly frustrated when re-creating shapes in the later stages of the fractal pattern, but spoke with confidence about both the exponential growth and the patterns present within fractals.

Figure 4. Student fractal re-creations: Squareflake and Sierpinski triangle



Formative and Summative Assessment

To assess students' understanding of concepts addressed in our work with fractals, we required each of them to take part in a one-on-one interview/performance assessment with their mathematics teacher. It should be noted that though students at the Calgary Girls' School complete a great deal of work in collaboration with their peers, group work is never assessed as such. Instead, students are assessed independently following collaborative group-work projects and activities. During the interview/performance assessments, students were presented with graph paper, a right angle, a protractor, a pencil and a ruler, and were asked to show how they re-created their fractal, discuss the reason behind their choice of dimensions, explain what made their pattern a fractal, and identify which fraction/percentage was applied in the re-creation of their pattern and how. Students were also asked to use benchmarks to identify the length and width of a square whose dimensions were 25 per cent or 300 per cent that of a 12 × 12 cm square. Although these interviews were time-consuming, the insight they provided into students' mathematical reasoning and conceptual understanding were well worth the effort. As a result of their work in re-creating fractals, it was apparent that many students not only solidified their understanding of percentage benchmarks but also developed their proficiency in working with fractions and their ability to identify patterns.

Along the way to these summative one-on-one assessments, students completed several formative assessments known as thinking-logs (see Figure 5), in which they were required to use benchmarks to solve authentic problems involving percentages. On these thinking logs, students were also required to assess their own understanding of the concept at hand by identifying themselves as red, yellow or green (see rubric in Figure 5 for additional details).

Figure 5. Percentages thinking log

Tom's Shoes Thinking-Log

TOMS

Shoes for Tomorrow

A pair of Tom's shoes originally-priced at $50 are on sale for 20% off. GST of 5% will be charged on the purchase of the shoes. How much will the shoes cost?

Please explain how you went about solving this question:

Please describe any challenges, if applicable, that you encountered in solving the question:

Identify your level of understanding by circling beside each question (How confident do you feel about your answers to these questions?)

[redacted] *Not confident*	YELLOW *Somewhat confident*	[redacted] *Confident*
• I don't get it • I don't know where to start • I cannot explain my answer • I need help • I am confused	• I am on my way to understanding • I can start but have some questions along the way • I need some help to explain my answer • I think my answer is correct but would like it to be checked	• I get it • I can solve it by myself • I can explain my answer • I feel confident in helping others

Curriculum Connections at Other Grade Levels

Although this project was given to students in Grade 8, it might make for a good fit in Grade 9, where students are required to "demonstrate an understanding of similarity of polygons" (Alberta Learning 2007, 152). Moreover, fractal patterns could do much to illustrate the nature of exponential growth to Grade 9 students. For example, the Sierpinski triangle in Figure 1 consists of 1 equilateral triangle in white in the second stage of the fractal, 3 smaller equilateral triangles in white in the next stage of the pattern, 9 smaller triangles in the third stage, 27 in the fourth stage and so on. Hence, the Sierpinski triangle provides an authentic example of the 1, 3, 9, 27 base-3 exponential pattern. The number of shapes at each successive stage of the X-Out fractal pattern is as follows: 1, 5, 25, 125 and so on (base-5 exponential pattern). Congruent shapes in several of the fractals discussed in this paper can also be related through translations, reflections and rotations, and so, this project could be adapted to fit an exploration of transformational geometry in any mathematics class at just about any level.

Although fractals are not explicitly identified as a topic of study in the Alberta program of studies for mathematics, an investigation of these complex, visually appealing mathematical patterns presents the opportunity to explore a wide variety of concepts. Our own particular exploration addressed scaling and the use of percentages greater and less than 100 per cent, converting fractions into equivalent percentages and vice versa, as well as the multiplication of fractions by whole numbers. This specific study was welcomed by both teachers and students alike for its hands-on nature and its ability to place students at the centre of their mathematics learning. Our investigation of fractals led to much discussion and the exploration of numerous other concepts (exponential growth, pattern recognition and extension, measurement, angles, similarity and congruence), which we hadn't planned on addressing with this work, yet were brought up along the way by our curious group of young mathematicians.

References

Alberta Learning. 2007. *Alberta K–9 Mathematics Program of Studies with Achievement Indicators.* Government of Alberta website. http://education.alberta.ca/teachers/program/math/educator/progstudy.aspx (accessed April 8, 2012).

Michael Jarry-Shore has taught mathematics and science for seven years to Grades 7–9 students in at the Calgary Girls' School. He is currently enrolled as a graduate student at McGill University, where he is working to complete his master's degree in education. The focus of his research is preservice teacher education in the area of mathematics. He is particularly interested in exploring the acquisition and development of mathematical knowledge for preservice elementary teachers. He is also interested in the establishment of formal coaching initiatives intended to support teachers new to the profession.