1964

ANNUAL

.RICULUM REVISION ISSUE

MCATA Publication of the Mathematics Council of The Alberta Teachers' Association

Mathematics Council of The Alberta Teachers' Association

1964 Annual Curriculum Revision Issue

EXECUTIVE COMMITTEE

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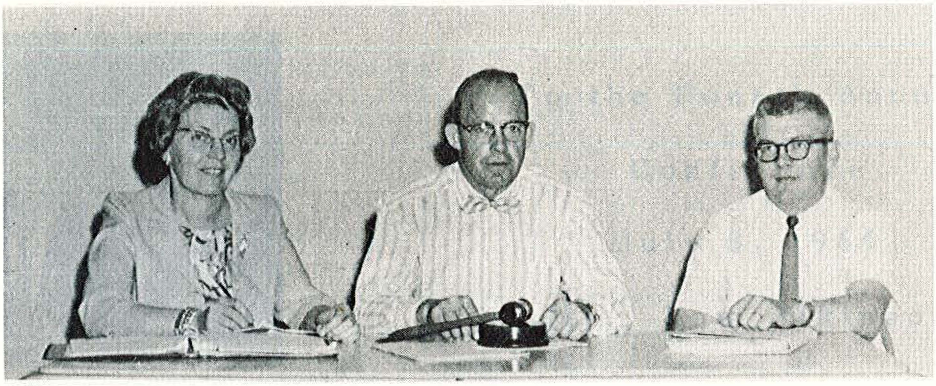
Membership in the MCATA is open to: (a) any member of the ATA covered by the TRF; (b) any member of the University of Alberta or the Department of Education. It is hoped that teachers of mathematics at all levels, elementary and senior high school, will take this opportunity to participate in professional development. In view of recent and forthcoming curriculum changes, the MCATA will be of considerable benefit to mathematics teachers. There is particular urgency on elementary and junior high mathematics teachers to join now since changes are presently being implemented in the curriculum at both these levels. All correspondence regarding the Annual should be addressed to MCATA Newsletter, J.Holditch, Editor, 11035-83 Avenue, Edmonton.

EXECUTIVE FOR 1964-1965



Elected Representatives for 1964-1965

The elected representatives for 1964-1965 are: Centre - L. C. Pallesen, Assistant Superintendent of Calgary Secondary Schools, President. Left - Jean Martin, Secretary. Right - Ted Rempel, Vice-President.

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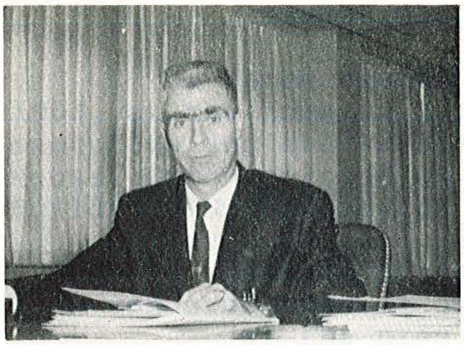
EDITORIAL:

This Annual was envisioned by the executive of the MCATA at its October meeting. At that time it was considered that a more extensive publication than the Newsletter - to be published at the conclusion of the year - would be especially useful to those members desirous of becoming familiar with local thought in mathematics education particularly as it applies to curriculum revision. Bearing this in mind, your editor has put together accounts of these developments along with articles based on the new authorizations in junior high school which become effective in September 1965. The STA series, which enjoys popularity with our members, has considerable analysis given to it by our writers. To these articles have been added some pertinent research accounts contributed by the leaders in mathematics education in Alberta. We rely on these people for our enlightenment.

The executive has given me a free hand and if readers do not find the contents in line with the aims of the publication then the editor must shoulder the blame. I hope I have chosen those things most useful at this time.

PRESIDENT'S REPORT:



To the Fourth Annual

Conference

July 8, 1964

Mr. Tom Atkinson

As recorded in the minutes of the 1963 annual general meeting, the executive for the year to come was elected at that time. The retiring executive was responsible for certain activities planned for the summer. The new executive gradually assumed control, with the transition being complete by about August 1.

The four seminars conducted in July, 1963, in Calgary, Red Deer, Edmonton and Grande Prairie were attended by a total of more than 200 persons. Those involved in these seminars - acting as planners, directors, lecturers and consultants - are too numerous to mention individually, but they and those who attended are to be commended for their service in improving the teaching of mathematics in the schools of our province. During my term of office:

1. Paid membership has been approximately 330, with an additional 20 complimentary memberships.

2. The executive has consisted of -
   Mr. Tarlton, Past President
   Mr. Atkinson, President
   Mr. Jepson, Vice-President
   Mrs. Martin, Secretary-Treasurer; elected by the members of the MCATA
   Mr. Sillito; appointed by the Executive of the ATA
   Mr. Massey, Elementary Consultant for Edmonton Public Schools
   Mrs. Hutchinson, Ottewell Junior High School, Edmonton
   Mr. Richards, Viscount Bennett High School, Calgary
   Dr. Nelson, Faculty of Education, University of Alberta, Edmonton
   Dr. Gibb, Faculty of Education, University of Alberta, Calgary
   Mr. Phibbs, Department of Mathematics, University of Alberta, Edmonton

Mr. Larson, President of CARMCATA
Mr. Rempel, President of EARMCATA; appointed by the president
Mr. Holditch, Editor, has been recognized as being a member also.

3. Three executive meetings were held. One on July 12, 1963, at the home of Mr. Tymchuk in Edmonton, attended by the 1963 executive and the President-Elect; on October 19, 1963, in Barnett House, Edmonton; and on February 29, 1964, in the Arts/Education Building, University of Alberta, Calgary.

4. Members of the executive, individually and in groups, have rendered special services, some of which are as follows:

(a) the president, secretary-treasurer and editor, accompanied by a representative from each of the two regional councils, attended the Banff Specialist Councils Seminar in August, 1963.

(b) Mr. Holditch prepared two bulletins for publication and is planning a year book.

(c) the president spoke at two teachers' conventions in October and November of 1963.

(d) Mr. Jepson, Dr. Gibb and Mr. Richards studied the Professional Load Report and submitted a criticism of it to the Executive Council of the ATA.

(e) Mr. Jepson publicized among members of the MCATA in the Calgary area an invitation issued by the Business Education Council to attend a one-day short course on Data Processing in Calgary.

(f) Mr. Larson rented three films from a set of thirty films prepared by SMSG and made them available for viewing in Red Deer, Lacombe, Calgary and Edmonton.

(g) Mr. Tarlton has acted as chairman of the Nominating Committee.

(h) three seminars are currently operating - again the persons involved in the various stages are too numerous to mention.

(i) several persons are assisting with the program today.

5. The MCATA is now affiliated with the NCTM.

6. Mr. Larson attended the 1963 summer meeting of the NCTM in Eugene, Oregon. He has contributed two articles to the bulletins pertaining to some of the concepts presented in talks there.

7. Both Regional Councils have carried out programs of in-service training during the winter. Mr. Larson and Mr. Rempel have given strong leadership to their respective organizations.

REPORT OF JUNIOR HIGH SCHOOL MATHEMATICS SUBCOMMITTEE

R. Plaxton

Editor's Note: R. Plaxton has been in the Junior High Mathematics Subcommittee since its appointment. He has written several published articles on the work of this committee and on experimental work prior to the Grade VII authorizations. Here he brings us up to date at the fourth Annual Meeting MCATA - July 8, 1964.

I very much appreciate the opportunity to speak to you today concerning the work of the Junior High School Mathematics Subcommittee. I know that a number of your members have spent the last two days attending the seminars sponsored by our Mathematics Council. The last issue of The ATA Magazine reports that the seminars have been recessed for one day in order that the participants might attend this annual conference. If you think as I do, the noun "recess" produces in the mind some picture of a joyous release from bondage. I have some difficulty fitting such a picture with the one that was produced by reading through your program for today.

Wide-eyed Idealists

My talk today has been entitled "Report from the Junior High School Mathematics Curriculum Subcommittee". Anyone who would attend a talk with that title at this hour of the morning has certainly gone beyond the call of duty. This is especially true when one considers that the decisions that I am going to report are already widely known for they were published by the Department two weeks ago today. Perhaps I can make your presence here seem a little more reasonable by telling you that my purpose is not so much to report our final decisions as to examine the process and thought that led to those decisions.

It sometimes appears that curriculum decisions, far reaching in their effects, are made and dictated to the teachers of the province, with little thought of the consequences. Some suggest that such decisions are made by wide-eyed idealists who have been outside the classroom for so many years that they have lost all contact with reality. Perhaps a look behind the official pronouncement of the Department of Education concerning junior high mathematics will assist you in making judgments in this area.

Moment of Decision

I have said that I am going to report a decision already widely published. The widely-published decision that I mentioned earlier is actually only one month old; but news of decisions of this kind, because the decisions affect almost every teacher of the subject in the province, travels fast. Nevertheless, in case there are some present who are not familiar with the decisions, I will review them briefly. Starting September 1965, not 1964 notice, two texts will be authorized for use in Grade VII classrooms in this province. The first is entitled Seeing Through Mathematics, Book One, and is published by Gage and Co. This is the same company that published Seeing Through Arithmetic, the book that has gained general acceptance in the elementary school. The second is entitled, Exploring Modern Mathematics, Book One, published by Holt, Rinehart and Winston, the publishers of the book presently authorized for Grade VII. Starting September, 1966, Book Two of Seeing Through Mathematics, often referred to as S.T.M. II, and Book Two of Exploring Modern Mathematics, usually referred to as E.M.M. II will be authorized. No recommendation for a Grade IX text or texts will be authorized for use in September, 1967. This means that students will write their first Grade IX examination on the new material in June, 1968. Senior high school teachers must be prepared to accept students with a very different mathematical background than previously, by September of that same year.

These decisions were not entirely unexpected. Some of you here today have already had rather extensive experience with one or other of these recommended series. Certain aspects of the decisions, however, require additional emphasis and explanation.

1. You will notice that this is a dual authorization. Either text series, or for that matter, both series may be used in our classrooms. As far as I know this is the first time that more than one mathematics text has been authorized for the same grade at the junior high school level.

2. Both series are entirely "modern" in their approach and content. The texts are very different from those presently authorized.

3. No text or texts, at the present time, have been recommended for authorization at the Grade IX level.

4. The books that I have with me today are not the authorized texts. Certain changes in the books in both series will have to be made before they can be used officially in our classrooms.

5. We are required to wait a full year after the books have been recommended for authorization before we can actually use them in our classrooms.

