THE OLD TEACHER AND THE NEW MATHEMATICS

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Editor's Note: William F. Coulson was on the staff of the University of Alberta during the 1962-63 session. He was editor of the MCATA Newsletter and a tireless worker on the MCATA executive. His address was given to the third annual meeting MCATA.

When New is Old

Before anybody gets too excited, let me do what every good "modern" mathematician does. In the title of this paper, I have used two familiar words. As adjectives I used the word "old" and the word "new". By putting the two words "old" and "teacher" together I am not implying that the teacher is ready for retirement to the rocking chair. My use of the word "old" here is simply a device to denote those individuals who received their preparation for teaching and who started teaching prior to the great rash of agitation for "modern" mathematics. The mathematics courses taken by these people, both in high school and post high school, were traditional in nature. Nor does this mean that they are any less effective as a teacher. It simply means that their academic background in a specific subject matter discipline is out of step with current thinking.

Let's take a quick look at the second of the two words I identified as special earlier. This is the word "new". I'm not trying to imply that the new mathematics is "younger" than the "old" teachers. With a disgustingly great degree of unanimity, this mathematics is older than any of us here in this room. By "new", I'm referring to those mathematical ideas that, until the last half decade, have met a closed door when they tried to be nice and respectable about getting into the senior high school mathematics curriculum. About five years ago all doors were opened wide and look what blew in. Every kind of idea conceivable was put into some kind of text by some kind of author dealing with some kind of publisher.

Christopher Sails the Blue

It has been said that if Columbus were to have the misfortune of being thrown out of heaven and sent back to earth he would find very little change in the mathematics curricula between 1492 and 1958. This may be a true statement, but, at any rate, very little of the mathematics developed in that span of time has been able to have any influence upon the high school curriculum. The "new" mathematics to which I am referring was developed by some free thinkers during the past three centuries.

Looking at the textbooks being published today, a forward-looking, non-ostrich type of teacher will be able to see some very plain handwriting on a very clean wall. This handwriting says, in no uncertain terms, that changes are being made. Who among us is going to say that just because the horse and buggy was good enough for grandpa, it's good enough for me. Society, industry, and life in general have progressed. Why not the senior high school mathematics curriculum?

Some very able mathematicians were educated in the past few decades. They were educated to fill a need and they were able to modify themselves to fill a still greater need. Is the need the same as it was thirty years ago? I say "no". We no longer need human computers. We need human thinkers who can tell a computing gadget what to do.

Mathematical Skeletons

How then must the "old" teacher change to fit into place with the "new" mathematics? First, this teacher must be able to think in terms of the structure of mathematics. Jerome Bruner, in The Process of Education, says that

> Algebra is a way of arranging knowns and unknowns in equations so that the unknowns are made knowable. The three fundamentals involved in working with these equations are commutation, distribution, and association. Once a student grasps the ideas embodied by these three fundamentals, he is in a position to recognize wherein "new" equations to be solved are not new at all, but variants on a familiar theme.

Thus, we see emphasis on structure, the building up from the foundation.

Let's take time out to consider a few specific examples. When we ask students to learn to solve equations, do we go back to what he had learned from his arithmetic? No! We take him to a pan balance and ask him to add to or subtract from the amount in each pan so that both pans are kept in balance. If the student is to find the value for which x is holding the place in the sentence, $x + 5 = 12$, we neglect the fact that this is related to a basic fact from second grade addition. He knows immediately that x is holding the place for 7. We make him go through a long rigamarole trying to find something that is perfectly obvious to him. Our time would be better spent asking the student to rename the 12 so that 5 is in the new name. This would give $12 = 7 + 5$. From this the student can see that the value of x is 7.

This idea can be applied by the student to the equation $x + 348 = 5623$. It is possible to rename 5623 so that 348 is a part of the new name? We can find the other part of the new name by first subtracting 348 from 5623 and find that the new name is 5275 + 348. After practising on a few more examples similar to these the student will be able to state the familiar generalization for the solution of equations of this type. Also, the student has now built a very simple structure.

Another example in which we ignore structure and in which the "old" teacher may have to modify habit is in the realm of the operations with signed numbers, or integers. Previously, these students have learned some fundamental ideas. They know how to operate with numbers zero or greater. Putting this with the idea of $a + x = 0$, where $a > 0$, what is the value for which x is holding the place, we can build all of the operations with integers. In the equation, $a + x = 0$, if $a = 8$, then x must equal -8. If $a = 12$, then $x = -12$. In other words, 8 and -8 are additive inverses of each other; 12 and -12 are also additive inverses of each other.

