# A MODERN APPROACH TO EUCLIDEAN GEOMETRY

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Editor's Note: Dr. Dulmage is the newly appointed head of the Mathematics Department at the University of Alberta. He brings a broad experience with him from the University of Manitoba and the Royal Military College at Kingston. Those who attended the fourth annual meeting were impressed with his intimate knowledge of mathematics education, as is shown in the article which follows.

At the turn of this century the mathematician David Hilbert wrote his Foundations of Geometry in which he developed plane Euclidean geometry as an abstract mathematical system. The undefined elements of this mathematical system are point and line and the undefined relations are incidence, betweenness and congruence. Most of the new geometry texts which are currently being written for our schools are more or less faithful imitations of Hilbert's book. In this talk I want to make some remarks about this modern approach to Euclid.

In introducing the student to geometry in school there are two almost contradictory paths which should be followed. In the beginning in junior high school the student's spatial intuition should be developed. The dots and marks which he makes on paper - which he calls points and lines - must have some kind of physical reality for the student. It is in exactly this way that geometry developed historically. It must be pointed out, however, that there is no satisfactory physical definition of a point or a line. In junior high school the student should discover, at least in some psychological sense, those things which he will be later asked to assume as axioms.

The high school student should be introduced to geometry in the Hilbert manner as a mathematical system in which point and line are the undefined elements. It is important to try to explain to the student that the reason why point and line are undefined elements is not merely because it is impossible to define them as physical entities in our world but rather because every mathematical system has undefined elements and one or more undefined relations. The same situation prevails in arithmetic or in algebra. And yet these undefined elements and undefined relations are really undefined initally only. The thing we do immediately after stating the names of or symbols for the undefined elements is to write down certain axioms involving them, and as soon as an axiom is stated, the undefined elements are no longer undefined. They are partially defined in an abstract sense by the axiom itself. The more axioms we state, the more we define the undefined elements and the relations. In fact, it is our goal in constructing a geometry as an abstract mathematical system to state sufficient axioms that the elements are completely defined in the sense that there is essentially only one set of points and lines satisfying them. When this goal is achieved, we have then given a complete abstract definition of the original undefined elements. The very important point of view to impart to the student in high school is that in proving theorems he must not make use of a property of the points and lines unless this property has been assumed as an axiom or has been proved as a theorem from these axioms. The properties which he feels that points and lines should have as a result of his spatial intuition in junior high school must, in high school, either be assumed as axioms or proved as theorems from the axioms. In developing Euclidean geometry it is necessary to introduce only three relations. These are incidence, betweenness and congruence. You will see references also to parallelism and continuity, but the parallel axiom of Euclidean geometry can be described in terms of the incidence relation; continuity can be described in terms of betweenness.

## The Incidence Relation

Instead of saying, "The point P is on the line p ", or, "The line p passes through the point P", we say, "The point P and the line p are incident." Instead of saying, "Lines p and q intersect at the point P", we say, "The point P is incident with both p and q ." We now state our first axiom:

> Corresponding to any two distinct points P and Q, there exists exactly one line p , which is incident with both P and Q. This line may be so designated as the line PQ.

## The Betweenness Relation

Examples of axioms involving the betweenness relation are the following.

If P, Q and R are three distinct points which are incident with the same line, then exactly one of P, Q, and R is between the other two.

If the point P is between the point Q and the point R then P, Q and R are distinct points which are incident with the same line.

There are two essential things that must be achieved by the betweenness axioms, either by explicitly stating them as axioms or by deducing them as theorems. The first is that if the point P is incident with the line p , there are two sets of points, $\alpha$ and $\beta$ , which are incident with p and have the following properties. P does not belong to $\alpha$ or to $\beta$ . If Q is incident with p , then either Q = P or Q belongs to $\alpha$ or Q belong to $\beta$ . The set intersection of $\alpha$ and $\beta$ is the null set. If Q and R are both in the same set (i.e., both in $\alpha$ or both in $\beta$ ) then either Q is between P and R or R is between P and Q. If Q and R are in different sets, then P is between Q and R. We may say that the sets $\alpha$ and $\beta$ are the two sides of P on the line p . The set union of set $\alpha$ and the point P is called a ray emanating from P. The set union of the set $\beta$ and the point P is another such ray. Thus associated with every point P incident with a line p , we have two rays.

The second essential thing which must be achieved by the betweenness axioms is the following. Let p be any line. We must achieve the existence of two sets $\alpha$ and $\beta$ of points which are not incident with p and have the following properties. The set intersection of $\alpha$ and $\beta$ is the null set. If P is any point, then either P is incident with p or P belongs to $\alpha$ or P belongs to $\beta$ . If Q and R are both in $\alpha$ or both in $\beta$ , then there is no point which is between Q and R and is incident with p . If Q and R are in different sets, then there exists a point P which is between Q and R and is incident with p . The two sets $\alpha$ and $\beta$ are the two sides of the line p in the plane.