No doubt these comments have already raised a number of questions in your minds. I hope to answer some of them by reviewing the work of the subcommittee since its inception. May I begin by taking a moment to discuss how curriculum change takes place in Alberta. The provincial Department of Education maintains a curriculum branch and four major curriculum committees: the General

Curriculum Committee, the High School Curriculum Committee, the Junior High School Curriculum Committee and the Elementary Curriculum Committee. One of the major functions of these committees is to keep abreast of contemporary educational thought and to develop curricula and authorize textbooks. The Alberta Teachers' Association has three or four teacher members on each of these committees. From time to time the committees appoint subcommittees to study particular subject areas. One such subcommittee, the Junior High School Mathematics Subcommittee was formed nearly three years ago with a membership of eleven, including the associate director of curriculum and the mathematics consultant from the Department, a high school inspector, a professor of mathematics, two professors from the Faculty of Education who specialize in mathematics, two city superintendents and three teachers of junior high school mathematics. You will notice that this subcommittee has good representation from those groups who are most concerned with the teaching of mathematics. I mention this fact because I have heard teachers say that those most concerned with teaching mathematics have no say in what will be taught. It is true that curriculum development in Alberta, as in every Canadian province, is highly centralized. However, teachers have quite good representation on all committees and subcommittees in all except one province. In that province the curriculum department consists of one man.

The Junior High School Mathematics Subcommittee was directed to study the courses in mathematics now offered with a view of determining whether any changes should be made in the courses in light of the fact that new textbooks had been authorized at the elementary school level, and in light of new thought in mathematics teaching now present throughout the western world. We began by taking a very close look at the present texts. We agreed that they should be replaced if we could find a series that we felt would serve the teachers and pupils in Alberta better. Our decision was based on a number of considerations.

1. The Grade VII text now in use does not constitute a reasonable sequel to the Seeing Through Arithmetic series now generally accepted throughout the province in the elementary school. Those of you who have students in Grade VII this year who have taken the STA VI course will know, for example, that percentage is taught earlier and by a different approach in the Grade VII text.

2. The present texts are repetitive. Mr. Robert E. Rourke, when speaking to the CTF Seminar on "New Thinking in Mathematics" had this to say:

> Practically everyone at every level agreed, that in the United States there was no place in the whole program that was more wasted on mathematics than Grade VII and VIII. It was a rehash of the same old stuff of the first six grades - more of the same - and large amounts of time

were devoted to the kind of arithmetic we were talking about yesterday, in which students were doing problems that were supposed to be of social significance to them.

We felt that much the same criticism could be applied to our courses in Alberta.

3. Finally there is much new material that was not included in the present text. According to C. Baley Price:

> The twentieth century has been the golden age of mathematics, since more mathematics and more profound mathematics have been created in this period than during all the rest of history.

Some of this new knowledge has finally filtered down to influence the junior high school programs. There is only so much time in which to teach mathematics. It seems reasonable to replace content no longer important to the learner. There is nothing wrong with the traditional program; it is simply outdated. The present texts have served well but we felt that the time for their retirement had come.

Thinking Men or Thinking Machines

Having made this decision we turned our attention to discovery of a kind of course that would likely be most valuable to the students of the next few years. No one can say with certainty what mathematics will be most important to students who will become productive members of society in the 1970's and '80's. I think, however, that certain trends are obvious. For example, the advent of calculating machines that carry out simple arithmetic computations faster and more accurately than man, have nearly eliminated the need for children performing computations with large numbers. It seems more important, therefore, to have children learn to operate the computing machine rather than have them learn to carry out operations of the sort which they may never perform again after leaving school. Conversely, we have not yet arrived at the stage where we can ignore the computational skills. No one has yet developed a pocket computor designed to compute problems as a watch tells time. We must continue to teach computation skills, but perhaps developments in technology have already reached the point where we should shift emphasis a little more toward basic understanding of mathematics. It has been a long time since we have added our grocery bill, but we still want to be able to do it if we must.

Another factor, in addition to technological advances, must be considered; there is more and more mathematics to learn as the years go by. We cannot meet the needs of mathematics by cramming more and more material into an old organization. Instead we have to cast out useless parts, make broader and more general approaches and use general and unifying concepts. This is what is really meant by "modern mathematics". There is one content and one organization of mathematics that is given that name. In fact, widely different courses have been given the same name. All, however, have attempted to include simple, clear, and more broadly-based ideas about number and space.

Throw Out the Baby with the Bath Water

Our wish to include new ideas is no reason to discard the best of the old. The movement to incorporate modern concepts in our mathematics courses has been called a revolution. Revolutions can be dangerous because they tend to go too far; to throw out the baby with the bath water. A well-balanced program, we felt, should combine some of the more recent developments with the best of the traditional methods and topics.

With these thoughts in mind we set about finding a text for Grade VII that would:

- follow smoothly from the Grade VI authorized texts, especially the STA series;
- develop mathematical insight, power and understanding;
- be based on the structure of mathematical rather than on the practical application of the subject;
- use practical applications to illustrate basic principles whenever possible;
- not be too difficult or include too much material for students eleven and twelve years of age;
- not be so different that a vast training program for teachers would have to be initiated before any further action could be taken;
- retain the best of the present course;
- read easily into the kind of program now being developed in the higher grades.

We were hopeful, too, that we could find an integrated series for all junior high school grades. We hoped to avoid a split in the series such as we now have between Grades VIII and IX.

Through Adversity to the Stars

We were hampered right from the start by lack of material.

New ideas in mathematics have a way of filtering down from the university level rather than building from the bottom up. A few books had been published using the new content and approach at the college and high school level, but three years ago there were virtually no published materials available at the Grade VII level. In the first year we worked from incomplete materials often in monograph form. This problem became less serious as time went on, but it is still causing us some difficulty. One of the major reasons that the Subcommittee has not recommended a text for use in Grade IX is that the Grade IX texts in both series became available only very recently. As a result there has been little opportunity for studying or for trying out these texts.

By October 1962 we were able to institute trials with books from two different publishers. These trials were run to determine student reaction, teacher reaction, adaptability of the material to different ability levels and so on. One of the books used in these first trials was Seeing Through Mathematics, Book One. The first trial involved thirty classrooms. Approximately that number of classrooms have been involved in trials each year since that time.

I will not go into specific details of the results of these first trials. It is sufficient to say that the reaction of both students and teachers was favorable, and contrary to our expectations, the precise language and increased use of symbols seemed to cause the students little difficulty. Most teachers felt the material was more challenging and as a result more interesting.

During the 1962-63 school term thirty-three classes, or about eleven hundred students in Grades VII and VIII were trying out six different texts. The trials in that year were backed up by a good testing program and a questionnaire designed to determine teacher reaction. As a result of these trials we were able to lay down guidelines indicating the content we would like to have included in any authorized texts. These guidelines included much material that is usually included in "modern" texts. We were also able on the basis of the test results to eliminate all but two series of texts from consideration. The remaining series, of course, are those to be authorized starting September 1965.

During the last school year a Master's student at the University of Alberta, in Edmonton carried out, with the assistance of the Subcommittee, a well controlled experiment with classes using both series, and control classes using the presently authorized Winston text. The study was used as the basis for a thesis and was, therefore, very carefully conducted. That is why I have called this study an experiment rather than a trial. I understand that those of you most interested in Junior High School Mathematics will be discussing these experiments and trials later today, so I will not attempt now to give you a complete run-down of the results. Two facts were of particular importance to us:

1. Both series made it possible for students to learn a significant amount of new mathematics without loss to their knowledge of conventional mathematics.

2. Test results on an instrument prepared by the Master's student to measure new mathematical ideas, indicated superior achievement by Grade VIII students of average and below average ability studying Exploring Modern Mathematics over similar groups studying Seeing Through Mathematics. The differences in achievement were not large, but they were statistically significant, in the sense that there was less than one chance in a hundred that such differences would occur by chance. There was no significant difference in the results in the case of superior students.

Replies to the teacher questionnaire provided additional information. One factor over which the researcher had no control was the training and experience of teachers. While the years of experience of the teachers were very similar for both groups in the study, the teacher of the E.M.M. course had considerably more training in mathematics. What influence this training had on the results is not known. The questionnaire also indicated that teachers of the S.T.M. series were more inclined to favor the course than those who taught the E.M.M. course.

Zeroing-In on a New Curriculum

These findings were added to the experience and information gathered over a period of three years to produce a recommendation for a dual authorization. This decision was based on a number of considerations:

1. With the trend toward decentralization of some curriculum decisions, it was felt that there was considerable merit in providing local areas with the opportunity to make a decision concerning the text to be used in their schools. The members of the Subcommittee felt that the study necessary to make the decision would in itself be valuable training for teaching the course. They expressed the hope that as many teachers as possible would have the opportunity to study both texts.

2. Although the subjective judgment of the teachers involved in the trials seemed to favor the S.T.M. series, the objective evidence from the Master's study favored the E.M.M. series.

3. It was generally agreed that while extra training was very desirable for all teachers teaching either series in 1965, the need might be less severe in the case of E.M.M. because the format of the text was closer to that to which the teachers were accustomed. Since the opportunities for extra training vary considerably throughout the province, it was felt that a dual authorization might permit a greater degree of flexibility.

4. The actual content of the two series is sufficiently parallel to permit a common Grade IX examination.

5. It was felt that the problems arising from a pupil transfer would not be too significant since most pupil transfers in the province take place within a particular division or county and because our experience in this regard in the trial classrooms indicates that pupils adjusted to quite different programs remarkably quickly.

6. Finally, the university members on the Subcommittee could see no serious problems arising from the dual authorization with regard to preparation of inservice and preservice training programs, since these programs tend to stress the mathematical concepts rather than the details of a specific text.

Be Alive by '65

I have made reference to some of the reasons that the Subcommittee decided to recommend texts using the modern approach and content, decided to recommend a dual authorization, and decided not to recommend texts for authorization at the Grade IX level at this time. I will turn now to a discussion of why it was thought advisable to delay the authorization for one year. There were two major reasons for this action. The first is a very practical one. We have asked for certain revisions in the texts in both series to bring them more in line with Alberta's needs. The E.M.M. series for example, does not yet have a Canadian edition. We have asked that the texts be completely Canadianized. Those of you who have had experience with the presently authorized texts will know that Canadianization was never carried out completely in that series. We have suggested also certain changes with respect to problem solving and the handling of percentage, that will make for better articulation with the Grade VI S.T.A. series. Experience with the S.T.M. series has taught us that many teachers have difficulty completing the course, especially in the first year it is taught. We have asked Gage and Co. to rebind the Grade VII and VIII texts so that the courses in both years will be reduced somewhat in length. These changes cannot be made in time for school opening in September.

