1965

ANNUAL

Elementary Mathematics Education Issue

MCATA Publication of the Mathematics Council of The Alberta Teachers' Association

In process,

MATHEMATICS COUNCIL OF THE ALBERTA TEACHERS' ASSOCIATION

1965 ANNUAL CURRICULUM REVISION ISSUE

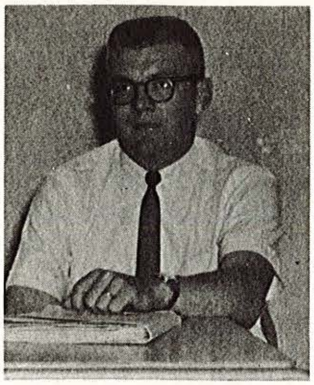
Contents

- continued

Contents (continued)

Membership in the MCATA is open to: (a) any member of the ATA covered by the TRF; (b) any member of the University of Alberta or the Department of Education. It is hoped that teachers of mathematics at all levels, elementary and senior high school, will take this opportunity to participate in professional development. As a member you receive all newsletters and the MCATA Annual which is a summary setting forth enriching material written by local and other writers. Even though our articles are predominantly of local origin our affiliation with NCTM has given us the choice of their best materials, allowing us to reach out for points of view of the writers in mathematics from other parts of the continent. All correspondence should be addressed to MCATA Annual, J. Holditch, Editor, 11035-83 Avenue, Edmonton.

EXECUTIVE FOR 1965-1966

The elected representatives for 1965-66 are: from left to right - Ted Rempel, Edmonton, President. Marshall Bye, Bowness, Vice-President. Mrs. Joan Kirkpatrick, Edmonton, Secretary-Treasurer.

EDITORIAL

MCATA is an affiliate of the National Council of Teachers of Mathematics. This year one of the major conventions of NCTM was held at Vancouver, British Columbia, August 26 to 28. Many of our members might benefit from a brief summary of the history, work and purposes of this organization.

History and Purpose

The National Council of Teachers of Mathematics was organized in 1920, incorporated in 1928, and became a department of the National Education Association in 1950. Its purpose - as stated in its charter - is to assist in promoting the interest of mathematics in America, especially in elementary and secondary fields and to vitalize and coordinate the work of local organizations of mathematics teachers.

The NCTM is the only organization of its type. It has had a major influence on the teaching of mathematics and the most outstanding persons in mathematics education have been leaders and participants in its programs. It serves as an agency for both the coordination and stimulation of major efforts to improve the teaching of mathematics.

Membership has grown continuously and rapidly. The total number of members has quintupled within only ten years. On December 1, 1964, there were 37,000 individual members, with 23,000 institutional subscriptions to the two official journals, the Mathematics Teacher and the Arithmetic Teacher.

Affiliation with other Organizations

The NCTM is a department of the National Education Association and an affiliate of the American Association for the Advancement of Science. It urges that its members also be members of these two organizations. The NCTM is a member of the Conference Board of the Mathematical Sciences.

Publications

Journals. The Mathematics Teacher, now in its 58th year of publication, is the oldest journal. It is published monthly except during June, July, August, and September and contains articles on mathematics and the teaching of mathematics at the secondary level by outstanding educators. It gives valuable information on such topics as testing, current practices, research, history of mathematics, reviews and evaluations, and tips for beginners.

The Arithmetic Teacher, first published in 1954, is concerned with the teaching of mathematics in kindergarten and in all the grades of the elementary school. It is also published monthly except during June, July, August and September. Special features include information on investigations and research, teaching and curriculum problems, testing and evaluation, classroom ideas, teaching aids and devices, and reviews.

Both the Mathematics Teacher and the Arithmetic Teacher are official journals. Membership dues include a subscription to either one, with the other journal being available for a small additional fee.

The Mathematics Student Journal, introduced in 1954, contains enrichment and recreational material for students in Grades VII through XII. A special feature is the problem section to which students may submit both problems and solutions. It is published four times a year (twice each semester) in November, January, March, and May. Both individual and group subscriptions are available for this journal.

Yearbooks. The yearbooks deal with timely problems in teaching of mathematics and have been published as needed since 1926. They have been outstanding contributions to the literature in their fields.

The 29th yearbook, Topics in Mathematics for Elementary

School Teachers, is the most recent. The two prior yearbooks presented enrichment materials for the grades and for high school. The next edition, now in preparation, will discuss the use of the history of mathematics in the teaching of mathematics. The cost of the books is not included in membership dues.

Supplementary Publications. The wide range of topics presented in these booklets and pamphlets is indicated by the fact that more than 40 publications are available that give information on teaching aids, tests, curriculum, teaching methods, study techniques, enrichment, and other subjects. Recent topics include the evaluation of textbooks, the twelfth-year mathematics program, utilizing computers in school mathematics, mathematics for elementary school teachers, and a compilation of challenging problems for students.

Conventions

Conventions are held in various areas in order to make them available to as many persons as possible. Meetings scheduled for 1965-66: Joint with NEA, New York, New York, June 29, 1965; Vancouver, British Columbia, August 26-28, 1965; Joint with AAAS, Berkeley, California, December 19, 1965; San Diego, California, March 11-13, 1966; Forty-Fourth Annual, New York, New York, April 13-16, 1966; Joint with NEA, Miami Beach, Florida, June 29, 1966.

Other Professional Activities

The NCTM is teacher oriented, exists to serve teachers, and derives its strength from the activities of its members. It serves the mathematics teaching profession through its many committees, panels, and projects and through joint action with other educational and professional groups. Examples of the many and varied activities of the NCTM are production of films for the in-service education of elementary school teachers, study of ways to improve mathematics instruction through a better use of educational media, and preparation of instructional materials for the non-college bound student. Assistance to affiliated professional groups is another avenue of service. There are now 117 area, state, and local groups throughout the United States and Canada affiliated with the NCTM. Both professional and financial assistance are provided to these affiliates to help them be even more effective in stimulating professional growth among mathematics teachers. We urge you to join the affiliated group(s) in your area.

Benefits of Membership in NCTM

You as a member receive, without additional cost, either the Mathematics Teacher or the Arithmetic Teacher, and the Newsletter.

You receive continuous information about publications, services, and meetings.

You receive, without additional cost, many professionally useful materials sent both by other organizations and, through special authorization of the Board of Directors, by the NCTM.

You pay lower registration fees when you attend national meetings.

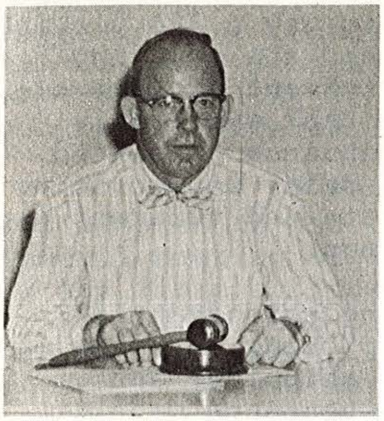
You are permitted to purchase at reduced fees one copy of certain yearbooks and other publications of the NCTM.

You have the privilege of participating in NCTM business, voting, and holding office. (Student members to not have this privilege.)

You identify yourself as a person of high professional ideals when you join with other frontier thinkers in mathematics education.

Your membership strengthens the NCTM and its program of service. Thus you make a definite contribution to the improvement of mathematics education.

PRESIDENT'S REPORT



To the Fifth Annual

Conference

August 27, 1965

Vancouver

L. C. Pallesen

Membership growth in MCATA, as reported by your secretary, is not so much a credit to the executive as it is a credit to members themselves. When changing curricula present a challenge, mathematics teachers rise to the occasion and become most active in their efforts to make certain they are as well prepared as possible to teach their classes. Participation in many activities indicates that Alberta mathematics teachers are sharply aware of the truism that no course or text can be effective without an enthusiastic, well-trained teacher. In addition to increased participation of teachers in MCATA projects, the manner in which enrollments in seminars, locally conducted inservice programs, conferences, and university courses have increased, reflects an almost phenomenal determination on the part of Alberta mathematics teachers to ensure that every Alberta student receives the best possible instruction.

During my term of office there were a number of opportunities to appraise the status of teacher preparation in other provinces and in several states. Nowhere have I felt there is a more enthusiastic body of teachers than in Alberta. I am not suggesting we have reached the point where we can complacently rest on our oars. Quite the contrary! It is vital that every individual who has had an opportunity to familiarize himself with recent developments should adopt a missionary attitude. There is a responsibility to help and enthuse colleagues who have not had similar opportunity to gain the necessary background that makes them feel at ease in their classroom.

The 1964 Annual General Meeting of MCATA instructed executive to investigate the possibility of securing films for distribution throughout the province. Arrangements were made with the University of Alberta, Calgary, to secure films prepared for the university course, Mathematics 341. Our agreement with the University was sufficiently flexible that MCATA did not stand to suffer financial loss regardless of the scope of the project. It was agreed to establish a viewing centre at any location where fifteen MCATA members indicated a willingness to pay a $20 viewing fee. (The regular fee for the course was $80 and the auditing fee was $40.) Early hopes were for the establishment of five or six centres, but the response was so favorable that eventually nineteen centres were arranged. The large attendance presented some difficulty in shortages of texts and some delay in providing brochures, however, members were most considerate and tolerant in their acceptance of these inevitable shortcomings.

Because of the necessity of centralizing arrangements your President handled scheduling of centres, picking up films and supplementary materials at the University, shipping, and collecting fees. At times the volume of correspondence made it difficult to be as prompt as desirable, but coordinators of the viewing centres were most competent and cooperative. The fact that membership fees to MCATA were, in many cases, included with viewing fees presented some problems to our secretary-treasurer. At the present time, three viewing centres have not made their final adjustments and it has not been possible to provide the treasurer with a statement to finalize the books. The sum of $7,849.50 was received from viewing centres of which $6,177.62 has been paid out. It is sincerely hoped that the 320 teachers who viewed the films at the various centres will find the hours spent have given them a sense of confidence and satisfaction in their teaching.

Through the generous cooperation of the Copp Clark Publishing Company, Saturday seminars were held in Calgary and Edmonton. Messrs. MacLean and Mumford, two of the authors of the newly authorized text for Mathematics 10 conducted these seminars at no cost to the Council. Because these speakers were provided by the publishing company, your executive decided to make the sessions open to all mathematics teachers with no registration fee. Arrangements for similar seminars in Peace River and Lacombe on October 1 and October 2 are in the process of being finalized.