What is 8 + (-5)? First the student must go back through the mental filing system he has developed to find the pieces of information which relate to this operation. One is that he can rename 8. Another is that $a + (-a) = 0$. Using these two concepts the student is now able to find a new name for this rather complex set of symbols. 8 is equal to 3 + 5. Thus the original phrase becomes (3 + 5) + (-5). Applying the associative property of our number system we get 3 + (5 + (-5)). But we know that (5 + (-5)) is another name for zero. Thus, 3 + (5 + (-5)) = 3 + 0 = 3. It won't take long for students to state the usual generalization for the addition of integers.

Let's go to multiplication. Traditionally, there has been a rather cumbersome attempt to develop this operation in the integers. All kinds of real life situations are thrown in with no significant degree of success. Looking at our structure, we find several ideas available to us. Basic are the elementary facts for multiplication and $a \cdot 0 = 0$. Zero can be represented by a vast number of different sets of symbols. For purposes of this illustration, I will arbitrarily choose one of the symbols; 7 + (-7) = 0.

We know that (5) (0) = 0 and from this we know that (5) (7 + (-7)) is equal to zero. The distributive property can be used here to multiply; (5) (7 + (-7)) = (5) (7) + (5) (-7) = 0 (5) (7) is known to equal 35. Thus 35 + (5) (-7) = 0. What must (5) (-7) equal? For all of the other properties of our structure to hold (5) (-7) must be equal to -35. A few more such examples will bring out the usual generalization.

Another example is (-5) (7 + (-7)) = 0. Applying the distributive property as before -

$$(-5)\ (7 + (-7)) = (-5)\ (7) + (-5)\ (-7) = 0.$$

But, we just indicated that (-5) (7) = 35. Then -

$$(-5)\ (7) + (-5)\ (-7) = -35 + (-5)\ (-7) \text{ must equal } 0$$

if all of the previous properties are to hold. Again, after a few examples, the usual generalization may be stated.

One further example from algebra before we turn to a quick look at geometry. The idea of two unknowns is a part of the mathematics course for the high schools. In my opinion this is very poorly done. Again, do we follow or attempt to develop a structure? Basically, no. We make the poor student jump in without even a straw to grasp let alone a life preserver. No attempt is made to analyze completely one sentence in two unknowns before the second one is put with the first.

If x + 6 = 7, what can we say? This is a set of ordered pairs of numbers. The numbers may be natural numbers, integers, rational numbers, or real numbers. Each gives us a different set of pairs. Each of you can think of quite a few possible solutions. If replacements are confined to the set of natural numbers, there are only six pairs of numbers which can satisfy this condition. These may be represented on the Cartesian co-ordinate system as six distinct points.

If our replacement set is any of the others, we have larger sets until the set of real numbers is reached. This set would give us an infinite set of ordered pairs which result in a straight line when graphed.

Traditionally, no attempt has been made to analyze each individual equation thoroughly before it is put with a second one and the student is asked to find the intersection of these two sets of ordered pairs.

Now, for a quick look at geometry. We always start out by saying that a point and line are intuitive notions and that they are the simplest of the geometric figures. But, then what do we do next? All of a sudden and for no obvious reason we talk about a figure involving a minimum of three lines and three points. Does this make good sense? What happened to those figures involving fewer than this? For instance, a point and a line, two lines, two lines and one or two points. Our structure is not very well built, is it?

Now to take a closer look at some of the geometric figures which are made up of fewer parts than is the triangle. Not many statements can be made about a point and a line taken as one figure. The point is either on the line or it is not on the line. Next, consider two lines. Immediately, several distinct possibilities arise. The lines intersect or they do not intersect. If they do, and the only point in common is the end point of each of the two lines, an angle is formed. A number of statements about conditions can be made. If the two lines intersect at the end point of one and some point other than the end point of the other, we have adjacent angles, supplementary angles, or possibly perpendicular lines. Still more statements about conditions can be made.

If the lines do not intersect, they may be a parallel. Still more statements. Now we can add a third line. I don't think I need to go into any more detail about the statements that the students can develop from this situation.

As can be seen, a very good geometry course could be built using no text. The teacher must have done some very careful studying so that each comment by the pupils can be analyzed for its full value in the sequence of statements. The teacher's main function here is to guide and direct the learning activities of the students.

Creative but Old

I hope I have not strayed from the topic as listed in the title. I have attempted to relate the "old" teacher and the "new" mathematics. The second element to be considered follows from what I said about teaching geometry. The teacher must show a creative, inquisitive attitude. This will make the subject come alive for the students. At a conference on the Canadian High School held in Banff last month, a number of the speakers hammered away at the idea of "teaching for creativity". Before one can teach for creativity, one must practise this himself. Irrespective of departmental examinations, one must permit the student to build ideas for himself without being put in harness and driven along a pre-determined path.

I'm begging the "old" teacher to remember that he or she is a teacher and that a modification of ideas and habits is not impossible. It may be extremely difficult, but the spirit of learning and thinking anew will help one leap over a number of hurdles.