But even if the books could have been made ready we would have recommended a delay of one year. This delay in the face of mounting pressure for the new courses is an indication of the importance that we attach to giving every teacher in the province an opportunity to prepare well before teaching this course. The teachers who have taken part in the experimental programs, almost as one, have indicated the value, perhaps necessity, of training in modern mathematics before one undertakes to teach these new courses. Appropriate training for teachers is the most conspicuous and difficult prerequisite for the introduction of this program. The Department, The Alberta Teachers' Association, the University and local school boards are all deeply concerned about this problem. I think it is safe to say that the change from the traditional course is more dramatic at the junior high school level than it has been at the elementary level or will be at the senior high level. I am concerned, if this course is introduced with too little warning and too little preparation on the part of teachers, that we might actually set mathematics teaching back a few years rather than advance it. It is easier to abuse these new courses than it was the traditional course, in much the same way as a poor mechanic can do more harm to a fine racing motor than to a tractor motor. There is no doubt in my mind, on the other hand, that one of the greatest values to arise from the introduction of this new course will come about because professionally-minded teachers will acquaint themselves with what is new in their field. I urge you to spread this message of the need for preparation to all your associates. This extra year gives time for much to be accomplished. The message need not be confined to junior high teachers. It is not too soon for senior high teachers to become acquainted with these materials. In order to ensure continuity three members of the junior high school subcommittee are also members of the senior high school subcommittee. These members tell me that activity in that committee will increase sharply now that the junior high school decisions are final.

I have discussed our decisions to date and some of the reasons for them. Perhaps some of you came to this gathering hopeful that I would describe the content included in the courses or present my impressions of the new texts. I have deliberately avoided these areas as much as possible. I could say little about content that would be useful to you. As teachers you are aware that there is really only one way to know a course. That way is to teach it. Even private study of the texts in their present form would serve you better than listening to a series of generalizations about content for forty-five minutes. There is really no substitute for individual effort in this regard.

As for my impressions of the course, I discovered after a few years of sitting where you are sitting today, that people closely associated with the development of a new program, in their attempts to enthuse others, become too enthusiastic themselves; and as a result leave the impression that the new developments will solve all our problems. We know that no course no matter how well developed can achieve the end. I am enthusiastic about the program. I think that teachers will find it more challenging to the intellect, more in harmony with contemporary thought, and more meaningful to students. I know that the ferment of new ideas caused by the introduction of the course will be good for the teaching of mathematics. The extra thought, study and preparation that each mathematics teacher will have to undertake will pay big dividends in broadening horizons and revitalizing teaching. But I also know that there will be some negative reaction. We feel comfortable with the <u>status quo</u>. Disturbing new elements make us uneasy; extra work annoys us. I know too, that actual use of the texts on a large scale will force retreat on certain fronts. No committee sitting around a table could possibly identify all the areas of difficulty. Time, experience and constructive criticism will bring about the necessary changes

And, of course, new texts are only the beginning. A new curriculum is no substitute for inspired teaching. In fact, it is the other way around. Good teaching would be a very satisfactory substitute for a new curriculum. The success of any program is determined in the heart of the school, in a classroom where a teacher is face-to-face with a learning child. The success of this new program depends on you.

WHAT MIGHT BE DONE IN MATHEMATICS

Professor E. Phibbs
Department of Mathematics
University of Alberta
Edmonton

Editor's Note: Professor Phibbs has shown interest in MCATA affairs, having served as Department of Mathematics' representative on our Council. Those who have taken courses from him attest to his ability to make what seems complex appear quite simple. We welcome his considered opinions about mathematics education.

New Views, New Language

The last few years have seen a revival of interest in the subject matter of the mathematics curriculum in the schools. The volume of mathematical research has increased enormously. The demand for mathematicians by our society is increasing rapidly; not only for defence purposes but also in the fields of sociology, economics, and business. This being so, it has become increasingly necessary to do all that is possible to train mathematicians quickly and efficiently.

To this end it is proposed that the less important subject matter be taken out of the curriculum which should concern itself with the basic concepts of mathematics. For example the solution of large numbers of numerical problems concerned with the calculation of the sides and angles of triangles is considered to be wasteful of time although of course a high degree of competence in the arithmetic involved is desirable. Again, while a familiarity with the idea of percentages is important, there seems to be no good reason why much time should be spent on its applications to business transactions. Instead it is considered to be of much more importance that the student should be presented with a picture of mathematical systems as instances of "sets" having "elements" which may be combined by certain "operations", the whole having a certain form or "structure". We find such words as "closure", "commutative", "distributive" appearing together with a symbolic notation which in some of the new texts is kept to a minimum, in others is allowed full play. Thus there will be not only a change in point of view but also a new language in which it is expressed.

Here certain difficulties may well appear. The new approach is more mature, mathematically, than the conventional one and it may be that the less able students will experience more difficulty with this approach. Perhaps the solution for this problem is to have some separation of the students at the Grade XI or XII levels; in one group to place those students who are evidently much interested in mathematics and who are likely to go on to use it at university; in the other those students who for various reasons do not have the same deep interest in the subject or who are planning to pursue their studies in other fields.

Teachers' Equipment

Another difficulty is that by and large the instruction in our schools and the training of our teachers have not been patterned on the new ideas. If instruction in the new mathematics is to be really effective then the relevant ideas and notation must become the automatic equipment of teachers. This, of course, involves very difficult readjustment for teachers to make. The answer, here, is to give intending teachers an acquaintance with what is required before they graduate, and already at the university this is being done. For others who have been teaching for some time the problem of readjustment is very real and troublesome. Perhaps it could be met with types of inservice action possibly it could be minimized by the use of carefully chosen textbooks.

Besides the change in point of view it is also proposed to introduce new subject matter which so far has been generally presented at the university. In high school, mostly at the Grade XII level, some elementary treatment of groups, linear transformations, matrices, determinants and allied topics are likely to make their appearance. High school algebra will remain much as it is at present except that where it is proper to do so its concepts will be expressed in the new language. The algebra remains because it is the necessary prerequisite for calculus.

Structure or Manipulation

The new material, besides being interesting in itself, provides the opportunity for the investigation of new systems with accompanying structures, elements and operations. Non-commutative operations make their appearance. It is at this stage that the importance of the notion of "structure" begins to appear. It becomes evident that systems which at first sight appear to be very different are yet absolutely identical as far as their "structure" is concerned.

It seems a pity that for the greater part of his school life the student would be familiar only with the ordinary number system and that only at the Grade XII level is his outlook really broadened. Surely early knowledge of more than one type of mathematical system is desirable quite soon in the lower grades - if we wish to keep the student's interest alive. Many simple systems of a kind which can be readily appreciated by quite young children exist and could be used. There are the symmetries of the square, equilateral triangle, the rotation groups and the additive groups of the integers relative to a given modulus which come to mind. An elementary contact with geometry is necessary - this at present seems to be first met with in junior high school. But the simple ideas needed for the above suggestions could easily be appreciated much earlier. The same applies to simple geometric constructions. The idea of loci could illustrate very effectively the ideas of set and function in a way which would, at an early age, drive home the full generality of these concepts, to be met at later stages of their training.

THE OLD TEACHER AND THE NEW MATHEMATICS

Wm. F. Coulson

Editor's Note: William F. Coulson was on the staff of the University of Alberta during the 1962-63 session. He was editor of the MCATA Newsletter and a tireless worker on the MCATA executive. His address was given to the third annual meeting MCATA.

When New is Old

Before anybody gets too excited, let me do what every good "modern" mathematician does. In the title of this paper, I have used two familiar words. As adjectives I used the word "old" and the word "new". By putting the two words "old" and "teacher" together I am not implying that the teacher is ready for retirement to the rocking chair. My use of the word "old" here is simply a device to denote those individuals who received their preparation for teaching and who started teaching prior to the great rash of agitation for "modern" mathematics. The mathematics courses taken by these people, both in high school and post high school, were traditional in nature. Nor does this mean that they are any less effective as a teacher. It simply means that their academic background in a specific subject matter discipline is out of step with current thinking.

Let's take a quick look at the second of the two words I identified as special earlier. This is the word "new". I'm not trying to imply that the new mathematics is "younger" than the "old" teachers. With a disgustingly great degree of unanimity, this mathematics is older than any of us here in this room. By "new", I'm referring to those mathematical ideas that, until the last half decade, have met a closed door when they tried to be nice and respectable about getting into the senior high school mathematics curriculum. About five years ago all doors were opened wide and look what blew in. Every kind of idea conceivable was put into some kind of text by some kind of author dealing with some kind of publisher.

Christopher Sails the Blue

It has been said that if Columbus were to have the misfortune of being thrown out of heaven and sent back to earth he would find very little change in the mathematics curricula between 1492 and 1958. This may be a true statement, but, at any rate, very little of the mathematics developed in that span of time has been able to have any influence upon the high school curriculum. The "new" mathematics to which I am referring was developed by some free thinkers during the past three centuries.

Looking at the textbooks being published today, a forward-looking, non-ostrich type of teacher will be able to see some very plain handwriting on a very clean wall. This handwriting says, in no uncertain terms, that changes are being made. Who among us is going to say that just because the horse and buggy was good enough for grandpa, it's good enough for me. Society, industry, and life in general have progressed. Why not the senior high school mathematics curriculum?

Some very able mathematicians were educated in the past few decades. They were educated to fill a need and they were able to modify themselves to fill a still greater need. Is the need the same as it was thirty years ago? I say "no". We no longer need human computers. We need human thinkers who can tell a computing gadget what to do.

Mathematical Skeletons

How then must the "old" teacher change to fit into place with the "new" mathematics? First, this teacher must be able to think in terms of the structure of mathematics. Jerome Bruner, in The Process of Education, says that

> Algebra is a way of arranging knowns and unknowns in equations so that the unknowns are made knowable. The three fundamentals involved in working with these equations are commutation, distribution, and association. Once a student grasps the ideas embodied by these three fundamentals, he is in a position to recognize wherein "new" equations to be solved are not new at all, but variants on a familiar theme.

Thus, we see emphasis on structure, the building up from the foundation.

Let's take time out to consider a few specific examples. When we ask students to learn to solve equations, do we go back to what he had learned from his arithmetic? No! We take him to a pan balance and ask him to add to or subtract from the amount in each pan so that both pans are kept in balance. If the student is to find the value for which x is holding the place in the sentence, $x + 5 = 12$, we neglect the fact that this is related to a basic fact from second grade addition. He knows immediately that x is holding the place for 7. We make him go through a long rigamarole trying to find something that is perfectly obvious to him. Our time would be better spent asking the student to rename the 12 so that 5 is in the new name. This would give $12 = 7 + 5$. From this the student can see that the value of x is 7.

This idea can be applied by the student to the equation $x + 348 = 5623$. It is possible to rename 5623 so that 348 is a part of the new name? We can find the other part of the new name by first subtracting 348 from 5623 and find that the new name is 5275 + 348. After practising on a few more examples similar to these the student will be able to state the familiar generalization for the solution of equations of this type. Also, the student has now built a very simple structure.