During the year the Council cooperated with several convention committees by providing speakers and assisting in planning programs. This appears to be a way in which MCATA can render continuing and increasing service to teachers in the future. At a recent executive meeting initial steps were taken to establish a list of resource personnel by geographic location so the Council will be able to provide the names of available people on request. Some consideration should be given to establishing policy regarding the degree to which the Council should be prepared to subsidize costs involved in providing speakers and consultants to conventions, conferences and workshops. Because of the many instances where MCATA members have assisted with local programs, and because I am probably not aware of all such activities, it is not possible in this report to commend these persons individually. Some of them have been rewarded generously and concretely by local groups; others have displayed the missionary spirit mentioned in my opening remarks and received little or no recompense. Whether handling of such activities can or should be standardized will be considered during this conference.

MCATA is now affiliated with the NCTM. Your president was honored to receive the charter of affiliation at the Annual General Meeting of NCTM in Detroit. As an affiliated body, one of the first actions of executive was to request that the Name of Site Committee of the NCTM consider sponsoring an August 1966 meeting in Alberta. This request was considered by the directors in Banff with Father Egsgaard of St. Michaels College, Toronto, and Dr. Lund, President of the California Mathematics Council designated to examine the facilities of the Banff Springs Hotel as a possible convention site. Certainly members who made the trip to Vancouver this year and experienced the revitalizing effect of the excellent program will encourage their colleagues to make plans to attend a similar conference in Alberta. It has been learned that the NCTM Name of Site Committee have decided against Banff as a locale but would view favorably an application for a convention in Calgary. Members will be asked to give executive direction in this regard.

Two regional mathematics councils, Central Alberta and Greater Edmonton, have been active during the year with the former concentrating on work supplementary to the Mathematics 341 films and the latter conducting a study at the elementary school level. This study will probably be reported in a later issue of the Newsletter.

The present year has seen the publication of three Newsletters and the <u>1964 Annual</u> - all of a calibre of which our membership might well be proud. Our affilication with NCTM facilitates the exchange of publications with other councils throughout Canada and the United States and the future will see an increase in this type of activity.

I would be most remiss if I did not take this opportunity to thank members of the executive for their many efforts to help me - a rank novice to the executive of MCATA - avoid innumerable mistakes. I have found the year most gratifying and hope that MCATA has, to some measure, fulfilled its role of aiding and encouraging better mathematics teaching in the province.

LET'S TAKE A LOOK AT THE DUAL AUTHORIZATION IN MATHEMATICS - GRADE VII, 1965

H. L. Larson
Superintendent of Schools
Athabasca

Editor's Note: Administrators have made the choice of a Grade VII mathematics text for use in their systems for 1965-66. In the light of the fact that MCATA publications have given extensive coverage of STA it might be contended that they have given more than ordinary encouragement to this series. As editor of some MCATA publications I feel our policies have contributed to the landslide of STA authorizations this year. Accordingly, H.L. Larson, who has experience with both series appeals to you to give EMM the chance it deserves.

Because I have had considerable experience with the EMM series since they were first published, both in teacher inservice work and in actual classroom situations, I feel somewhat obligated to make some subjective remarks. I am in no way connected with the publishers, nor will I be in the future. I have subdivided my remarks into various headings, none of which are distinct or unique.

1. Rationale Building

EMM develops each concept in such a way that the student and teacher will soon wish to leave the confines of the text and search the literature for further understanding. This is in contrast to semi-programmed type materials that remind me of conducted tours. Each rationale is brief and to the point, using suspense and discovery to a very high degree. It is this aspect that seems to underly the general feeling of enthusiasm in classrooms that use this type of development.

Of course, I am mindful of the fact that much more depends upon the background and enthusiasm of the teacher. But teachers also like some elbow room in developing their own methods or operation and what teacher will not be learning as well as teaching these new materials for a long time to come?

2. Definitions, Assumptions and Theorems

These are kept to a minimum. Sophistication, in deductive proofs, is balanced nicely with intuitive and inductive processes for this age group of students. The major structures are not obscured by multitudinous directives. As one proceeds through the text, he feels he is riding the high-level mountain road rather than the detailed bushy pathways at the river's edge.

3. Symbolism

There are no unique symbolisms. They consist only of those that have met with general agreement in most of the contemporary math literature. This applies also to the hatch marks for Venn diagrams.

4. Zero a Natural Number?

Practically all modern texts define natural numbers as the set of counting numbers. Zero is included in the set of whole numbers. This emphasizes the distinctive nature and historical development of zero. EMM follows contemporary definitions in this respect so that students are less confused as they read beyond the text.

5. Problem Solving

The three major aspects of mathematics have always been and still are:

1. Conceptualization.
2. Computation.
3. Application.

Modern mathematics makes a radical shift in emphasis from application to conceptualization.

Traditionally, we have been concerned with the pragmatic notion that problems to be solved were mainly those of the workaday world in our immediate surroundings. More and more so-called "problems" were included in our math texts which might have been better placed in the study of community economics.

The only justification for any trumped-up problem situation is that by its solution we are contributing to the structural development of mathematics. For example:

"Mary had 6 dolls and 4 doll beds."

Our first step is to see that the student can take raw data of the physical world and recognize a problem. From the above data it would be fairly obvious to the average student, age 7, that Mary needs more beds. But the student should be trained to recognize the problem at this point and from his mental construct, state it in English. In the past and, I fear in the present, both teachers and authors have been all too eager to perform this task for the student, thus robbing him of the joy of discovery. The problem above then is "How many more beds does Mary need?"

The second phase of this problem is to translate this physical "mental construct" or model into a pure mathematical abstract sentence. This should be done in easy stages depending upon the maturity of the student - from the real object, to pictures, to blackboard representations, to symbols. The sentence should read "4 + n = 6". Notice that the translation should be from a complete English sentence to a complete math sentence. We can do this best by using algebraic symbolism.

Now we are ready for the third stage of our fictitious problem. Traditionally, we have placed all our emphasis on this aspect of the total process - called "computation". In this particular problem we would use one-to-one correspondence, inverse of addition and subtraction of whole numbers. These are three basic notions, the first two of which have been badly neglected in the past, especially the notion of inverse:

"4 + n = 6" or "n = 6 - 4".

The fourth step is to bring the problem out of the abstract math world back to the physical world with the desired data all intact. This we might call "interpretation".

All problem solving essentially consists of these four steps. However, in our zealousness to salve our traditional mass conscience, we still expect the authors to conjure up the perfect problem samplings of everyday living and many modern texts reflect this public expectation. Both the selection of, and the statement of the problems reflect mainly the degree of the author's ability to fictionalize. Is there anyone today any more capable of selecting truly representative samplings of problems from real life than his predecessors? Not only do many of these problems contribute little to the student's general math understanding, but they crowd out much that might be done by way of structural development.

I believe that this public expectation of problem solving will be a real deterrent to good textbook building in the future, except in school systems that have already made considerable strides in conceptual methodology.

I commend the efforts of the authors of EMM to break through this tradition, but I suspect that they will suffer in popularity because of it. The long-range view of EMM is that with a good background of structure, the student will be in a much better position to recognize and solve problems than if he should be taken by the hand and led in and out of a few pathways around the edge of the forest.

If EMM, 1963 issue, appears to be weak in the aspect of application (problem solving), I suggest this is more apparent than real. Just as piano players are not trained merely by playing a large variety of tunes, neither are mathematics learned by large samplings of applied problems.

In any event the Canadian edition should provide some improvement in problem solving if indeed such is needed at the Grade VII level.

I hope these observations will stimulate some second looks at EMM.

COMPUTATIONAL SKILLS IN ARITHMETIC "STA" VERSUS "TRADITIONAL"

Ted Rempel
MCATA President
1965-66

Editor's Note: We have Jim Wood, Joan Fitzpatrick and Ted Rempel to thank for this refreshing investigation. A careful perusal of these statistics may go far in allaying the vague fear that losses are being sustained in computational efficiency when students are taught under the STA program.

These are the Problems

Can the student entering Grade VII from an STA program add, subtract, multiply and divide as rapidly and accurately as the student coming from a traditional arithmetic? Is the STA program meeting the requirements "set out" by junior high school teachers? Will the Grade VII teacher have to spend the first five weeks of the school year drilling the so-called fundamental operations? These questions and others were submitted to an MCATA executive meeting, showing that there is some concern regarding the new math. At the meeting everyone felt that STA was doing as good a job as the traditional program had, but no one had any concrete evidence to back up his beliefs.

A Basis of Comparison

It was brought to light that the Edmonton Public School System was using a survey test during the fall term of each school year and that a part of this test dealt with the fundamental operations. It was agreed that this would not answer all the questions that had been asked and that would likely be asked in the future, but it was at least one step up the ladder of comparison. Permission was received from the Board to produce some statistics and compare them with similar statistics that had been produced some years back.

To make the comparison meaningful it is necessary to discuss the test, the history of STA in the Edmonton system, and the statistics obtained from the two testings.

The Test

Teachers of mathematics in the Greater Edmonton Public Schools had been administering a survey test to the Grade VII students for a number of years prior to 1961. Early in 1961, the Junior High School Mathematics Committee decided to revise the old survey test so that it would be applicable to both Grade VII and VIII levels. The revised version consisted of twelve pages of fundamental operations, four pages each of whole numbers, common fractions and decimal fractions. Each of these was subdivided into four sections, one each of addition, subtraction, multiplication, and division. Six other pages of the test dealt with language and operations. For this particular comparison only the first twelve pages of the test will be considered and discussed.

History of STA

The STA program was first taught in Edmonton in 1958-59. During that year one elementary school put some of its students on the new program. The next year another school joined the experimental ranks and in 1960-61 there were eight different schools teaching STA. The year of the big switch was 1961-62 for during that year 43 elementary schools were teaching the new math, some at all levels and some at only a few or one level. In 1962-63, 27 schools added an STA program and by the year 1963-64 a total of 79 schools were using the series. Students entering Grade VII in the fall of 1961 were almost entirely from a traditional program, whereas the group entering Grade VII in the fall of 1964 were almost entirely from the STA program.