An angle can now be defined as a pair of distinct rays on the same or on different lines, emanating from the same point.

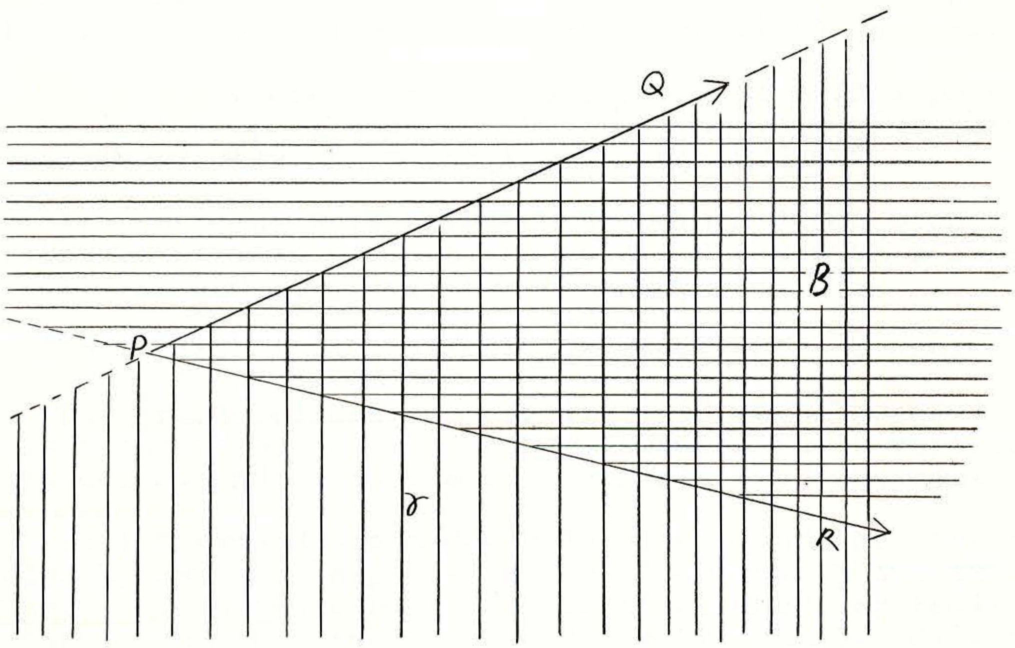


Figure 1

The interior of an angle can be defined as follows. Let the angle consist of the two rays PQ and PR emanating from P as in Figure 1. Let $\gamma$ be the set consisting of all the points which are on the same side of PQ as the point R which is indicated by the vertical shading. Let $\beta$ be the set consisting of all the points which are on the same side of the line PR as the point Q, as indicated by the horizontal shading. The interior of the angle QPR is the set intersection of $\gamma$ and $\beta$.

If P and Q are distinct points, we define the segment PQ to consist of the points P and Q and all the points (incident with the line PQ) which are between P and Q.

If P, Q and R are three distinct points which are not incident with the same line then the set union of the segments PQ, QR and RP is called the triangle PQR. The points P, Q and R are called the vertices of the triangle.

The Congruence Relation

At the junior high level it is important to have the student realize that just as the idea of a one-to-one correspondence is more fundamental than counting, so the notion of congruence is more fundamental than length. Students should be encouraged to use a compass to decide whether or not two segments are congruent and it should be pointed out that this decision can be made without knowing the length of either segment.

There are two congruence relations, congruence for segments and congruence for angles. We denote the relation by the symbol $\cong$. The following are two examples of the axioms.

If P and Q are distinct points and if R is a point incident with a line p, then on the line p on a given side of R there exists exactly one point S such that $PQ \cong RS$.

If the point R is between the points P and Q on a line p and if the point T is between the points S and U on a line q and if $PR \cong ST$ and $RQ \cong TU$, then $PT \cong RU$.

There are similar axioms for congruence of angles.

Two triangles ABC and DEF are said to be congruent if there is a one-to-one correspondence between their vertices such that the corresponding angles and segments are congruent.

Euclid proved his side-angle-side congruence theorem for triangles by using superposition i.e., picking up one triangle and placing it on the other. In his proof there is really the tacit assumption that the triangles are congruent. Hilbert gets around this difficulty by taking the side-angle-side theorem as an axiom. It is possible then, using this one axiom to prove the other familiar theorems concerning congruence of triangles. Some of the new texts on geometry assume all the congruence theorems as axioms. It is true that this approach may be all right if "you prefer thieving to hard labour" as the mathematician Betrand Russell once remarked in a similar connection. But from a logical point of view, it is considered undesirable to assume as an axiom a result which can be proved from the axioms we already have.