Another example in which we ignore structure and in which the "old" teacher may have to modify habit is in the realm of the operations with signed numbers, or integers. Previously, these students have learned some fundamental ideas. They know how to operate with numbers zero or greater. Putting this with the idea of $a + x = 0$, where $a > 0$, what is the value for which x is holding the place, we can build all of the operations with integers. In the equation, $a + x = 0$, if $a = 8$, then x must equal -8. If $a = 12$, then $x = -12$. In other words, 8 and -8 are additive inverses of each other; 12 and -12 are also additive inverses of each other.

What is $8 + (-5)$? First the student must go back through the mental filing system he has developed to find the pieces of information which relate to this operation. One is that he can rename 8. Another is that $a + (-a) = 0$. Using these two concepts the student is now able to find a new name for this rather complex set of symbols. 8 is equal to $3 + 5$. Thus the original phrase becomes $(3 + 5) + (-5)$. Applying the associative property of our number system we get $3 + (5 + (-5))$. But we know that $(5 + (-5))$ is another name for zero. Thus, $3 + (5 + (-5)) = 3 + 0 = 3$. It won't take long for students to state the usual generalization for the addition of integers.

Let's go to multiplication. Traditionally, there has been a rather cumbersome attempt to develop this operation in the integers. All kinds of real life situations are thrown in with no significant degree of success. Looking at our structure, we find several ideas available to us. Basic are the elementary facts for multiplication and $a. 0 = 0$. Zero can be represented by a vast number of different sets of symbols. For purposes of this illustration, I will arbitrarily choose one of the symbols; $7 + (-7) = 0$.

We know that $(5)\ (0) = 0$ and from this we know that $(5)\ (7 + (-7))$ is equal to zero. The distributive property can be used here to multiply; $(5)\ (7 + (-7)) = (5)\ (7) + (5)\ (-7) = 0$ $(5)\ (7)$ is known to equal 35. Thus $35 + (5)\ (-7) = 0$. What must $(5)\ (-7)$ equal? For all of the other properties of our structure to hold $(5)\ (-7)$ must be equal to -35. A few more such examples will bring out the usual generalization.

Another example is $(-5)\ (7 + (-7)) = 0$. Applying the distributive property as before -

$$(-5)\ (7 + (-7)) = (-5)\ (7) + (-5)\ (-7) = 0.$$

But, we just indicated that (-5) (7) = 35. Then -

$$(-5)\ (7) + (-5)\ (-7) = -35 + (-5)\ (-7) \text{ must equal } 0$$

if all of the previous properties are to hold. Again, after a few examples, the usual generalization may be stated.

One further example from algebra before we turn to a quick look at geometry. The idea of two unknowns is a part of the mathematics course for the high schools. In my opinion this is very poorly done. Again, do we follow or attempt to develop a structure? Basically, no. We make the poor student jump in without even a straw to grasp let alone a life preserver. No attempt is made to analyze completely one sentence in two unknowns before the second one is put with the first.

If $x + 6 = 7$, what can we say? This is a set of ordered pairs of numbers. The numbers may be natural numbers, integers, rational numbers, or real numbers. Each gives us a different set of pairs. Each of you can think of quite a few possible solutions. If replacements are confined to the set of natural numbers, there are only six pairs of numbers which can satisfy this condition. These may be represented on the Cartesian co-ordinate system as six distinct points.

If our replacement set is any of the others, we have larger sets until the set of real numbers is reached. This set would give us an infinite set of ordered pairs which result in a straight line when graphed.

Traditionally, no attempt has been made to analyze each individual equation thoroughly before it is put with a second one and the student is asked to find the intersection of these two sets of ordered pairs.

Now, for a quick look at geometry. We always start out by saying that a point and line are intuitive notions and that they are the simplest of the geometric figures. But, then what do we do next? All of a sudden and for no obvious reason we talk about a figure involving a minimum of three lines and three points. Does this make good sense? What happened to those figures involving fewer than this? For instance, a point and a line, two lines, two lines and one or two points. Our structure is not very well built, is it?

Now to take a closer look at some of the geometric figures which are made up of fewer parts than is the triangle. Not many statements can be made about a point and a line taken as one figure. The point is either on the line or it is not on the line. Next, consider two lines. Immediately, several distinct possibilities arise. The lines intersect or they do not intersect. If they do, and the only point in common is the end point of each of the two lines, an angle is formed. A number of statements about conditions can be made. If the two lines intersect at the end point of one and some point other than the end point of the other, we have adjacent angles, supplementary angles, or possibly perpendicular lines. Still more statements about conditions can be made.

If the lines do not intersect, they may be a parallel. Still more statements. Now we can add a third line. I don't think I need to go into any more detail about the statements that the students can develop from this situation.

As can be seen, a very good geometry course could be built using no text. The teacher must have done some very careful studying so that each comment by the pupils can be analyzed for its full value in the sequence of statements. The teacher's main function here is to guide and direct the learning activities of the students.

Creative but Old

I hope I have not strayed from the topic as listed in the title. I have attempted to relate the "old" teacher and the "new" mathematics. The second element to be considered follows from what I said about teaching geometry. The teacher must show a creative, inquisitive attitude. This will make the subject come alive for the students. At a conference on the Canadian High School held in Banff last month, a number of the speakers hammered away at the idea of "teaching for creativity". Before one can teach for creativity, one must practise this himself. Irrespective of departmental examinations, one must permit the student to build ideas for himself without being put in harness and driven along a pre-determined path.

I'm begging the "old" teacher to remember that he or she is a teacher and that a modification of ideas and habits is not impossible. It may be extremely difficult, but the spirit of learning and thinking anew will help one leap over a number of hurdles.

I SHALL TEACH ARITHMETIC

Derek V. Morris
Supervisor, Division II
Calgary School Board

> When called before the Tribunal of the French Revolution to state what useful thing he would do to deserve life, Legrange answered, "I shall teach arithmetic."

Some Basic Questions

Educational man has long wrestled with the age old problems of WHAT to teach, WHEN to teach it, HOW to teach it, and TO WHOM to teach it. Incidentally, too, he has grappled with the concomitant problem of WHO SHOULD DECIDE these major issues.

As examples of such problems, I draw your attention to the following highly controversial issues:

What? 1. Should religion be taught in the schools?

When? 2. At what age in a child's life should instruction in reading begin?

How? 3. Is the enterprise method an effective one for teaching social studies?

To Whom? 4. Should the children in the lowest twenty percent on a standard intelligence scale receive the same type of instruction in the same form of curriculum as the children in the highest twenty percent on this scale?

Who? 5. As an example of the concomitant problem of WHO SHOULD DECIDE, I urge you to consider the present issue of accreditation.

You and I have been faced with a fait accompli as far as the introduction of modern mathematics in the elementary school is concerned. That is, decisions have been made as follows:

What? 1. The STA program embodies the "what" that is necessary in arithmetic.

When? 2. The STA program has an approved sequence of learnings.

How? 3. The STA manuals outline adequate and acceptable techniques of instruction.

To Whom? 4. The STA program is for all children, with their wide range of individual differences, in our elementary schools.

Who? 5. Furthermore, the decision as to WHO DECIDES has been taken: the Minister and his Department!

Before I continue, however, I must interject a note of caution that I have neither stated nor implied a value judgment as to the rightness or wrongness of the *fait accompli* of the WHAT, WHEN, HOW, TO WHOM and DECIDED BY WHOM questions noted earlier; I simply comment that, like the recently elected government, it is now with us.

The Problem of Re-selection of Content

Emphasis Upon Basic Intellectual Aims. When one looks even casually at the STA program, one is struck by the fact that there is undoubtedly an emphasis upon intellectual aims. In essence this forms my first postulate: the STA course is essentially directed towards intellectual rather than personal, social, societal, cultural, or vocational aims; and this is in marked contrast to traditional courses.

That this is in keeping with the expressed desires of Albertans is clearly shown by the expression of such wishes as revealed in two key studies - the Andrew study, and the Downey study, both of which showed that the intellectual aims rank first in both public and professional opinion of the task of our schools in Alberta.

Application of Knowledge of the Processes of Intellectual Development in Children. My second postulate is that modern courses and their methods of instruction must be and are in keeping with what we know of the processes of intellectual development in children. The STA course does, in fact, do more than pay lip service to this fundamental principle.

The work of Jean Piaget has indicated that there are five discernible stages in perception in the growing child. They are as follows:

1. The sensory-motor stage, from birth to about 2 years of age.

2. The stage of pre-operational thought, from about 2 years to about 4 years.

3. The stage of intuitive thought at the kindergarten and primary level.

4. The concrete operations stage in the balance of the elementary grades.

5. The formal operations stage, which begins in the late elementary and continues through the junior high school grades.

As stages three to five are of major concern to us, I propose now to outline these stages in slightly more detail.

Stage Three - The child lives in a world of symbols. Through word and image symbols he has created for himself a stable internal world in sharp contrast to the shifting, changing world of perceptions previously known.

From immediate imitation of present sounds, actions and beings, he has progressed to deferred imitation of absent things. The abstraction from action to thought has been accomplished; yet but the first faltering steps have been taken in the long walk from the particular to the general. He still relies upon transductive reasoning, that is, he still holds that as he knows the results to be right, then so must the means or process of arriving at the results be right!

His notions of groups and especially of groups of operations are still hazy, and ideas of invariance are just beginning to take root. Soon, he will be thinking operationally in constructing concepts of groups of operations with invariant features.

Stage Four - In stage four he has progressed to the stage of operational groupings. He puts classes together mentally, classifies objects and actions, and forms more inclusive classes from the combination of several sub-classes. He begins to serialize asymmetrical relations of "greater than" and "less than". He begins to be capable of understanding that number systems are products of classification and seriation, or ordering.

Ideas of space, time, number, and of the material world around him flood into his mind, but these are complex ideas, and he is, so far, only capable of comprehending them in more-or-less concrete terms.

He is beginning to be less ego-centric in his attitudes and behavior and the elements of detached logical reasoning emerge slowly and often painfully.

Stage Five - By the end of the elementary school, the child's ability to perform abstract operations becomes apparent. The child of the earlier stages has been concerned with action-in-progress, with the here-and-now. At this stage of the pre-adolescent, he thinks beyond the present, back into the past and forward into the future. The historical sense emerges. Hypotheses begin to be formed and tested. The elements of postulational thinking and of logical deduction struggle for form and function.

My second postulate, then, is that the STA course is consonant with knowledge of the development of thought in children.

Perception and Perceivers. If perception is considered as the process of organizing and interpreting sensations received through the senses, then it can be shown that there is evidence for the existence of characteristic modes of perceiving and for the existence of certain types of perceivers.