Statistics Obtained

When the revised edition of the survey test was administered to 3,700 Grade VII students in October of 1961, results were forwarded to the Mathematics Committee. A sub-committee then analyzed the results and compiled modes, medians, means, and standard deviations for each sub-division of the test. Similar statistics were compiled when the same test was administered to about 5,000 Grade VII students in October of 1964. Using these statistics a detailed comparison of the 1961 results and the 1964 results follows in chart form.

1961 Results were recorded on 3700 students taught traditionally

Chart Showing Comparative Results of 1961 and 1964 Survey Test of Computational Skills

1964 Results were recorded on 5000 students taught with STA

Elements of Test	Nature of Computation	Time Minutes	Results 1961 Mean	Results 1961 S. D.	Results 1964 Mean	Results 1964 S. D.
1. Whole Number - Addition	12 vertical columns of 2 or more digit and 2 horizontal questions	5	5.45	2.13	5.30	2.14
2. Whole Number - Subtraction	20 vertical questions each consisting of numerals with 2 or more digits (greatest number had 9 digits)	5	8.26	3.52	9.53	3.47
3. Whole Number - Multiplication	14 questions all arranged vertically; multipliers ranging from 1 to 3 digits	5	4.90	2.23	4.72	2.29
4. Whole Number - Division	12 questions all set up in the traditional form with 1, 2, and 3-digit divisors	5	5.04	2.27	5.32	2.28
5. Common Fractions - Addition	17 questions of vertical type and 3 set up in horizontal method; all answers reduced to lowest terms	4	9.70	3.13	10.01	3.13
6. Common Fractions - Subtraction	20 questions, horizontal, mostly mixed numerals; all answers to be reduced to lowest terms	4	11.50	5.30	11.71	4.49
7. Common Fractions - Multiplication	18 questions, about 1/2 containing mixed numerals; 16 of the questions set up horizontally	4	7.33	3.41	6.93	4.03
8. Common Fractions - Division	18 questions are all horizontal in form and at least 1/2 the questions contain a whole number	4	8.56	5.32	6.24	5.44

Chart (Continued)

9.	Decimal Fractions - Addition	15 questions of which 2 involve dollars and cents and are set up horizontally	4	8.12	2.71	9.11	2.49
10.	Decimal Fractions - Subtraction	16 questions, 10 of them vertically set up	4	7.66	3.62	8.69	3.21
11.	Decimal Fractions - Multiplication	16 questions of which 6 questions had multipliers of 10, 100, or 1000	4	6.88	2.85	7.21	3.31
12.	Decimal Fractions - Division	20 questions with divisors of 1 or 2 digits; traditional format	4	7.80	3.35	7.14	3.41

Summary

Summarizing the results we see in 1, 3, 4, 5, 6, 7, 9, 11, 12 only very slight differences occur, but those which do occur favor the 1964 results. Where significant differences occur in 2 and 8, one favors 1961, and one 1964. Investigation shows that the groups were comparable and of sufficient size to form a reasonably good sample. It would appear on the basis of the above results that certain processes of division may have suffered, but that subtraction offset this to some degree. It must also be remembered that the original test was designed to test "traditional mathematics." One would have to say that results of computational skills in 1964 of STA-trained students compare favorably with students taught under the traditional program of 1961.

THE LABORATORY APPROACH TO THE TEACHING OF LOW ACHIEVERS

Shirley Myers
Associate Editor
The California Mathematics Council, Elementary Division
Southern Section

The laboratory arrangement efficiently accomplishes much of what is needed at the elementary level, to teach mathematics to the low achiever. It stimulates the student's interest, aids him to develop the arithmetic skills by doing, enables him to engage in creative activities in accordance with his own interest and talents, and allows exploration of appropriate portions of a selected and organized sequence of study.

The laboratory is characterized by giving attention to individual differences, use of audio-visual materials, pupil-teacher planning, and problem solving activities. The most modest effort is an improvement over the regular classroom. All of the new, and modern arrangements are ideal, but do not delay the program if money is the reason for putting if off. To begin, place the students' desks in clusters in an area around the demonstration table and overhead projector (if you have one). Establish a pupil-teacher work center, a library center, and a problem solving center where various problems may be worked out. Unused corners can be set up with various other available A-V materials. From this plan, a more elaborate mathematics laboratory may be built.

The overhead projector helps capture the attention of the student and allows the teacher to continually watch the reaction of the whole class. Student-made transparencies are an asset, for they challenge the learner to organize his ideas and recognize him for his accomplishments.

The filmstrip projectors and individual viewers naturally have their place in the lab. Many fine filmstrips are available for all levels of understanding. The film can be cast on the wall for easy viewing.

Teacher-made flannel boards of various sizes should be used and these materials made accessible to the student for repeat study of work already demonstrated. An additional benefit would be "class-made" flannel board items.

An area should be set up for extra text books and supplementary books of all types and levels, shapes and sizes. When available, answer books should be placed in this area. Allow students the freedom to use this area as they wish. The pamphlet from the National Council of Teachers of Mathematics entitled, "Library Books for Elementary Libraries" is a tremendous help in setting up a good varied selection for this "must" in a laboratory.

Encourage and stimulate students to create, design and put up room bulletin boards. This competition is healthy and a real experience for the students. All types of pupil inspired work are of the greatest value and teach far more than we can evaluate at the time.

The author saves all the used purple carbons of the ditto masters, staples a new piece of paper over each, and allows students to make student-designed material. After evaluation with the student, personal copies are run. When the master has class-wide value, a copy is made for each student.

As the mathematics laboratory develops, so will many new ideas. Students, books, articles, and daily experiences will be a continual source of ideas for laboratory teaching.

The environment of the mathematics laboratory stimulates and creates desirable attitudes. Further, it furnishes the thrill that comes with a sense of accomplishment.

DISCOVERY AS A METHOD OF TEACHING MATHEMATICS: A CRITICISM OF FIVE ARTICLES AND COMMENTS

R. Kraft

Edited by
Dr S. E. Sigurdson

Editor's Note: During 1964-65 R. Kraft was attending the University of Alberta as a fourth-year student. He has had experience in teaching at elementary school level in Saskatchewan for nine years. He was a candidate for the B.Ed. degree during 1964-65.

Discovering a Discoverer

I goes without saying that, left to himself, the child will go about discovering things for himself within limits. It also goes without saying that there are certain forms of child rearing, certain home atmospheres that lead some children to be their own discoverers more than other children. These topics are of great interest, but I shall not be discussing them here. Rather, I should like to confine myself to the consideration of discovery and "finding-out-for-oneself" within an educational setting - specifically the school - more specifically in the area of mathematics.

The articles reviewed in this report cover a method of mathematics instruction known as "discovery". Five such articles will be reviewed and commented on.

The first article I read was entitled "A Look at Discovery" by James Bolding of the Appalachian State Teachers College, Boon, North Carolina. Bolding feels the student discovers by finding the desired conclusion rather than having the conclusion pointed out for his acceptance. By asking carefully phrased questions a student is set to thinking.

Discovering a property has a distinct advantage over being told. Beginning teachers are usually anxious to explain everything in detail. This is not always a satisfactory technique. Students are not eager to be told; they are eager to find out for themselves. By rediscovering mathematics with the assistance of a teacher, three thousand years of mathematical development are cut short to the students' years at school.

When the proof of a theorem is very difficult, one can precede it by two or three easy lemmas or a group of exercises which lead the student to discover a proof of the theorem. This work need not necessarily be written. By careful questioning the student can finally solve the problem by himself. (Cite the classic example of Socrates finding twice the area of a given square.)

Fortunately, this was the first article I read. Because it was so general and said so little it aroused my curiosity. It served to raise a series of questions in my mind and thus provided the motivation to read on.

Discovery as Old as the Hills

It seemed to me that the definition of discovery was very weak. To what degree does the teacher play an active part in the discovery method, if indeed he plays any part? What technique is used in its presentation? How does one recognize the point at which a student has discovered the generalization? The point that did provide some element of surprise was the claim that in reality the discovery method is not a new instructional device but has a history traced back to Socrates. Certainly, he was the master of questioning and it's very true that his followers were led to discover the solution to their physical and metaphysical problems.

I think perhaps the best point in this article is that suggested lastly and only very briefly - that of incentive or motivation. As I see it, the discovery method serves to enlist a competence motive which becomes the driving force behind behavior. Once competence is internalized, the child is in a vastly improved position from several obvious points of view - notably that he is able to go beyond the information he has been given to generate additional ideas. But over and beyond that, the child is now in a position to experience success and failure as information to be used in future problem solving. In other words, what I am suggesting is that through the development of competence, extrinsic behavior no longer plays such a dominant part in the child's learning of mathematics. Instead, he will act from an intrinsic motivation.

But what does discovery really imply? And if the discovery method is really not such a new development after all, are we not presently providing for some elements in this theory in our teaching of mathematics? The answer to the latter question is apparently in the resounding negative according to an article entitled "What Do We Mean By Discovery?" by Harry C. Johnson, University of Minnesota. He states that educational literature is heavily charged with statements that learning, to be effective, must be meaningfully presented, and that teachers must modify their practices if appropriate changes in procedures are to be brought about.

Is the Element of Discovery a Criterion?

The newer programs also emphasize the importance of discovery - pupil self-discovery - as a criterion for judging the quality of a method of teaching or a method of presenting new materials. Almost all of the newer programs also insist that particular scope and sequence use the avenue of discovery to arrive at their designated goals.

Johnson goes on to say that we should examine whether or not our teaching really provides or allows for pupil discovery after all. According to him, pupil discovery in its literal sense should mean exactly what it says. Pupils discover for themselves. They are not told. They are not asked to trace a series of steps. They are not asked to do experiments with materials according to well formulated sequences. They are not asked to say "yes" to questions such as, "Do you see that the number 2 must satisfy the value of $\underline{n}$ in the number sentence $n + 5 = 7$?"

He says, it is perhaps better to admit that we are not really providing for pupil self-discovery after all in most of our teaching.