## Parallelism

As a preamble to parallelism we can prove that the exterior angle QRS in Figure 2 is "greater than" the angle PQR. To see this

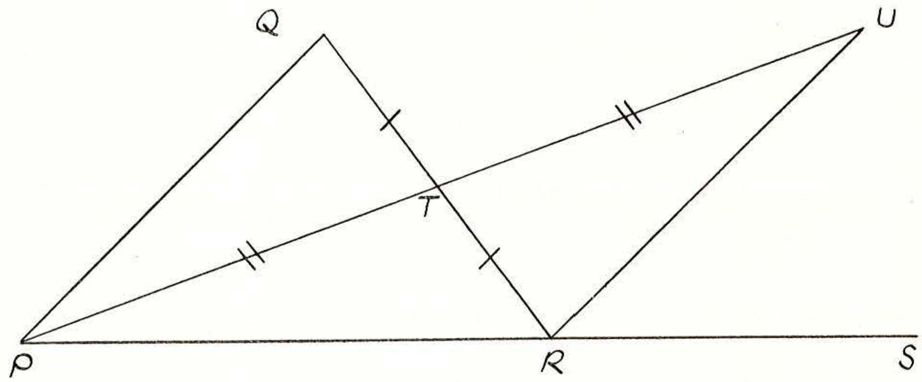


Figure 2

we bisect the segment QR at T and take PT $\cong$ TU as indicated. Using betweenness we see that U is in the interior of angle QRS. Triangles QPT and RUT are congruent and hence angle PQT is congruent to angle TRU.

We now prove the following important theorem which we will refer to as Theorem A.

> If the point P is not incident with the line p, then there is at least one line q which is incident with P and has the property that no point is incident with both p and q.

Proof: As in Figure 3. let Q be any point which is incident with p and let S be a point such that angle RPS is congruent with angle PQT with P between R and Q and S and T on the same side of PQ. Denote PS by q. If there exists a point U which is incident with both p and q, there are two cases to consider. First let us suppose that U is on the same side of PQ as S and T are. Then angle RPS is greater than angle RQT by the previous theorem. This gives us a contradiction and we get a similar

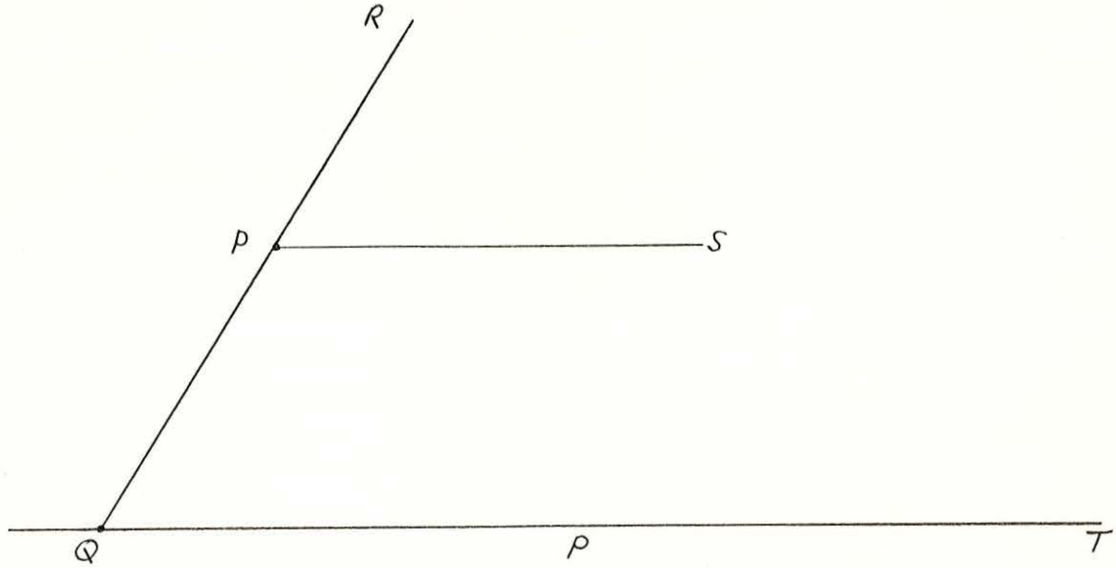


Figure 3

contradiction if U is on the other side of PQ. Since U cannot be incident with PQ, we see that no point U exists that is incident with p and q .

We now state the parallel axiom of Euclidean geometry.