Some people may have a preferred sense: the visiles, those who perceive best visually; the audiles, those who perceive best auditorily; and the tactiles, those who perceive best through kinaesthetic means.

Two main types of perceivers have been postulated, the analytic and the synthetic. The analytic tends to concentrate upon isolated detail, rarely seeing the total patterns in a situation at first, but gradually synthesizing the detail into a whole. The synthetic, on the other hand, sees the total field as an integrated whole which he later analyzes in order to perceive the details. These ideas have import for classroom teachers in that they suggest that our grouping procedures, introductory work with symbols at all levels of complexity, provision for individual differences, and our remedial work should be planned with the ideas of the different modes of perceiving and the different types of perceivers in mind. This, in essence, is my third postulate.

Scientific Method and Problem Solving. In traditional arithmetic programs, it was assumed that the fundamental processes are best learned through association and drill, and so there has been a tendency to confine thinking in this area to associative thinking alone, even in the so-called problem-solving activities within the program. This solving of problems through the use of techniques of associative thinking is in sharp contrast to the problem-solving activities of STA. Here the definition of problem-solving is closer to "the process of overcoming difficulties encountered in the attainment of objectives". The sequence of steps used is:

1. Comprehension - read the problem carefully.
2. Translation - write the equation that represents the action in the problem.
3. Computation - do the computation that accompanies the equation.
4. Interpretation - write the statement that answers the problem.

The thought processes involved then are not merely those of associative thinking but are extended to convergent thinking.

No longer do we have to rely upon key words and gimmick phrases such as: "Look at the numbers in the problem. Decide whether to add, subtract, multiply, or divide." - without ever being shown how to decide! In other words, the STA program invites a higher form of thinking in its problem-solving techniques, and makes use of two very powerful and refined aspects of mathematical technology, one of which is the equation. This is my fourth postulate.

Emphasis Upon the Contribution of Knowledge to a Recognized Subject Matter Discipline. Among the many criteria that have been used by curriculum workers in recent years - and these include "survival", "utility", "interest", "social significance", and so on - the contribution that a particular subject makes to an organized field of knowledge has tended to become the major one employed. It is not hard to see that traditional courses in elementary arithmetic have failed to meet this criterion, as they were, even in their problem-solving aspects, concerned mainly with computational speed and accuracy. Children were required to jump immediately from the statement of the problem to the computation which solved it. In the STA program, the intermediate step of translating the problem into the appropriate equation deduced from the mathematical action perceived in the problem, makes a definite contribution to the study of more advanced mathematics, and hence contributes something to a recognized subject matter discipline.

The essence of the application of this criterion in other aspects of the STA program is the emphasis upon structure: that is upon the basic principles of and the patterns of relationships within the subject. The STA program has reselected and replaced content and method in order to reveal clearly to the pupils the underlying principles and relationships that give mathematics its structure.

My fifth postulate, then, is that modern courses should - and STA does - contribute knowledge to a recognized subject matter discipline and reveal the structure of the discipline to the pupils.

The Seeing Through Arithmetic Program

From the five postulates I have just discussed with you, one could derive certain theorems which would indicate that the STA program is constructed on the following principles:

- It emphasizes the mathematical values and aims of arithmetic, rather than the social or other aims and applications, thus contributing to the basic intellectual objectives of modern education.

- The STA course recognizes and employs the most recent research findings in developmental psychology.

- In its methodology, the STA course attempts to appeal to all types of perceivers through all modes of perception.

- The STA program moves beyond associative type thinking to at least convergent thinking in its problem-solving aspects.

- The STA program emphasizes the structure of mathematics, employs several powerful aspects of mathematical technology, and thus contributes greatly to the recognized discipline of mathematics.

Deductive Nature of the STA Program

Few would deny that the Seeing Through Arithmetic and other modern approaches to elementary mathematical arithmetic are vast improvements over traditional arithmetic programs. It is my contention, however, that improved as they may be, they still suffer from a serious flaw, and strangely, this flaw arises from what I consider to be too restrictive an interpretation of a current definition of mathematics, namely that "mathematics is the search for patterns".

In the STA program, this is interpreted to mean: "Mathematicians have found certain patterns in elementary problem-solving, and have invented certain equations to describe and to reveal these patterns." Two of the equations so used can be shown generically as follows:

(1) $\boxed{a \ o \ b \ R \ c}$

(2) $\boxed{a : b \ R \ c : d}$ (rate-comparison)

Examples of the first equation are:

$$5 - 3 = 2$$
$$12 + n = 4$$
$$3 \times n = 20$$

Examples of the second are:

$$15/3 = n/1$$
$$n/20 = 35/100$$
$$3/4 = 4/n$$

As, in the first equation, the "unknown" n can appear in any one of three places, replacing either a, or b, or c, and o can be any one of "add", "subtract", "multiply", or "divide", whilst, for almost all problems, R is restricted to "equals", it then follows that there are

3 x 4 x 1 = 12 basic types of non-comparative problems.

In the case of the second, n can replace, a, b, c, or d, and R is again generally restricted to "equals", thus there are 4 x 1 = 4 basic types of rate-comparison problems.

This means that our pupils sequentially, developmentally, and systematically are taught to translate all their problems into one of the 16 basic equations, are taught a corresponding process to solve the equation, and are thus restricted to a convergent type of thought process.

The problem-solving approach, too, is necessarily deductive in nature:

- Comprehend the problem.

- From the structure of the problem, determine the structure of the equation that represents it.

- Perform the standard computation for this equation-form.

- Interpret the results.

Now this is, as I have said before, a tremendous advance over traditional courses, and a most welcome one. My plea to you, however, is for the introduction of certain inductive methods to supplement this deductive approach, to broaden problem-solving, to involve more than associative and convergent thinking, to involve divergent, inductive thought in our mathematical problem-solving activities. So often, in life's problems, there is no standard, ready-made pattern of solution: one has to wrestle with the problem inductively, and the pattern only emerges after strenuous divergent thought. Let us equip our pupils with at least the readiness steps for this scientific mode of thinking.

Inductive Patterns

(Note:- the following dialogue is based upon ideas presented by Professor G. Polya in his two-volume series entitled *Mathematics and Plausible Reasoning*, published by the Princeton University Press, 1954.)

Dialogue to Illustrate the Sequence of Steps in Inductive Problem-Solving

Teacher: - (holding up before the class a regulation chess-board). How many squares are there on this chess-board?

Pupil: - (counting). Eight rows of eight . . . sixty-four.

T. - Is that all you can see?

P. - Yes, thirty-two are black and thirty-two are white.

T. - Good. What are the dimensions of each of the squares that you can see?

P. - Oh, about two inches by two inches, I guess.

T. - That seems a reasonable estimate, but - (holding up a second then a third board) - what are the dimensions of this, and this?

P. - Oh! I see. I think you can say that each of the squares is one unit in area.

T. - Good. For each board, although they are different in area, any one of the squares you can see can be said to be one unit in area.

P. - Oh! Wait! I see now! The whole board is a square and (excitedly) . . . there are some squares two units by two units . . . some three-by-three . . . There are hundreds of squares!

T. - Now you are beginning to see that the question I asked wasn't really an idle one; but <u>are</u> there actually hundreds?

P. - Well - there certainly are a lot, and (ruefully) they are very hard to see!

T. - So! They are hard for the eye to see, so let us try to "see" them in a different way - through a mathematical microscope, so to speak.

P. - How can you do that, sir?

T. - The way we have "seen" through other problems: let us <u>order</u> what we see. Let us find some pattern in the quantities before us.

P. - What sort of patterns do you mean, sir?

T. - (smiling). Look above you at the acoustic tile in the ceiling. Look beneath your feet at the tiled floor. Suppose, now, that the ceiling and the floor were like this chess-board, each a square structure of squares. Now! How could I rephrase my original question, "How many squares are there on the chess-board?" so that it is more general, so that it could refer to the chess-boards, the ceiling, the floor, in fact to any square structure of squares?

P. - (after several false starts) . . . How many squares are in an n by n square structure of unit squares?

T. - Very good. Now instead of confining our attention merely to an 8 x 8 board, we have now generalized our problem, namely, we are considering an n by n board. We have created many more problems than the very restricted one we started with. Often the solution to a set of problems is much simpler to find than is the solution to one member of the set. So let us now solve the new general problem.

P. - I know, sir! We have done this type of problem before!

T. - Good. Then what is the next step?

P. (1) - Find the simplest case . . .

P. (2) - Find the first case . . .

P. (3) - Use one . . .

T. - Good! Good! Now take it easy! Yes, you are all correct. What is the specialization, the simplest form of our new general problem?

P. - How many squares on a one by one, sir? The answer is one.

T. - Yes, of course. Now what?

P. (1) - Try a two by two, sir . . .

P. (2) - The second simplest . . .

P. (3) - Two units by two units . . .

T. - I'll need a two by two to clobber you if you shout so exitedly!

P. - There are five squares now: one big one and four little ones.

T. - Good. Now what?

P. - A three by three has ten squares.

T. - How many unit squares has it?

P. - Nine, sir . . . oh! I see, there are nine one by one's and . . four two by two's and one three by three . .

T. - Excellent, now you are beginning to see through your mathematical microscope! In fact, you are well on your way through the inductive process . . .

Now let us put some order into our observations. (writes).

n x n	No. of squares
1 x 1	1
2 x 2	1 + 4
3 x 3	1 + 4 + 9

Can you see the pattern?

P. - Yes sir, the next should be 1 + 4 + 9 + 16, 30, sir!

T. - You have formed a conjecture about an unfolding pattern. Conjectures are tricky things. Can you verify it?

P. - Yes sir! On a four by four board there will be 16 little squares, then there will be four two by two's and, let's see now . . . nine three by three's . . . Yes sir, it is verified.

T. - Yes, for one case, but is this enough?

P. - We could try it for five by five.

T. - Of course, and for other cases too, but will we then have proved our conjecture?

P. - I guess not sir, but if every one we try is true, then

T. - Well?

P. - I see sir, we cannot prove it, I guess.

T. - Later on you will find a way to prove conjectures of this nature, it is called mathematical induction. However, we will now assume our conjecture to be true if it checks out on one or two more.

P. (1) - It does for five, sir.

P. (2) - And for six . . .

T. - Very good, but let us not forget our problem! What was it?

P. (1) - How many squares are there on a chess board?

P. (2) - . . . on any square board!

P. (3) - . . . on an n by n . . .

T. - Good! Let us look again at our numbers.

n x n	Number
1 x 1	1
2 x 2	1 + 4
3 x 3	1 + 4 + 9
4 x 4	1 + 4 + 9 + 16

What do you notice in the series under "number"?