I cannot imagine a classroom of Grade X mathematics students turned loose on a program of undirected or unguided, fumbling and faltering discovery. I don't think this is being advocated in the article but nevertheless this seems to be his frame of reference towards the very definition of discovery. If this is discovery, I want no part of it. It seems to me, the author resents having such methods as stimulating thinking, directing the understanding, or providing for insightful experiences, known as discovery. If by discovery is meant a mere accidental acquisition of some form of knowledge, then, I feel, it has very little place in a classroom. I have found that there are not many Columbuses who would accidentally discover America without some guidance. For the person to search out and find regularities and relationships in his environment, he must be armed with an expectancy that there will be something to find and once aroused by expectancy, he must devise ways of searching and finding. But from whence comes this spark of expectancy and who helps to steer the searching if not some conscientious and dynamic teacher, or textbook?

Blind Leading the Blind

The next article was "Discovery" by Stephan S. Willoughby, University of Wisconsin. The theme of this article is that the discovery method may be more effective when the teacher does not know precisely where he is going and how to get there. Usually the discovery method is used to help pupils find out for themselves something that the teacher already knows. Although such knowledge in the mind of the teacher increases the efficiency of the process, some of the potential good in the discovery method may be missed if the teacher always knows precisely what he is looking for and so does not encourage the pupils to continue along lines that are unfamiliar to both pupils and teacher.

On occasion, pupils ought to be allowed and encouraged to go through many of the activities carried on by mathematicians - including following a blind alley to its end. By fostering such activities on the part of the pupils we can help them to understand the job of the professional mathematician.

He goes on to illustrate a particular example carried on at a laboratory school of the University of Wisconsin. The class was a group of high school pupils planning to continue their education following high school but not in the area of science or mathematics. The average grade of this class was slightly below C. All had taken geometry and half the class had taken a second year algebra course.

The purpose of the exercise was two-fold. First, to become acquainted with some of the things mathematicians do. Second, to expose pupils to mathematics useful for vocations.

The problem was the twelve-coin problem. Given 12 coins, one of which is counterfeit, find the counterfeit coin in 3 weighings with a balance scales and tell whether the counterfeit coin is heavier or lighter than the others. You know that all the good coins have the same weight and that the counterfeit is either heavier or lighter than the others.

Many attempts were made to solve the problem, but after a good many failures and some suggestions from the teacher, one boy did suggest the usual solution. Having come this far, the pupils were asked to generalize the solution. Could they figure out a different problem which would involve a similar solution? Several problems were suggested. They decided to work on finding out how many coins could be involved if four weighings were allowed.

Many solutions were suggested. The largest number proposed was 48. This was tried experimentally but was found to be incorrect. The next largest number was 36 which was also tried. Through variations they found it was possible to use 39 coins. The general pattern was later discovered.

During the process (3 class periods) the pupils were very interested; they learned something about making and testing conjectures; they discovered some mathematics that was new to them and to their teacher. For the first time, they decided mathematics was a fascinating subject.

I think that possibly the idea suggested has some merit. However, the topic of the example seems rather trivial, and I don't feel that many topics would lend themselves to this type of discovery. I suppose the way in which the problem was solved is the significant thing. A group of pupils of relatively low mathematical ability had shown what seemed to be very high ability and a great deal of interest in mathematics.

If a problem does arise in a classroom to which the teacher does not know the answer, the teacher ought to admit that he doesn't know. Through his example and questions the teacher can show the pupils how a person with some mathematical training tries to find an answer. Thus, the pupils will learn that mathematics is more than just a body of knowledge and more than memorizing theorems and proofs created by others.

Theoretically, in our mathematics classes we ought to concentrate less on covering a certain body of knowledge and more on thinking about what we have done, how that can be generalized and applied to other problems. But let's be practical. Students are not tested on the amount of creative thinking and generalizing they can do but on the amount of their acquired knowledge. It would seem to me to be quite unwise (practically speaking) to spend too much time on this sort of discovery unless the problem can be made more realistic and applicable to the course of studies.

The next article answered many of the questions I still entertained about the discovery method. These questions came to mind: (a) How much guidance should the teacher give to the student if he wants the student to understand the materials and ideas at hand and not merely to parrot them back? (b) In what areas of learning is the discovery method superior to other methods? (c) For what educational objectives might the discovery method be most useful?

Discover "Discovery" Objectively

This article is entitled "The Motivating Effect of Learning by Directed Study" by Bert Y. Kersh, Teaching Research, Oregon State System of Higher Education. In this article Kersh's research shows that many of the claims made of the discovery method do not receive unanimous agreement. Kersh bases his study on two factors: retention and transfer. He states that the following four claims have been made of the discovery method:

(a) Increases the learner's ability to learn related material;

(b) fosters an interest in the activity itself rather than in the rewards which may follow from the learning;

(c) develops ability to approach problems in a way that will more likely lead to a solution;

(d) tends to make the material that is learned easier to retrieve or reconstruct.

Research evidence does not entirely support these claims. Kersh's study consisted of learning the following two rules:

1. Odd Numbers rule: the sum of any series of consecutive odd numbers beginning with 1 is equal to the square of the square of the number of figures in the series.

2. Constant Difference rule: the sum of any series of numbers in which the difference between the numbers is constant is equal to one-half the product of the number of figures and the sum of last and first numbers.

A total of 90 high school geometry students took part in the experiment. The entire sample was told the rules and given practice in their application. They were taught by a programmed booklet procedure. Then the total of 90 was sub-divided into three groups of 30 each.

One group, known as the Directed Learning Group, was taught the rules entirely by a programmed learning technique. The second group, called the Guided Discovery Group, was required to discover the explanation with guidance from the experimenter. This group was taught through questioning which required each student to perform specific algebraic manipulations and to make inferences without help. The final group was called the Rote Learning Group. For this group explanation of the rules was omitted.

A test of recall and transfer was given to each subgroup after 3 days, 2 weeks, and 6 weeks. Perhaps the most striking finding in this study is that the Rote Learning Group was found to be consistently superior in every respect to the other groups. The Guided Discovery Group used the rules after the learning period more frequently than the subjects in the Directed Learning Group; therefore, transfer was higher. With respect to the permanence of the retention and increased transfer effects, the Guided Discovery Group was clearly superior to the Directed Learning Group.

The data of this experiment seems to suggest that under certain conditions of learning, the lecture-drill techniques produce better results than the techniques which attempt to develop understanding. Another fact that stands out is that initial achievement was very high and then dropped to where only about half of each group was able to recall and apply the rules after 4 to 6 weeks. Also, the results leave not doubt that there is a tendency for interest to accrue as a result of learning by discovery. And finally, the results failed to support the notion that attempts to provide added meaning will necessarily prolong memory for rules and procedures and will enhance their transfer.

This does not mean that rote learning is superior to learning with understanding. Rather, it means that we need to know much more than we do about meaningful learning and how we achieve it.

When is Learning Meaningful?

I was quite astonished to read these comments on the meaningfulness of material learned. I had never really questioned that this type of learning was superior to rote memorization. It seems to me that if one grasps the meaningfulness of the material learned, he should be capable of making applications to related areas more freely than one who merely memorizes. I have read a substantial amount of literature on this topic and rarely have I found that the value of meaningful learning is questioned. But again linking this matter back to the classroom situation, we must admit that our testing program is weighted heavily upon the element of recall. It is difficult to see that memorization produces results which at least equal the efforts of meaningful learning.

Regardless whether the programs are of Skinner's linear type or of Crowder's branching variety, I think there are four major difficulties arising out of this method of instruction. First, there is a danger that by merely selecting answers from multiple choice questions or filling in the blanks, our students are not building up their communication skills. Second, I think there is a danger that students will fail to make applications and correlations into related subject areas. Third, there is a complete lack of teacher-student and student-student interaction. It seems like a cold mechanical process of learning. Lastly, while programmed learning provides for many individual differences, I cannot see how it can possibly provide for all such differences. I do not disapprove of programmed learning; I am merely suggesting that it be used wisely as a supplement to the teacher's efforts rather than replace the teacher in the classroom.

Since I have already stated my views on Guided Discovery, there is no point in repeating them here except to say that wise guidance leading to pupil discovery is surely essential.

"Learning by Discovery" by Gertrude Hendrix, University of Illinois, was an analysis of some methods of teaching, concerning the question "What is the discovery method of teaching?" As a matter of fact, she suggests that there are indeed three varying forms of discovery, and she hints at a fourth. According to Hendrix, there is no such thing as "the" discovery method. She illustrates how a discovery lesson would be taught based on the three interpretations of the discovery method; (a) inductive method, (b) nonverbal awareness method, (c) incidental method.

Inductive Method

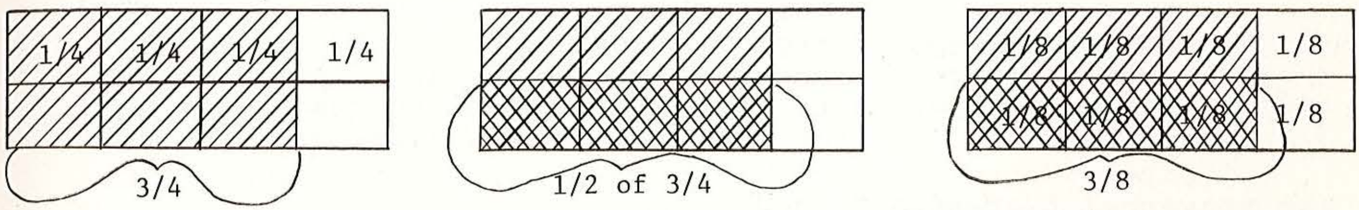
First, the class reviews

(a) fractional parts of rectangles;

(b) when rectangles are divided into unit squares, the resulting squares can be thought of, for example, as two of three or as three of two;

(c) the number of units in a rectangle as described above is 2 x 3 or 3 x 2.

Second, the problem is stated: 1/2 x 3/4 = ? The teacher helps the students to find the answer by making appropriate drawings (as below) and then counting. A rectangle is divided into fourths and three of them are shaded. Next the shaded part is divided into halves, and cross shading is used to identify one-half of the three fourths of the original.



1/2 of 3/4 = 3/8

Finally, each part of the cross-shaded area is found by counting, to be one-eighth of the original rectangle. Then, by counting, the children see that one-half of three-fourths of the rectangle is three-eighths of the rectangle. This is recorded on the chalkboard as: 1/2 of 3/4 = 3/8. Refering back to review they restate the problem as 3/4 x 1/2 = 3/8.