> If the point P is not incident with the line p , then there is, at most, one line q which is incident with P and has the property that no point is incident with both p and q .

We are now in a position to prove the following important theorem.

> Let P be any point incident with the line q and let Q be any point incident with the line p . Let P be between Q and R. Let S be incident with q and T incident with p and let S and T be on the same side of PQ. Then there is no point incident with both p and q if and only if angle RPS is congruent to angle PQT.

Proof: If angle RPS is congruent to angle PQT we have seen in Theorem A that there is no point incident with both p and q .

Conversely, if there is no point incident with p and q, let U be a point on the same side of QR as S and T so that angle RPU is congruent to angle PQT as in Figure 4. According to Theorem A there is no point incident with the line PU and the line p.

But we are given that there is no point incident with p and q . Thus, by the parallel axiom we see that PU and q are the same line. Thus angle RPS is congruent to angle PQT, as required.

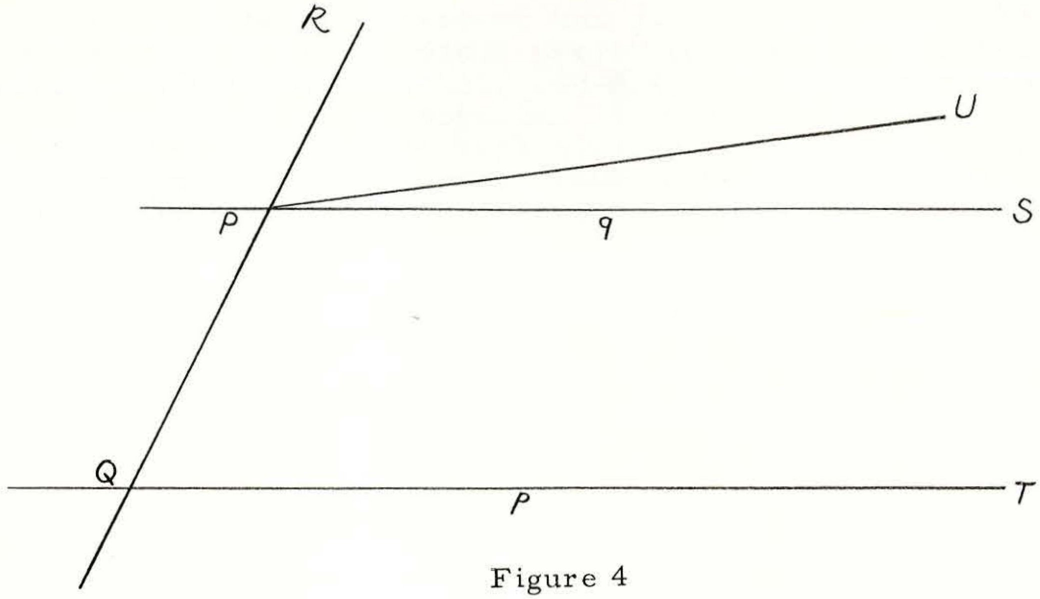


Figure 4

It is important to realize that there are non-Euclidean geometries which result from choosing different parallel axioms. One such axiom is that if the point P and the line p are not incident then there is more than one line q incident with P such that there is no point incident with q and p .

The geometry which results from this axiom is known as Hyperbolic geometry. On the other hand, if we assume that corresponding to any two distinct lines p and q , there exists exactly one point P which is incident with both p and q , then we get a geometry known as Elliptic geometry. In Euclidean geometry the sum of the angles in a triangle is two right angles. This sum is less than two right angles in Hyperbolic geometry and is greater than two right angles in Elliptic geometry. These geometries were discovered in the first half of the nineteenth century. In Elliptic geometry we do not have a betweenness relation. A relation of separation is introduced instead. Theorem A would seem, at first glance, to be a contradiction to the axiom above, which distinguishes Elliptic geometry. However, Theorem A was proved by using the exterior angle theorem which in turn was proved using betweenness.

In junior high school it is important to develop the student's spatial intuition. However, this should be done with the use of appropriate words so that the transition to the axioms introduced in high school is a smooth one. For this reason it is important that every teacher of geometry, even at the most elementary level, has a perspective in which he appreciates Euclidean geometry as an abstract mathematical system and realizes that there are non-Euclidean geometries which are equally satisfactory abstractions.

One of the very best of the current crop of books on school geometry is the Addison Wesley publication, "Geometry", by C.F. Brumfiel, R.E. Eichalz and M.E. Shanks. This book is intended for use in high schools. I would also like to recommend for your consideration the book by the same authors entitled "Introduction to Mathematics". This latter book contains an excellent introduction to geometry at the junior high level.