P. - They're squares, sir!

T. - Excellent!
(writes).

n x n	Number	Series
1 x 1	1	1^2
2 x 2	1 + 4	$1^2 + 2^2$
3 x 3	1 + 4 + 9	$1^2 + 2^2 + 3^2$
4 x 4	1 + 4 + 9 + 16	$1^2 + 2^2 + 3^2 + 4^2$

What would be the series for an n by n?

P. $1^2 + 2^2 + 3^2$ . . . up to n^2?

T. - Excellent! Now - if our conjecture is true, we have solved our general problem, and along the way, our special problem of the chess board. Does anyone have the answer to our original problem?

P. - (chorus) - Yes, sir. Two-hundred four!

T. - Excellent . . . ah! there's the bell.

. . . .

> To instruct and educate a child, therefore, does not mean to overwhelm him with bits of knowledge and with precepts, but to provide him, in proportion to his capacity and his needs, with such nourishment as he is capable of assimilating; to educate is not an effort from the outside to impose behavior or knowledge; it is to put the child in a position where he can make that effort himself ("Children are springs, not wells", a Belgian minister once wisely observed).[1]

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[1] Robert Dottrens, The Primary School Curriculum. Unesco, Place de Fontenoy, Paris - 7e, 1962. (p. 152).

Summary of Steps (At an Adult Level)

How many oranges are there in a pyramid with an equilateral triangular base if the smallest number of full dozens of oranges is used?

Generalize: How many oranges are there in a regular triangular pyramid?

Specialize to simplest cases: How many in the top layer? (1), in the next layer? (3), in the next? (6).

Conjecture the pattern: 1 + 3 + 6 + 10 + . . .

Verify for several cases: e.g., the fifth term in the series should be 10 + 5 = 15.