This process for making and examining appropriate drawings is repeated for at least three more examples. Each is properly recorded on the chalkboard.

Then if set theory is sought, the teacher can again go back to the review section and establish that 3/4 of 1/2 = 3/8, or 1/2 x 3/4 = 3/8. This is done only after the appropriate drawings have been constructed.

Now the inductive method generally proceeds to examining each pair of fractions and the fraction which names the product, to try to tell how the product could be found without the aid of diagrams. In the inductive method the ability to be able to "tell how to do it" is the criterion by which the teacher recognizes that discovery has taken place.

Linguistics: A Problem

This verbalization of the concept involved can lead to enormous confusion and frustration, as the concept must be stated precisely. Children at this age do not have the linguistic ability for stating precisely what they mean even if they have discovered the rule. This usually leads to replies such as "multiply the top numbers, and multiply the bottom numbers ..." or "you get the numerator of the answer by multiplying the numerator of the first fraction by the numerator of the second fraction ...". If the teacher is acquainted with the modern mathematics program, he will insist on the distinction between number and numeral. After all this, the pupils might well have forgotten what they have really discovered.

Finally, an acceptable statement is formulated. Each student writes it in his notebook or checks the text where he will find the rule already stated (if he has not found it there previously). The students are then considered ready to work a set of practice exercises.

This method gives the children a crutch to fall back upon in that they can prepare diagrams whenever they find themselves in doubt about the rule. According to the author, the inductive method is vastly superior to the authoritarian approach, but the inductive method is inferior to the nonverbal awareness approach.

The Nonverbal Awareness Method

The author contrasts this method to the inductive method by using the same illustration as before. In this method, however, the problem is stated prior to the review. The lesson then proceeds similarly up to a point. Now, when the child has mastered the drawing-and-counting device for finding products, he is given a list of problems to work. As far as the student knows he is to work every problem on the list by making appropriate drawings.

The class activity thus shifts to individual seat work. What is not done in class time, is done for homework. To give pause to the student who might work out the whole list by making drawings just because that is what the directions are, examples like 17/21 x 11/19 should be included.

Students accustomed to this kind of discovery approach will be on the look out for a short cut, particularly if they know their teacher approves of this. If this teacher approval is not recognized from previous experience, instructions should be given to raise the hand when he can write down the answers without the accompanying diagram. When a hand goes up the teacher should check the answers; if correct, offer praise, if incorrect, ask him to make more experimental drawings.

How does a teacher know when discovery has taken place? A flush of excitement, and a student begins to write answers as fast as he can put them on paper, is ample evidence that discovery has taken place. To call a halt, now, in his progress for a long process of verbalization attempt would frustrate the excitement of the discovery. On the following day examples such as: 5/3 x 7/8 or 6/5 x 9/4 or 6/2 x 21/7 are presented. This will correct any false impressions that may have been created. It is recognition of the nonverbal awareness stage that converts the classroom experience into that of actual discovery.

The Incidental Method

If the "multiplying fractions" rule is left to incidental discovery a practical problem is presented. He is taken through the drawing experiment until he finds the answer to his problem. No other instances of multiplying fractions are included in that project. Later, however, he may again confront such a problem. Again the experimental diagrams are used to find a solution. From such examples widely distributed over time and space, the probability that generalization will emerge is next to nil.

The writer feels great disdain for authoritarian presentations, but even these are better than the incidental method of learning.

Hendrix now states her case for the deductive method. This is a case in which one begins with some sentences which express observations of a drawing. Starting with these sentences as premises, pure deduction takes over and out comes the desired formula. This, too, is a discovery method.

Upon reading this article, I found myself analyzing my own methods of instruction. Frankly, I must confess that I have never utilized the nonverbal awareness method, though it certainly appears to be a valid approach. I'd be interested to see the results of a study conducted comparing this method as opposed to the inductive and deductive methods. My major concern here is in the field of retention and transfer. I can see where substantial transfer of training might occur, but I'm somewhat hesitant in regard to retention. It seems to me that once a rule has been formalized and memorized it is more easily recalled over an appreciable period of time than if the generalization is discovered but not strictly stated. This is only a personal viewpoint, of course.

I have used both the inductive and deductive approaches extensively and am aware of their disadvantages. I think the modern mathematics programs provide for much more deductive reasoning than did the traditional mathematics program.

I was keenly interested in the classifications established in this article. I had never quite thought of discovery in these four ways. Perhaps my enthusiasm shows in the length of the commentary. I think this was by far the best of the five articles I read.

Finally, I shall try to enumerate some of the conclusions I have reached as a result of my readings for this article.

1. There is no one method that can be called "the" discovery method.

2. Discovery is in its essence a matter of "finding out" rather than "learning about" and rearranging this evidence in such a way that one is enabled to go beyond the evidence to additional new insights.

3. Communication usually plays an important part in setting the stage for discovery in the teaching of mathematics, but that part is quite different from actually communicating the new thing to be learned.

4. The learner must experience success when learning by discovery.

5. The learner may acquire effective techniques of problem solving because he has the opportunity to practise different techniques.

6. Learning by discovery is not necessarily the most effective learning procedure for all teaching objectives.

7. Perhaps the greatest advantage is that interest is aroused in what the student is learning (intrinsic motivation).

8. Guided discovery seems to offer a happy medium between independent discovery and highly directed learning.

9. And in the famous last words of most psychology articles, the ideas developed are not conclusive but require further study.

Two supporters of a discovery-type method of teaching are Jerome S. Bruner and Z.P. Dienes. Bruner in the Process of Education claims a discovery method is important in "the development of an attitude toward learning and inquiry, toward guessing and hunches, toward the possibility of solving problems on one's own." Dienes only says that a "certain freedom to 'fiddle around' was essential to successful formation of mathematical concepts." Dienes equates this to play, a natural phenomenon in children.

Robert P. Davis, director of The Madison Project in St. Louis, Missouri and Max Beberman of the University of Illinois are two math curriculum workers vitally interested in discovery. Other than the work of these two men rather little has been done to experiment with or define discovery. In spite of the general lack of experimental evidence one finds no shortage of statements in modern day textbooks in mathematics, like the following: "To be truly modern, a mathematics program must be modern in spirit. It must be a program that emphasizes inquiry, exploration, and discovery." In spite of the lack of experimental evidence many educators and psychologists are saying "The effective teacher of mathematics encourages creativity by helping pupils discover the basic ideas, laws or principles of mathematics."

One argument advanced by Robert Gagne against the use of the discovery method is that "practice does not necessarily make perfect." If we want children to learn discovery, "practicing discovery" in the classroom is not necessarily the best way to develop discovery habits. In fact, the argument goes, it is most often desirable to break down the skill or learning into its simple components and work on these simpler aspects. The very term "practicing discovery" is an irony.

Perhaps the most enthusiastic opponent of discovery and its implications is David P. Ausubel, professor of education from the University of Illinois. He implies that it can be used most profitably in the "unsophisticated stages of learning", in the elementary school. Ausubel's thesis is largely that problem solving is not a "primary" aim of education but transmission of subject matter is a "primary" aim. Ausubel insists that "Didactic exposition was and always will be ... used to transmit ... large bodies of knowledge." Finally Ausubel comments on the quality and kind of research done in this area. He is not impressed with this research.

Some Ideas

Although not always recognized, the discovery method means placing more emphasis on the individual learner - not in a group situation. "Class discovery" should not be equated with "individual discovery".

There is a tendency to dismiss the discovery method as "time-consuming". This, however, is irrelevant! More important is to determine what purpose it serves. One suggestion has been that the learnings in a subject - arbitrary associations, concepts, and understandings - can be developed most efficiently by various methods; discovery being, perhaps, one of these.

Motivation ranks high as a reason for using the discovery method. Can anything be justified simply on the basis that pupils like doing it? Others say discovery requires pupils who are already highly motivated.

Regardless of the importance of the discovery method taken as a single package, every aspect of the method has implications for general teaching procedure. Every teacher, regardless of the method being used, should be aware of such ideas as non-verbal awareness, motivation through achievement, individual learning, exploratory activity, problem-solving as an educational goal, general disposition or view toward a subject or a discipline, and the like.

At best the discovery can re-orient the entire mathematics course in secondary school; at worst, it can be an addition to our repertoire of teaching techniques.

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THE NEW LOOK IN ELEMENTARY MATHEMATICS: LET'S TEACH GEOMETRY

Mrs. Joan Kirkpatrick

"If nothing else, teach children geometry to open their eyes - to help them see patterns, to observe symmetry." As an example of what children can see when they "open their eyes", this was written by a pupil after completing a geometry unit.

Open the window - What do you see?
Geometrical figures!
The wings on a butterfly, wings on a bee,
Geometrical figures!
What are the church steeples? What are the vines?
Pyramids, triangles, squares, and curved lines!
Look all around you! What do you find?
Geometrical figures!
There's a line and a point and intersection set,
There's nothing anywhere that something hasn't met.
Look all around you! What do you get?
Geometrical figures!
- Wendy Waites, Hattiesburg, Mississippi

The new look of elementary mathematics is implied in the shift of terminology from elementary arithmetic to elementary mathematics. The program for elementary school was formerly limited to one branch of mathematics - arithmetic. With the introduction of more algebra and geometry into the elementary curriculum, the program has been broadened and the emphasis is now on elementary mathematics.

One of the significant changes in elementary mathematics is the inclusion of Geometry. The questions which immediately arise from this statement are "Why the new emphasis on geometry in elementary mathematics?" and "What geometry can be taught to elementary pupils?" These questions arise because this branch of mathematics has long been neglected in the primary grades of the elementary school and most people think that geometry is the formal high school geometry with its emphasis on proof. Conversely, the geometry of the elementary school is intuitive, descriptive geometry.

Why Teach Geometry in the Elementary Grades?

The wave of curriculum reform which has swept the school systems over the past decade is not diminishing but spreading in scope to almost all the disciplines in both elementary and secondary school curriculums. These reforms have been induced by skilled teachers and university scholars, who have led others in the preparation of materials for use in the schools. Mathematics has always been at the forefront of curriculum reform, yet it is still being kept under constant scrutiny in order that it continues to fulfill the needs of an advancing society.

One of the early groups to influence the mathematics curriculum was the Woods Hole Conference in 1959, at which many leading educators and scholars met for a conference on new educational methods. At this conference, Bruner challenged educators and teachers to present the structure and form of a subject to their students so that they may gain intuitive understanding at an early age as a basis for learning in later years. A similar position, specifically for mathematics, was taken by the authors of the Twenty-Fourth Yearbook of the National Council of Teachers of Mathematics, "Growth in Mathematical Ideas, K-12", (1959). The participants in the Canadian Teachers' Federation Seminar on "New Thinking in School Mathematics" held in 1960, expressed the view that intuitive, descriptive geometry should begin in the elementary school.

Probably one of the most recent meetings has been the Cambridge Conference on School Mathematics attended by a group of twenty-five outstanding mathematicians and mathematics-users from university and industry, who met during the summer of 1963 to review school mathematics and to establish goals for mathematics education. Their report, "Goals in School Mathematics" sets forth their views on the nature of a sound mathematics curriculum. They have outlined programs for the future so that we teachers may have some idea of the steps we should take now to prepare our pupils for a rapidly changing technological society. Their objective for mathematics instruction in the elementary grades is that the pupils become familiar with the real number system and the main ideas of geometry. One of the aims of the study of geometry is to "develop the planar and spatial intuition of the pupils and to serve as a model for that branch of natural science which investigates physical space by mathematical methods." Their outline of the topics and experiences in geometry for the elementary grades reads like the index of a secondary school text! They also come out strongly in favor of the "spiral" approach to the mathematics curriculum.

Whether or not we agree with any of their suggestions for curriculum reform, they do provide us with many unanswered questions which should lead to new and profitable research. And, much of this research is being carried on by experimental groups which are trying to identify the unifying ideas pervading all mathematics in an effort to determine ways of beginning the development of these ideas in the elementary school. Reports from some of these groups: School Mathematics Study Group, University of Illinois Arithmetic Project, Greater Cleveland Mathematics Program, Madison Project, Stanford Project, and Minnesota Elementary Curriculum Project, indicate they have all included geometric concepts among many other new ideas for elementary mathematics.

Many arguments for including geometry instruction in the primary grades have been advanced by both teachers and teacher-educators at conferences, conventions, inservice sessions and in articles. Two of these seem to me to "sum up" the thinking of these people:

1. geometry is especially useful as an effective and impressive introduction to analytic and creative thinking, as well as the concept of precision;

2. geometry, with its shapes, forms, sizes, measurements, and opportunities for analytic and creative thinking, will often arouse the interest of students who are "left cold" by arithmetic.

The point here is that everywhere you look, leaders in the fields of education, mathematics and industry - people who have some idea of what our pupils will have to know in order to carry out the technological and scientific revolution which is taking place - are all suggesting that basic geometric ideas and experiences should begin in the earliest grades.

While there is still little evidence available to indicate which concepts of geometry should be included in the primary grades, there have been several experiments which have shown that much more geometry <u>can</u> be taught at this level than <u>is</u> being done at present. There is also some evidence that children during their first years at school develop certain ideas and concepts fundamental to geometry long before they understand one-to-one correspondence, or before they understand the more abstract work with the operations of addition and subtraction.

Piaget's experiments led him to conclude that a child's first mathematical concepts are topological. (Characteristics of topological space are proximity, separation, order, enclosure, continuity). After this stage, pupils can begin classifying common shapes, and proceed from the topological to projective and then to euclidian relationships. He found that even a three-year old child could distinguish a simple closed curve from a non-simple closed curve. If Piaget is correct in these findings then there are definite sequences in children's ability to understand geometric concepts and these concepts can be developed in children at a much younger age than is required by our present curricula. Suppes and Hawley, who worked with young children, reported that many of Euclid's concepts could be discovered by construction and are teachable at the primary level. Admittedly there is a need for much more research in this field, and Jeremy Kilpatrick, in a paper to the Conference on Cognitive Studies and Curriculum Development (1964), stated that considerable research in the near future is certain to be centered on the issue of finding "elastic limits" in children's perceptive processes. The main point, is that whether we agree or disagree with the ideas of Piaget and others doing similar work, we certainly cannot afford to completely disregard what they have done.

A further reason for including geometry in the elementary school is a practical one: it has been stated that the new textbooks would devote twenty-five per cent of their program in elementary mathematics to geometry. Examination of the contents of most recent publications makes clear that expansion is being made of the geometry in the mathematics curriculum at every grade level. So, whether we like it or not, geometry _is_ being included in the new elementary mathematics programs.

What Geometry Should be Taught in Elementary Schools?

The geometry being introduced in many of the new elementary programs is the nonmetric geometry of space; it deals with such ideas as points, lines, planes and space, all organized in terms of sets of points. The presentation of these ideas in the various programs is informal, intuitive and descriptive but is developed in such a way that later understanding of precise mathematical ideas should occur. Thus, pupils are given an introduction to some very basic geometric ideas, which will help them recognize similarities and distinctions among the numerous geometric figures they will study in later years.

An examination of a few of the new texts, mainly for Grades I and II, gives us an idea of what geometry is being included for primary pupils:

(1) _Seeing Through Arithmetic_, Books One and Two, W.J. Gage and Co. Hartung, VanEngen, Gibb, Stochl, Knowles, Walch and Evenson.

Recognition of simple closed and open curves, inside (interior) and outside (exterior) of closed curves, betweenness for simple open curves, line as a special curve, naming points, line segments and naming line segments, intersecting lines and parallel lines, points of intersection, introduction of polygons as special closed curves, identification of vertices and sides of polygons.

(2) _Mathematics We Need_, Books One and Two, Ginn and Company. Brownell and Weaver

Paths, open curves, simple closed curves, inside and outside of closed curves, points, line segments, measuring line segments, rays, angles.

(3) _Discovering Mathematics_, Thomas Nelson. Vere deVault and Osborn

Line, point, plane, identification of shapes, basic constructions, angles, relationships of lines.

(4) <u>One by One; Two by Two</u>, Harcourt, Brace & World.
Clark, Beatty, Payne, Spooner, Clark.

Point as a place in space, line as a set of many points, recognition of shapes, regions, closed paths, drawing models, plotting points as an introduction to coordinate geometry, circle as a set of points, interior and exterior of closed curves.

(5) <u>Math Workshop for Children</u>, Encyclopedia Britannica Films, Inc. Wirtz, Botel, Sawyer.

Points, lines, points on a line, connecting two points with a line, drawing lines that cross or do not cross, properties of geometric figures, size in terms of square units and linear units.

(6) <u>Elementary School Mathematics</u>, Addison-Wesley.
Eicholz, O'Doffer, Brumfield, Shanks.

Using compass to construct circles, families of lines through a point, regions of a plane, inside and outside, radius and diameter, concepts of angle, triangle, sum of angles of a triangle, construction of right angles, right triangles, rectangles, quadrilaterals.

The geometric ideas which are common to these texts: points, closed and open curves, inside and outside, betweenness, line segments, points of intersection, polygons, are particularly suitable for the first stage of instruction in mathematics because they are less abstract than number ideas and can be more easily related to the experience of the pupils. These are ideas much closer to the pupils' interest and related to their intuitive insights than are the formal rules and definitions regarding circles, triangles and squares. All children have intuitive notions about the space in which they live. Their understanding is vague, but before they come to school they have some ideas about points, lines, betweenness, crosses, corners (inside and outside). From their earliest experiences with blocks and puzzles, children work informally with shapes and forms. While points, lines, planes, etc., are ideas, their physical representations in pictures or in the real environment are concrete. Thus, it seems only sensible to begin the study of geometry with some of the basic geometrical ideas that can be related to the physical world of the pupils.

This approach to geometry gets the pupils started on a basic set of ideas that may be built upon, and regularly broadened and deepened as the pupils progress through school. The need for readiness for the deeper study of mathematical ideas in the upper grades <u>requires</u> that children be introduced to some of the fundamental mathematical concepts in the elementary grades. In order that pupils need not "unlearn" mathematically incorrect ideas, they must be exposed to basic, mathematically correct ideas right from the start. They are not asked to learn precise mathematical definitions, but do learn mathematically meaningful ways of describing these basic ideas. For example, in order to understand polygons - such as a triangle as the union of the three segments determined by any three noncollinear points - pupils must have the idea of line segments and points. Inside and outside, or interior and exterior, of simple closed curves, is necessary readiness for the concept of area - the area of a region being the union of a simple closed curve and its interior. To facilitate understanding of collinear points and build a definition for a line segment, an understanding of the concept of betweenness is necessary.

Thus, in the primary grades is laid a foundation of mathematical concepts on which the program for later grades can be built. This groundwork is laid so that eventually a mathematical as well as an intuitive understanding of geometry will result. In addition to laying a mathematical foundation for later work, the understanding of these concepts enables a child to describe mathematically the world he lives in, which in turn means he can "communicate ideas about the world to his fellow human beings much more efficiently."

What does all this mean for the elementary teacher? Well, to this elementary teacher it means first of all - excitement - the fun of a whole new field opening up! Secondly, of course, it means work. We must prepare ourselves in order to teach these concepts. This, by the way, can be part of the fun. If I have been able to pass along some of my enthusiasm for this new venture, I would suggest you work through the geometry units in STM 1 and 2 to provide yourself with some background. Then, a thorough study of the Teacher's Guide accompanying any one of the new texts should get you well on your way.

> Geometry is the most challenging topic in school mathematics, and to meet this challenge in the light of "new thinking" is one of the most interesting and exciting tasks for all concerned.

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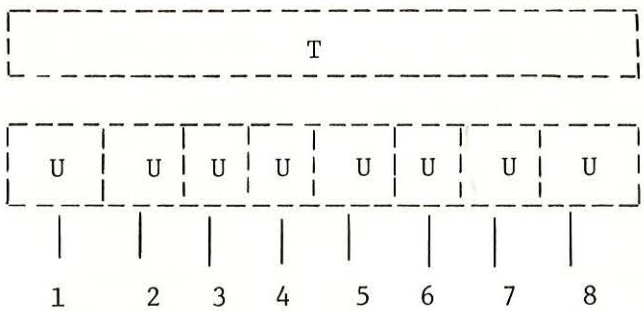
ESTABLISHING READINESS FOR LINEAR MEASUREMENT IN FIRST GRADE

L. D. Nelson

Editor's Note: L.D. Nelson reports further work in his unique studies and researches in measuring the mathematical concepts of young children. We are pleased to present this second article designed to increase our understanding of these very interesting young people.