Verification:

```
0
00
000
0000
00000      (verified)
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State the general problem: What is the sum of the first n triangular numbers?

Search for patterns within the pattern:

First term		Second term		Third term	
0	1	0		0	
		00	1 + 2	00	
				000	1 + 2 + 3

n^{th} term: 1 + 2 + 3 + ... n

Now, look closer!

1 + 1 + 1 + ... + 1 Sum = $\frac{n}{1}$

1 + 2 + 3 + ... + n Sum = $\frac{n(n+1)}{1 \times 2}$

1 + 3 + 6 + ... + $\frac{n(n+1)}{1 \times 2}$ Sum = ?

Conjecture:

$$\frac{n}{1} \qquad \frac{n(n+1)}{1 \times 2} \qquad \frac{n(n+1)(n+2)}{1 \times 2 \times 3} \qquad (?)$$

Verify for several cases:

One layer $\frac{n(n+1)\ (n+2)}{1 \times 2 \times 3} = \frac{1 \times 2 \times 3}{1 \times 2 \times 3} = 1$ (✓)

Two layers $\frac{n(n+1)\ (n+2)}{1 \times 2 \times 3} = \frac{2 \times 3 \times 4}{1 \times 2 \times 3} = 4$ (✓)

etc.

State the conjectured solution: The general solution is that there are $\frac{n(n+1)\ (n+2)}{1 \times 2 \times 3}$ oranges in a total of n layers.

Solution of special problem: The solution to the special problem is:

$$1 + 3 + 6 + \ldots + \frac{(n)(n+1)}{1 \times 2} = \frac{n(n+1)\ (n+2)}{1 \times 2 \times 3} = 12\ p$$

New problem: What is the smallest value of n to satisfy (in integers)

$$\frac{n\ (n+1)\ (n+2)}{1 \times 2 \times 3} = 12\ p \quad ?$$

Solution:

$$\begin{aligned} n(n+1)\ (n+2) &= 1 \times 2 \times 3 \times 12 \times p \\ &= 1 \times 2 \times 3 \times 2 \times 2 \times 3 \times p \\ &= (2 \times 2 \times 2) \times (3 \times 3) \times p \\ n(n+1)\ (n+2) &= p \times 8 \times 9 \end{aligned}$$

(intuition? or persistence?)

Therefore the smallest value of n is n = 7.

Check:

Layer 1 2 3 4 5 6 7
Number 1+3 +6+10+15+21+28 = 84 = 12 x 7

There are 7 dozen oranges in the structure

Recommendation

This pattern of elementary inductive problem-solving is fundamental to mathematical and scientific modes of thinking. Every attempt should be made to establish at least the readiness steps for such thinking in the elementary school. The application of this recommendation is left to the ingenuity of the arithmetic teacher.

Summary of Pattern

1. State the given, specific problem.
2. Generalize the problem.

   > This procedure of heaping up new problems may seem foolish to the uninitiated. But some experience in solving problems may teach us that many problems together may be easier to solve than just one of them - if the many problems are well co-ordinated, and the one problem by itself is isolated.[2]

3. Specialize to the simplest case, or to the simplest analogous form.
4. Begin the "inductive" process.
5. Form the conjecture through seeking out the patterns, the structures of the problem.
6. Verify the conjecture for specific cases.
7. If possible, prove the conjecture, (perhaps by mathematical induction).
8. Solve the general problem.
9. Solve the specific problem.

For the Thoughtful Reader

Try the pattern on this question:

Into how many portions is space divided by 5 planes in the "general" position?

Hint: Start with points dividing lines, lines dividing areas, then to planes dividing space. Good Luck!

[2] G. Polya, Induction and Analogy in Mathematics, Princeton University Press, Princeton, New Jersey, 1954 (p. 47).

THE PRE-SCHOOL CHILD'S CONCEPT OF NUMBER

Alec Brace
and
Dr. L. Doyal Nelson

Mr. Holditch has abstracted the following article from a long and complicated masters thesis by Alec Brace under the direction of Dr. L. Doyal Nelson. In 1962-63, Mr. Brace, who has now left us for duties at Memorial University, St. John's, Newfoundland, conducted a research of elemental importance to all primary teachers of number. Administrators, principals, and supervisors, have traditionally followed practices of admittance and grouping of beginners which might well be re-examined in the light of this research.

Rote or Concept?

In this research, the basic number knowledge of children was investigated. The researchers assume that rote memorization of symbols and facts forms a poor basis for early growth of number concepts. Based on this idea, the writers endeavored to find what school beginners know. In this way the most important factors would be found which influence early growth of number knowledge. The influence of sex, older brothers and sisters, social, economic and age factors on play school children in the Edmonton area were assessed in this study.

Accepting these as the influencing factors, the following questions were investigated:

(a) Was there a relationship between rote counting and number concept?

(b) Was there a difference in number knowledge of pre-school children coming from homes of high and low socio-economic levels?

(c) Was there a difference in number knowledge between boys and girls?

(d) Was there a difference in number knowledge attributable to older siblings in school?

(e) Was there a difference in number knowledge in children from age five and one-half to age six and children from age six to six and one-half?

Background of the Study

What children know about numbers in early years has been a controversial point amongst writers on the subject; ranging from the point of view that such knowledge is meagre to the opinion that it is extensive. Piaget set forth the idea that there were three stages in the development of number concept and accordingly made the following classifications.

(a) Pre-operational Stage (age 4 to 5) - children have not developed an idea of invariance; i.e., objects in a group change when arrangements of the objects change.

(b) Intuitive (age 5 to 6) - children are incapable of reasoning logically with number, but vague ideas are evident.

(c) Operational Stage (age 6-1/2 to 8) - concrete reasoning is possible, but abstract reasoning develops at a later age.

Although Piaget was critized for the way he found his samples and some of the odd ways in which he handled his statistics, those who followed in this study of children's thoughts about numbers, such as Dodwell, Lunzer and Holmes still accepted Piaget's classification. Estes, however, took an opposite point of view about "number concept". Further investigations along this line by such investigators as Brownell, Gunderson and Mott do not apply since their work was carried out using verbal responses. Piaget, Dodwell and the others had always confined investigations strictly to observation of things done by the children they were investigating, as opposed to verbal responses.

Experimental Design

Play school children attend three two-hour sessions each week in Edmonton. No formal instruction is permitted in this program. The program included children of the required ages for the year 1962-63. Included in the sampling were forty-nine of age five and a half and under, forty between ages of five and a half and six years and thirty-five of age six and over. There were sixty-three boys and sixty-one girls. The sample was considered representative although admittedly urban. Pilot tests were prepared and used to develop a test of fifty-four items which was divided into six sub-tests:

1. Rational Counting - this was designed to test one-to-one correspondence of names and objects counted. Discs or sticks were recognized in ones, twos, fives and tens.

2. Comparisons - groups of colored blocks, toy horses, dolls or rattles were compared for equalities and inequalities of numbers of related objects:- i.e., Is there a horse for each rider?

3. Conservation of Number - variance and invariance of totals were tested. This test was carried out as follows:

Blue and red blocks were first placed side by side in one-to-one correspondence; then the red blocks were equally spaced but with greater spacing between them than between the blue blocks. The child was then asked if the number was the same or different. In a similar manner the equalities of colored liquids in different sizes of containers were compared.

4. Cardinal Property of Number - recognition of objects in a group without counting was tested with cards of colored buttons varying in the number of buttons from two to nine. Both patterned and random groups were used for identification.

5. Ordinal Property of Number - ten toy horses arranged in single file (running a race) were employed: i.e., Which horse is in third place? Similarly firemen climbed to the fifth step of a ladder or a car was parked on a lot in the third place of the second row.

6. Place Value - concept of place value was tested with bundles of ten sticks. To these bundles additional single sticks were added to make groups of eleven, nineteen, twenty-three and so forth.

Administration, Scoring and Analysis

The tests were administered from March 15 to April 30, 1963, individually to children by one of the writers, with an assistant who recorded results. Of the fifty-five items tested the first four questions involving counting were recorded as logarithms of the highest number reached in each item. These logs were each multiplied by ten and then summed as a composite counting score. The remaining questions (5-55) were assigned a score of one or zero.

A factor analysis was made of the intercorrelations of items five to fifty-three, revealing the fact that seven separate factors were being tested by forty-nine items. Item 5 as tested consisted of two factors. Item 4 (cardinal numbers) also consisted of two separate factors. Only those with a factor loading of 4 or above were used in defining a factor. Below this loading they were discarded.

When the researchers had analyzed the factors to see which were the same in part or wholly, they found that they were really only testing about seven items:

1. Ordinal number
2. Conservation of number
3. Place value
4. Recognition of groups to five

5. Recognition of groups above five
6. Comparisons
7. Ordinal number

Omitting all the statistical details we can say certain things about these items as shown in this study. The business of counting "one", "two", "three", etc., is always easier for these beginners than counting "two", "four", "six" - or any other way. Indeed, with these beginners it was found that almost three-quarters of them were unable to count by 5's or 10's. Strangely enough it was also found that those who were able to spot the correct number of colored blocks, regardless of how they were arranged, could also recognize the number of buttons on a card without laboriously counting them out. If a little fellow could easily pick out numbers on a card, the chances are that he would also know which horse was in third place.

As well as being able to count "one", "two", "three", beginners could also tell,when looking at groups of buttons on cards, which cards held the largest number of buttons. This was done correctly more often by sight than by count. Older children did not exhibit this ability as well as the younger pre-schoolers. Four-fifths of the children were deceived when a quantity of liquid was poured from a thin narrow vessel to a wide mouthed one; they thought the amount of liquid had changed. Recognizing the numbers one to five on a card was easy; beyond five was difficult. Girls and boys could do all these things with equal ease; however, those children who came from families in the higher income brackets obtained the best scores.

Consequently, if you are trying to decide whether your pre-schooler is likely to do well at arithmetic don't decide on the basis of the highest number to which he can count. If a child says "nine" it does not necessarily follow that he knows what "nine" means or how it compares with "ten". It is significant if he knows whether he is on the seventh or even the fourth rung of a ladder. A more signigicant fact yet would be a proven ability to add three's to "ten" to make thirteen or two's to make twelve. Some of these skills which seem to be impossible for him at five-and-a-half might well develop when he reaches age six. It will still be very surprising if a little fellow, even after several years, can recognize the fact that when you pour liquid from a saucepan into a graduated beaker you have the same amount in each vessel.

REFERENCES

(1) Brownell, W.A. "Arithmetic in Grades I and II". Duke University Studies in Education No. 6. Durham, North Carolina: Duke University Press, 1941.

(2) Dodwell, P.C. "Children's Understanding of Number and Related Concepts". Canadian Journal of Psychology, XIV, 1960, pp. 191-205.

(3) Estes, B.W. "Some Mathematical and Logical Concepts in Children". Journal of Genetic Psychology, LXXXVIII, 1956, pp. 219-222.

(4) Gunderson, A.G. "Number Concepts Held by Seven-year-olds". The Mathematics Teacher, XXXIII, 1940, pp. 18-24.

(5) Holmes, E.E. "What do Pre-First Grade Children Know about Number". Elementary School Journal, April 1963, pp. 397-403.

(6) Lunzer, E.A. "Recent Studies in Britain Based on the Work of Jean Piaget". Occasional Publication No. 4. The National Foundation for Education Research in England and Wales, 1962.

(7) Mott, S.M. "Number Concepts of Small Children". The Mathematics Teacher, XXXVIII, 1945, pp. 291-301.

(8) Piaget, Jean. "The Child's Concept of Number". (Translated by C. Gattegno and F.M. Hodgson.) Routledge, 1952.

(9) Fisher, R.A. "Contributions to Mathematical Statistics". Article 32. New York: John Wiley & Sons, 1950.

(10) Wohlwill, J.F. "A Study of the Development of the Number Concepts by Scalogram Analysis". Journal of Genetic Psychology, Vol. XCVI - XCVII, 1960, pp. 345-347.

RESEARCH IN PROGRESS

At present the following research projects are in progress at the University of Alberta and you should hear of them in later publications.

Pelletier, Marcel - "The Measurement Concepts of Grade I Children".

Seward, Ronald - "The Relationship between Mathematical and Verbal Analysis".

Harrison, Bruce - "An Attempt to Compare the Understanding of Mathematical Ideas of Grade VIII Students in Various 'Modern' and 'Traditional' Programs".

Chamchuk, Nick - "Comparison of Mathematics Achievement of Grade IX Students who had STM Training in Grade VII and Grade VIII with Students who were Taught the Traditional Program"; the criterion in this last study will be the Grade IX Departmental Examination.