Modern programs in early elementary school mathematics usually provide a good variety of activities designed to establish counting as a meaningful process for beginning school children. The ideas basic to counting: one-to-one correspondence, cardinal number, and ordinal number are in general, carefully fostered both through exercises in printed materials and through detailed instructions to teachers. There is another mathematical process, however, which is usually introduced in the first grade, but which by comparison is treated in a completely unsatisfactory manner. This process is called measurement.

In its simplest form, the process of measurement requires that the child isolate and identify a single property of an object and then use number to describe that property. The diagram below illustrates the essential features of the process when length is the property to be measured

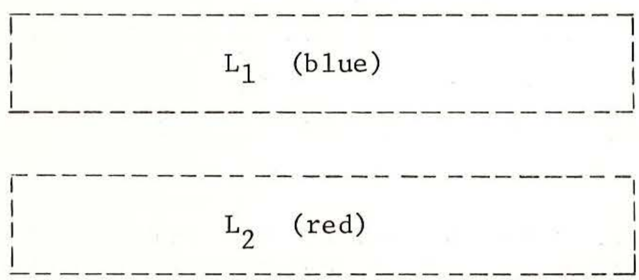


A given unit, U, is iterated along the length of T and a one-to-one correspondence is set up between these units and a sub-set of the counting numbers. We use number to describe the length of T as follows: "The measure of the length of T, when U is the unit, is 8." (The child should not proceed far into the problem of measuring before he discovers that the set of counting numbers is entirely inadequate for his new purpose. This discovery on the child's part should certainly be an important aim in the early stages of teaching measurement but will not be the main concern here.)

From the very beginning we must satisfy ourselves as teachers that the child can indeed readily isolate the physical property of an object which we want him to measure. Measuring the circumference of a circle would present a different problem from measuring the edge of a cube to some mature people. There is considerable evidence to indicate that such transformations as these are very troublesome problems for the young child, but as yet, their significance has been overlooked in most investigations. Evidence shows that in a similar way the young child can be confused between such simple properties as color, length and position. We should keep in mind that all the teaching he has had up to now has been arranged to help him see that number is independent of any of the physical attributes of the object in a set he is numbering; in other words, we encourage him to dissociate physical properties from number. Now in measuring the lengths of his yellow and red pencils we ask him to use number to describe a physical property of a particular object. If this shift in associating numbers and physical properties is not to create confusion in the mind of the child, it must be made with extraordinary care.

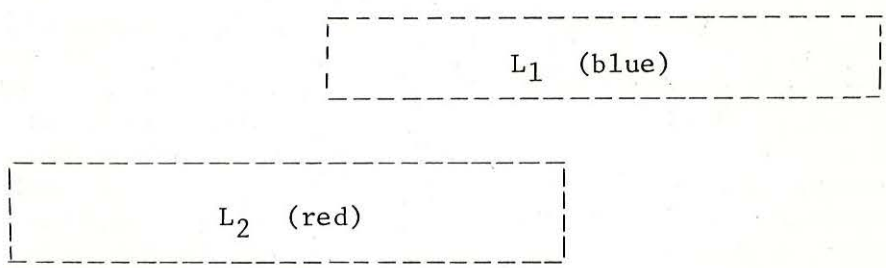
So it is, with this in mind, that I suggest some activities which might precede the introduction of actual measuring activities. These activities are designed to provide a readiness for the measurement of length. While it is recognized that the measurement of length is not the only kind of measuring activity provided in first grade mathematics programs, it will certainly play a dominant role in the introductory stages. The basis for these suggestions for activities is to be found in the work of Jean Piaget (1) and Marcel Pelletier (2).

In the very early stages of measuring, we should expect a child to be able to make crude comparisons of the length of similar objects. Consider the following:



Suppose that L_1 and L_2 are identically shaped sticks, about 6 inches long, painted blue and red respectively. Suppose further that they are presented side by side to the child with their ends along a vertical line. If the child is asked if the blue stick is longer than the red stick, the red stick is longer than the blue stick; or if they are the same, it is likely that he will judge them to be the same. (At this stage it is probably best in the interests of good communication to avoid use of the word, "length". However, once the concept has become more firmly established, there should be no hesitation about introducing the term.) Now, while the child watches, move L_1 along its length about one inch and repeat the question about their relative lengths. If the child has an understanding of the true nature of this property, he should see that a transformation such as this does not change the relationship. Most Grade I children when faced with this problem will say that the blue stick is now longer than the red stick.

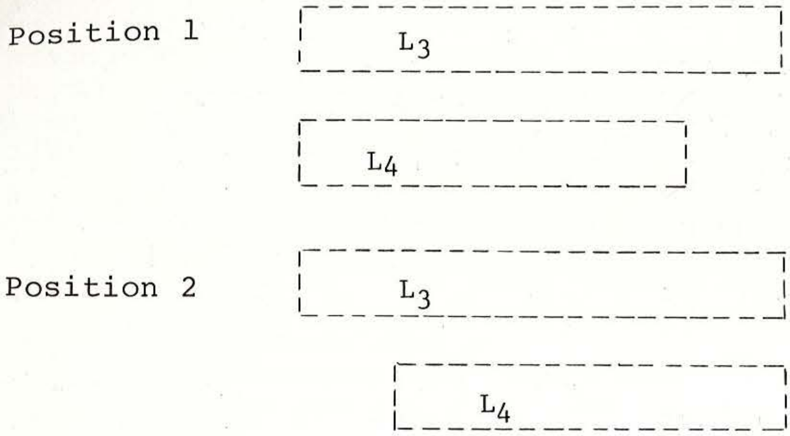
Why, when the two sticks are moved into the position shown below, do children tend to judge one longer than the other?



Piaget has noted that children tend to centre their attention at one end in making a comparison and cannot at the same time see that there is a compensatory difference at the other end.

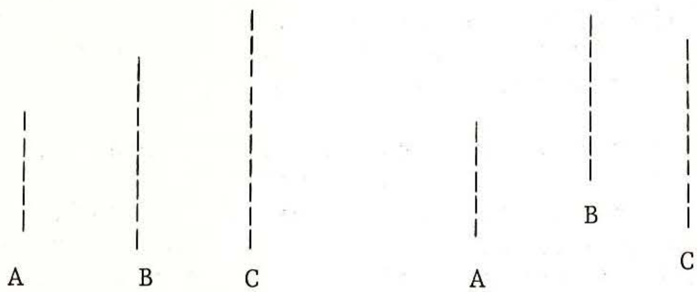
This is such an obvious perceptual blunder to the adult that the tendency for many teachers is to attempt to "tell" the child why he is wrong and "explain" how to correct his judgment. It appears, however, that experience with a great variety of objects in situations similar to this is the only way to ensure that the child has learned to make correct judgments in the face of what is apparently conflicting perceptual information. The teacher can explain with one set of materials and finally get correct responses only to find that the responses are incorrect as soon as different materials are used.

There is a great variety of activities which arise out of this basic situation. It has already been suggested that a variety of objects besides sticks should be used. They should vary one from another in color, weight, width, texture, and so forth. The kind of transformations should also be varied. Moves should be to the left as well as the right. The bottom object should be moved as often as the top one. The moves should not only be simple translations along a length, but the objects should be moved to form an angle with each other or to form a T. Indeed, the lengths of the objects to be compared should as often as not be different. What will the reaction be to the following?



Does moving L_4 from position 1 to position 2 apparently change the length of L_4 in the mind of the child? Material should be collected and stored in some kind of container and made readily accessible to students and teacher. Allowing children a great variety of easily accessible materials will enable them to separate the relevancies from the irrelevancies in learning how to deal with properties they are to measure. Only a small proportion of the teacher's own teaching time has to be devoted to directing these activities. Once the child is guided to ask the questions, his own curiosity can then lead him to independent discovery of important measurement ideas. In any case, it is fruitless to allow the child to proceed with the actual process of measuring before he understands completely the nature of the property he is to measure.

A variation of the activity suggested above is to present the child with several objects whose lengths are different. Exercises in ordering them according to length may be provided. However, a word of caution has to be introduced here. Look at the following example:



Suppose the child has ordered three objects as shown. Then suppose the middle one is placed upon a brick or some similar pedestal. Does he need to re-order the objects? Or does he assume that B is now longer because it is higher? (Perhaps 'taller' is a better word.) There is considerable confusion in the language which must be cleared away. For example, a child stands on a chair and claims to be "bigger" than his mother. Sufficient experience must be gained in a readiness program so that the child realizes he does not actually change his length just by standing on a chair.

The transformations discussed so far are those which do not change the property called length. However, there are others which do and which the child should come to recognize before he is required to proceed with the actual process of measuring length. Suppose we start with two strips of ticket board, one orange, one green and both about 12 inches long. The child judges them to be the same when they are placed so that their endpoints are coterminous. But what does he say if one of them is cut into two equal pieces and these pieces laid against the uncut piece? The property has been changed, but the length of the two pieces together is the same as if it were unchanged. Some children claim that the one having two pieces is longer than the uncut piece and others make the opposite claim. The reasons they give for these judgments are most interesting. The significance of this activity can be seen when one considers that in measuring exercises later, children will have to iterate units along the length of the object being measured. They should be convinced in their own minds that once they have said that an object has such and such a length that they are really saying the combined length of the units is the same as the length of the object whose length is being measured. The activity suggested here to prevent difficulties in this aspect of measuring can be varied in a great number of ways. The cuts can be as few or as many as desired. The pieces may be the same size or may differ one from another. The units into which objects are cut can be arranged in a different way. For example, is a pile of blocks as long as a row of the same number of similar blocks?

The key to providing adequate readiness for the process of measurement is the provision of adequate manipulative materials with which the child can explore.

Only a few activities have been detailed here, but the first grade teacher who recognizes the basic problems children have in learning about measurement can develop as great a variety of activities as she wishes.

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THE "NEW MATHEMATICS" - SCHOOL MATHEMATICS FOR THE SPACE AGE?