A MODERN APPROACH TO EUCLIDEAN GEOMETRY

A. L. Dulmage

Editor's Note: Dr. Dulmage is the newly appointed head of the Mathematics Department at the University of Alberta. He brings a broad experience with him from the University of Manitoba and the Royal Military College at Kingston. Those who attended the fourth annual meeting were impressed with his intimate knowledge of mathematics education, as is shown in the article which follows.

At the turn of this century the mathematician David Hilbert wrote his Foundations of Geometry in which he developed plane Euclidean geometry as an abstract mathematical system. The undefined elements of this mathematical system are point and line and the undefined relations are incidence, betweenness and congruence. Most of the new geometry texts which are currently being written for our schools are more or less faithful imitations of Hilbert's book. In this talk I want to make some remarks about this modern approach to Euclid.

In introducing the student to geometry in school there are two almost contradictory paths which should be followed. In the beginning in junior high school the student's spatial intuition should be developed. The dots and marks which he makes on paper - which he calls points and lines - must have some kind of physical reality for the student. It is in exactly this way that geometry developed historically. It must be pointed out, however, that there is no satisfactory physical definition of a point or a line. In junior high school the student should discover, at least in some psychological sense, those things which he will be later asked to assume as axioms.

The high school student should be introduced to geometry in the Hilbert manner as a mathematical system in which point and line are the undefined elements. It is important to try to explain to the student that the reason why point and line are undefined elements is not merely because it is impossible to define them as physical entities in our world but rather because every mathematical system has undefined elements and one or more undefined relations. The same situation prevails in arithmetic or in algebra. And yet these undefined elements and undefined relations are really undefined initally only. The thing we do immediately after stating the names of or symbols for the undefined elements is to write down certain axioms involving them, and as soon as an axiom is stated, the undefined elements are no longer undefined. They are partially defined in an abstract sense by the axiom itself. The more axioms we state, the more we define the undefined elements and the relations. In fact, it is our goal in constructing a geometry as an abstract mathematical system to state sufficient axioms that the elements are completely defined in the sense that there is essentially only one set of points and lines satisfying them. When this goal is achieved, we have then given a complete abstract definition of the original undefined elements. The very important point of view to impart to the student in high school is that in proving theorems he must not make use of a property of the points and lines unless this property has been assumed as an axiom or has been proved as a theorem from these axioms. The properties which he feels that <u>points</u> and <u>lines</u> should have as a result of his spatial intuition in junior high school must, in high school, either be assumed as axioms or proved as theorems from the axioms. In developing Euclidean geometry it is necessary to introduce only three relations. These are <u>incidence</u>, <u>betweenness</u> and <u>congruence</u>. You will see references also to parallelism and continuity, but the parallel axiom of Euclidean geometry can be described in terms of the incidence relation; continuity can be described in terms of betweenness.

The Incidence Relation

Instead of saying, "The point P is on the line p ", or, "The line p passes through the point P", we say, "The point P and the line p are incident." Instead of saying, "Lines p and q intersect at the point P", we say, "The point P is incident with both p and q ." We now state our first axiom:

> Corresponding to any two distinct points P and Q, there exists exactly one line p , which is incident with both P and Q. This line may be so designated as the line PQ.

The Betweenness Relation

Examples of axioms involving the betweenness relation are the following.

If P, Q and R are three distinct points which are incident with the same line, then exactly one of P, Q, and R is between the other two.

If the point P is between the point Q and the point R then P, Q and R are distinct points which are incident with the same line.

There are two essential things that must be achieved by the betweenness axioms, either by explicitly stating them as axioms or by deducing them as theorems. The first is that if the point P is incident with the line p , there are two sets of points, α and β , which are incident with p and have the following properties. P does not belong to α or to β . If Q is incident with p , then either Q = P or Q belongs to α or Q belong to β . The set intersection of α and β is the null set. If Q and R are both in the same set (i.e., both in α or both in β) then either Q is between P and R or R is between P and Q. If Q and R are in different sets, then P is between Q and R. We may say that the sets α and β are the two sides of P on the line p . The set union of set α and the point P is called a ray emanating from P. The set union of the set β and the point P is another such ray. Thus associated with every point P incident with a line p , we have two rays.

The second essential thing which must be achieved by the betweenness axioms is the following. Let p be any line. We must achieve the existence of two sets α and β of points which are not incident with p and have the following properties. The set intersection of α and β is the null set. If P is any point, then either P is incident with p or P belongs to α or P belongs to β . If Q and R are both in α or both in β , then there is no point which is between Q and R and is incident with p . If Q and R are in different sets, then there exists a point P which is between Q and R and is incident with p . The two sets α and β are the two sides of the line p in the plane.

An angle can now be defined as a pair of distinct rays on the same or on different lines, emanating from the same point.

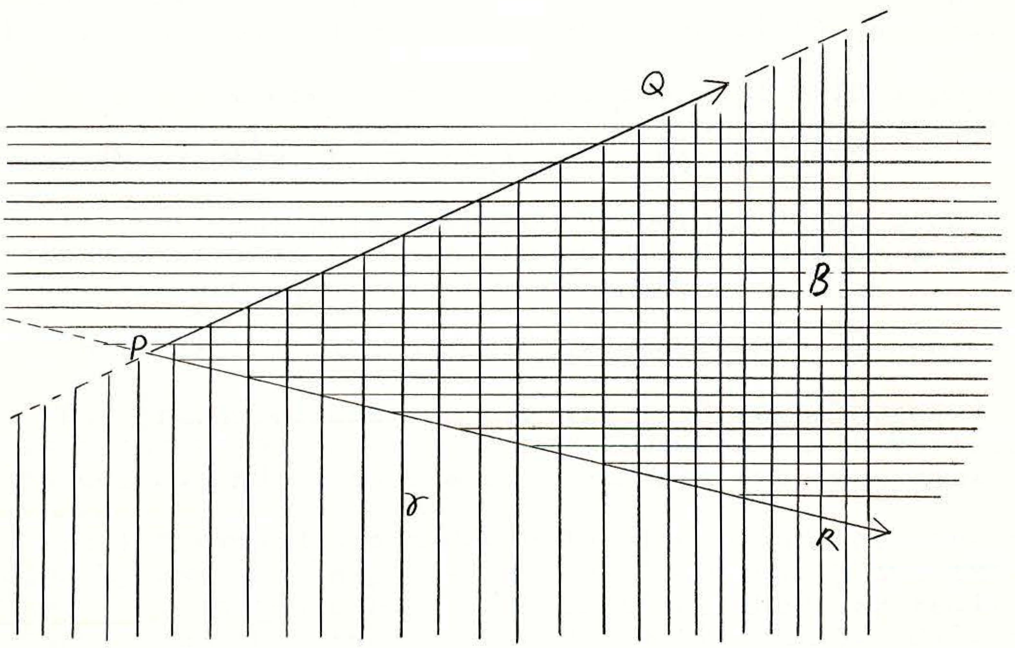

Figure 1

The interior of an angle can be defined as follows. Let the angle consist of the two rays PQ and PR emanating from P as in Figure 1. Let γ be the set consisting of all the points which are on the same side of PQ as the point R which is indicated by the vertical shading. Let β be the set consisting of all the points which are on the same side of the line PR as the point Q, as indicated by the horizontal shading. The interior of the angle QPR is the set intersection of γ and β.

If P and Q are distinct points, we define the segment PQ to consist of the points P and Q and all the points (incident with the line PQ) which are between P and Q.

If P, Q and R are three distinct points which are not incident with the same line then the set union of the segments PQ, QR and RP is called the triangle PQR. The points P, Q and R are called the vertices of the triangle.

<u>The Congruence Relation</u>

At the junior high level it is important to have the student realize that just as the idea of a one-to-one correspondence is more fundamental than counting, so the notion of congruence is more fundamental than length. Students should be encouraged to use a compass to decide whether or not two segments are congruent and it should be pointed out that this decision can be made without knowing the length of either segment.

There are two congruence relations, congruence for segments and congruence for angles. We denote the relation by the symbol $\cong$. The following are two examples of the axioms.

If P and Q are distinct points and if R is a point incident with a line p, then on the line p on a given side of R there exists exactly one point S such that $PQ \cong RS$.

If the point R is between the points P and Q on a line p and if the point T is between the points S and U on a line q and if $PR \cong ST$ and $RQ \cong TU$, then $PT \cong RU$.

There are similar axioms for congruence of angles.

Two triangles ABC and DEF are said to be congruent if there is a one-to-one correspondence between their vertices such that the corresponding angles and segments are congruent.

Euclid proved his side-angle-side congruence theorem for triangles by using superposition i.e., picking up one triangle and placing it on the other. In his proof there is really the tacit assumption that the triangles are congruent. Hilbert gets around this difficulty by taking the side-angle-side theorem as an axiom. It is possible then, using this one axiom to prove the other familiar theorems concerning congruence of triangles. Some of the new texts on geometry assume all the congruence theorems as axioms. It is true that this approach may be all right if "you prefer thieving to hard labour" as the mathematician Betrand Russell once remarked in a similar connection. But from a logical point of view, it is considered undesirable to assume as an axiom a result which can be proved from the axioms we already have.

Parallelism

As a preamble to parallelism we can prove that the exterior angle QRS in Figure 2 is "greater than" the angle PQR. To see this

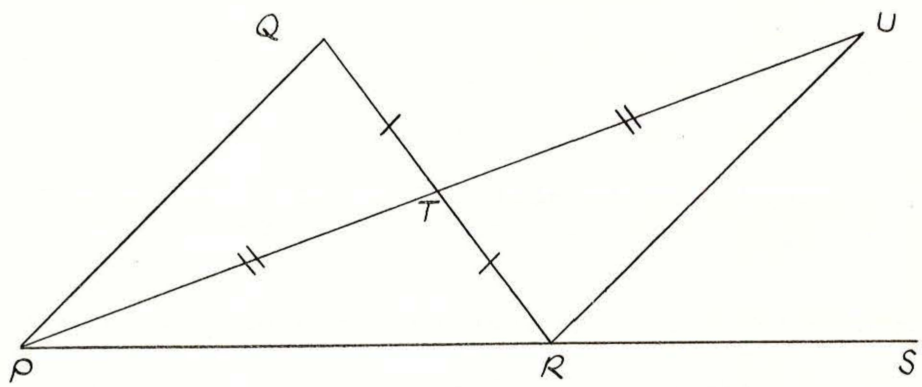

Figure 2

we bisect the segment QR at T and take PT $\cong$ TU as indicated. Using betweenness we see that U is in the interior of angle QRS. Triangles QPT and RUT are congruent and hence angle PQT is congruent to angle TRU.

We now prove the following important theorem which we will refer to as <u>Theorem A</u>.

> If the point P is not incident with the line p, then there is at least one line q which is incident with P and has the property that no point is incident with both p and q.

<u>Proof:</u> As in Figure 3. let Q be any point which is incident with p and let S be a point such that angle RPS is congruent with angle PQT with P between R and Q and S and T on the same side of PQ. Denote PS by q. If there exists a point U which is incident with both p and q, there are two cases to consider. First let us suppose that U is on the same side of PQ as S and T

are. Then angle RPS is greater than angle RQT by the previous theorem. This gives us a contradiction and we get a similar

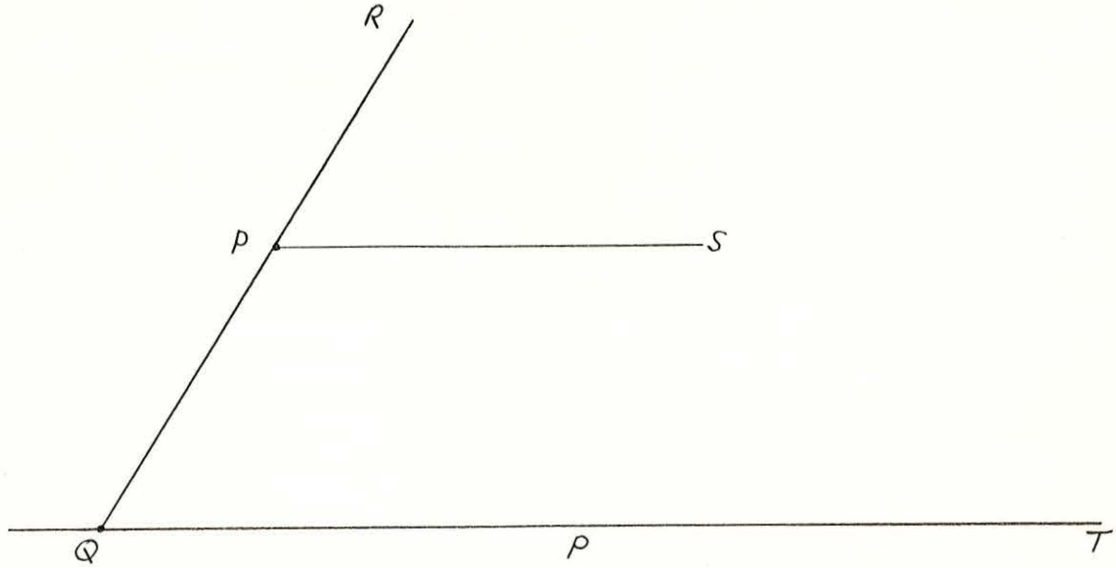

Figure 3

contradiction if U is on the other side of PQ. Since U cannot be incident with PQ, we see that no point U exists that is incident with p and q .

We now state the parallel axiom of Euclidean geometry.

> If the point P is not incident with the line p , then there is, at most, one line q which is incident with P and has the property that no point is incident with both p and q .

We are now in a position to prove the following important theorem.

> Let P be any point incident with the line q and let Q be any point incident with the line p . Let P be between Q and R. Let S be incident with q and T incident with p and let S and T be on the same side of PQ. Then there is no point incident with both p and q if and only if angle RPS is congruent to angle PQT.

Proof: If angle RPS is congruent to angle PQT we have seen in Theorem A that there is no point incident with both p and q .

Conversely, if there is no point incident with p and q, let U be a point on the same side of QR as S and T so that angle RPU is congruent to angle PQT as in Figure 4. According to Theorem A there is no point incident with the line PU and the line p.

But we are given that there is no point incident with p and q . Thus, by the parallel axiom we see that PU and q are the same line. Thus angle RPS is congruent to angle PQT, as required.

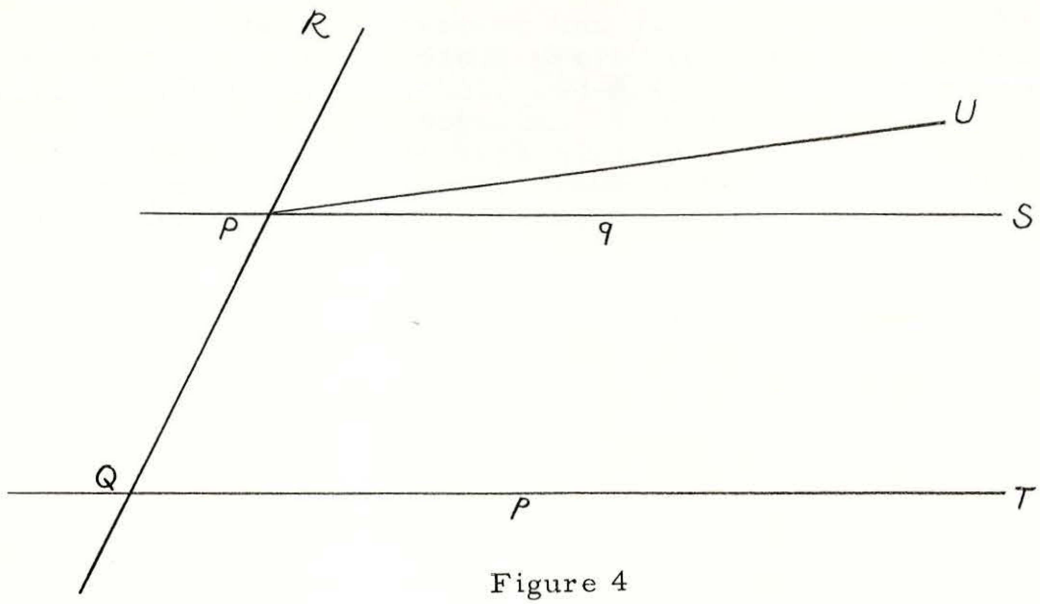

Figure 4

It is important to realize that there are non-Euclidean geometries which result from choosing different parallel axioms. One such axiom is that if the point P and the line p are not incident then there is more than one line q incident with P such that there is no point incident with q and p .

The geometry which results from this axiom is known as Hyperbolic geometry. On the other hand, if we assume that corresponding to any two distinct lines p and q , there exists exactly one point P which is incident with both p and q , then we get a geometry known as Elliptic geometry. In Euclidean geometry the sum of the angles in a triangle is two right angles. This sum is less than two right angles in Hyperbolic geometry and is greater than two right angles in Elliptic geometry. These geometries were discovered in the first half of the nineteenth century. In Elliptic geometry we do not have a betweenness relation. A relation of separation is introduced instead. Theorem A would seem, at first glance, to be a contradiction to the axiom above, which distinguishes Elliptic geometry. However, Theorem A was proved by using the exterior angle theorem which in turn was proved using betweenness.

In junior high school it is important to develop the student's spatial intuition. However, this should be done with the use of appropriate words so that the transition to the axioms introduced in high school is a smooth one. For this reason it is important that every teacher of geometry, even at the most elementary level, has a perspective in which he appreciates Euclidean geometry as an abstract mathematical system and realizes that there are non-Euclidean geometries which are equally satisfactory abstractions.

One of the very best of the current crop of books on school geometry is the Addison Wesley publication, "Geometry", by C.F. Brumfiel, R.E. Eichalz and M.E. Shanks. This book is intended for use in high schools. I would also like to recommend for your consideration the book by the same authors entitled "Introduction to Mathematics". This latter book contains an excellent introduction to geometry at the junior high level.