Gerard Hanson
Comments by
Dr. S. E. Sigurdson

Editor's Note: Gerard Hanson is now teaching in Westminster Junior High School in Edmonton. He obtained his B.Sc. in Mathematics from the University of Alberta, after receiving his early education in England. In this article he makes a case for the alliance of mathematics and science.

New or Modern?

We hear a great deal these days about "modern mathematics" in the schools, or to use an expression that is to be found in the jargon of modern-day educators, the "new mathematics". This term "new mathematics" has become the symbol for the school mathematics as necessary for the "space age". The "new mathematics" we are told, will enable college bound students to enter college well prepared to meet the demands that this technological age has placed on university curriculums. This, in turn, will allow us to compete with the Russians whose launching of Sputnik in 1957 caused Americans to ask just who was leading in the race for technological excellence.

These new programs in mathematics have been around a sufficient length of time that the following questions seem valid. Just what is the "new mathematics"? And, is it doing the job that the professional educator and the writer of articles in the popular periodicals would have us believe? That is, has it given the school curriculum the content necessary to allow its graduates to meet the demands of the age in which we live?

How Pure is Pure Mathematics?

In answer to the first question a discussion of some of the trends and attitudes that have arisen in mathematics and in particular "pure mathematics" over the last century is helpful. Since about 1870 mathematics has been concerned with a critical self-evaluation. It was found that much of the earlier mathematics developed through the intuition of its authors had lacked in logical rigour. It is not to be assumed that this earlier mathematics was poor or incorrect; it did, however, lack the formal structure and solid foundations from which its propositions could be deducted. Mathematicians over the past one hundred years have concerned themselves with putting in this foundation and structure. Much of this work was centered around the concepts of number and space. This led to generalizations of arithmetic from which arose purely formal abstract algebras. Non-Euclidean geometries were developed that gave rise to geometric spaces that were pure abstractions where logic replaced intuition and axioms replaced absolute truth. These efforts led to the unique and distinctly twentieth century viewpoint described as postulational method, or axiomatics. However, it should be noted that in doing this work modern mathematics is not replacing the old, rather reaffirming it, while at the same time adding to it.

The implications of this work for the school curriculum are many. It has been the experience of all school children to have to learn arithmetic in some form or other. The point now arises, if we are going to teach arithmetic should we not do it in the most mathematically sound way available? The general concensus is "yes". Armed with the psychology of men such as Bruner who stress fundamentals and structure when teaching a subject, and assisted by the willingness of mathematicians such as topologist Begle of Stanford University to write text books, educators are now bringing the "new" approach to mathematics into the schools. Therefore, in answer to our question, "What is the 'new mathematics'?" we find that as presently taught in the schools it is not "new mathematics" as such but rather a new approach, based on contemporary thinking in mathematics, to what has been basic to mathematics curricula for many years.

Point of View is Important

When considering this "new approach" it is important to keep in mind that it is not the only approach that mathematics offers. The layman tends to see mathematicians as all alike, however false this may be. Mathematics has many aspects, and mathematicians, as a result, have different tastes. The "new mathematics" borrows heavily from the language and findings of that area of mathematics known as "pure mathematics". This area of mathematics emphasized mathematics as a game with arbitrary rules where the principle consideration is to stick to the rules of the game. This is not the view of mathematics held by the "applied mathematician" who sees mathematics more as a unifying force among the sciences. It is not even to be construed that the language of "pure mathematics" and hence the language of the "new mathematics" is universal. R.A. Stoal, commenting in the Canadian Mathematical Bulletin has this to say about the "new mathematics". "... this renaissance has proceeded at a rate which has unfortunately, left most of the older (30 years or over) practising mathematicians far behind. This is a conclusion which has been reached, reluctantly, on searching the better known journals for examples of modern usage and finding them lamentably lacking except in certain fields (departments of education have seemingly been more progressive in this regard)." He further points out, "... a characteristic (of the new mathematics) is the appreciation of the importance of terminology .... Indeed there is an obvious correlation between the use of certain words and new-mathematics content."

Speak Your Own Jargon

I believe that the last remark is important when considering the fast rise in popularity of the pure approach to mathematics in the school. The "scientific educator" like his cousins, the psychologist and sociologist, is very fond of terminology that will allow him to give his academic domain an esoteric flavor. The language of pure mathematics does just this. No longer is the mathematics of the grade school intelligible to the layman. Couched in its new terminology, it requires the presence of a specialist for its interpretation.

We now turn to the second question, "Is the 'new mathematics' doing the job of educating students for the technological age we live in?" Unfortunately, the answer has to be "no". The reason is obvious; the "new mathematics" has added very little of the content necessary for training students to meet the demands of the space age. Contemporary educators may beat their chests proudly and say, "Look! We have brought modern pure mathematics into the schools." But modern pure mathematics and the applied mathematics so necessary for technology are not one and the same thing.

We Make the Rules

Much of the abstract modern mathematics is a very sophisticated game played for its own sake and the pleasure of those who take part. Its concern is definitely not with applying its findings to concrete situations. If by chance a use can be found for it, as was the case for group theory, that is very nice but this is not a criterion for its existence. The point to be realized here, is that pure mathematics, although a valid area of intellectual inquiry, is not the mathematics that is allowing successful competition with the Russians. In general, pure mathematics is somewhat isolated from the main streams of science. Therefore its influence on the curriculum is tending to isolate the mathematics taught in the schools from the other branches of science.

It was this fact that caused seventy five American mathematicians to issue a memorandum that included the following statement. "What is bad in the present high school mathematics curriculum is not so much the subject matter presented as the isolation of mathematics from other domains of knowledge and inquiry, especially the physical sciences." A further point made in the memorandum, that should serve as a warning to the teachers of the "new mathematics", was that "... the spirit of mathematics cannot be taught by merely repeating its terminology."

H. Resenberg, writing on pure and applied mathematics in the school curriculum, has the following to say: "As current efforts are made to increase the content of pure school mathematics within the allotted time limits, it becomes apparent that some aspect of school mathematics must give. There are those who believe that the concepts of applied mathematics should be among the first victims on the sacrifice list." Much has been written on this dichotomy in mathematics. The theme is basic, a concern that the encroachment of pure mathematics into the school curriculum will cause the student to perceive mathematics as a subject apart from the main body of scientific thought. It must be realized that a serious threat to science is implied if it is viewed only as a system of conclusions drawn from definitions and postulates that must be consistent, but otherwise may be created by the will of the mathematician.

Content Should be Allied to Science

If, as has been said, the "new mathematics" has not given to the school curriculum the necessary content, then the question arises as to what that content should be. A survey of four hundred chairmen of mathematics departments and directors of admission in more than 300 colleges and universities throughout America suggests that the most preferred courses for the college bound science students are analytic geometry, calculus, and probability and statistics. The Commission on Mathematics for the College Entrance Examination Board also suggests "strong preparation both in concepts and skills, for college mathematics, at the level of calculus and analytic geometry." It is to be noted that these topics come from the domain of applied mathematics.

Raymond H. Wilson, Chief of Applied Mathematics, National Aeronautics and Space Administration, points out that, "almost all the developments which have produced the space age depend more or less directly on mathematical and physical principles invented or at least collected and synthesized by Sir Isaac Newton just before 1700." Sir Isaac Newton, as most people know, was the father of the calculus and the system of mechanics that bears his name.

However, these subjects are not taught in most of our schools, and, as a result, students entering programs at the university in engineering, physics, and chemistry find that for the first two years their knowledge of mathematics is inadequate for the courses - physics in particular - that they must take. As an excuse for not giving an adequate program in mathematics, it is often said that the student is too immature to handle mathematical concepts at the level of the calculus, however, in rebuttal to this argument, the Rev. D. Smith, a teacher from England now teaching in America, comments, "I know from experience that able and well prepared ninth graders are capable of beginning calculus, and not as a meaningless mechanical procedure, but of studying it with a reasonable degree of accurate statement .... in their later high school careers they are capable of advanced work in both the pure and analytic geometry."

Untrained Teachers and Frustrated Students

If we admit that such subjects as analytic geometry and calculus can be taught, and should be taught to capable high school students, one may wonder why it is not being done. It has been suggested that a lack of qualified teachers to do the job is the reason. However, an examination of the mathematics programs designed for prospective teachers indicates that no real attempt is being made to train them. For example, at the University of Alberta, the mathematics program for students taking the Bachelor of Education degree consists of seven courses, five of which come from the area of pure mathematics. No courses in physics, chemistry or mechanics are required in this major.

I am at a loss to understand how high school teachers with this specific mathematical and non-scientific background will be able to teach mathematics with the wide perspective that is so necessary, if mathematics is to fulfill its role in this technological age. It would seem that some educators are forgetting that students studying mathematics in the schools will be entering into all areas in the scientific domain. I believe, however, that part of the reason that an adequate program is not taught in applied mathematics in the schools, arises from the fact that if the courses of mathematics that have been suggested should be taught, not all students will have the capacity to handle them. Possibly only 20 per cent of the students would benefit from taking such courses. In North America such a situation in the schools would be considered undemocratic. I conclude therefore that the "new mathematics" being taught is a compromise. On the one hand it satisfies, to some extent, in its borrowings from pure mathematics, the demand for college level mathematics in the schools, and yet at the same time does not place any additional demands that cannot be met by most students.

In conclusion I would like to make an appeal to those responsible for the school mathematics curriculum. Be honest and recognize that the "new mathematics" is not space age mathematics, even though the propaganda may argue that this is so. This is just a cover up for a job that is not being done. Find out what is really needed for the space age and incorporate it into your program. The going may be painful at first, but there are some teachers and potential teachers around, who are anxious and capable enough to teach good mathematics courses, suitable for high school students who are bound for the wide variety of science programs offered in the universities.

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EMERGING TWELFTH-GRADE MATHEMATICS PROGRAMS

Dr. Lauren Woodby

Editor's Note: We are presently in the process of high school course revision which should be completed at the Grade XII level in from three to five years. Some light is shed on recent trends by a condensation of Dr. Woodby's report which covers a widely obtained body of information from high schools in twenty states.

1. The individual teacher is the most important factor in the development of a strong mathematics program. National Science Foundation Institutes have been influential in preparing teachers already in service to develop and teach the emerging twelfth-grade courses. Even so, inadequately prepared mathematics teachers are still an obstacle to the development of fourth and fifth-year courses.

2. There is lack of agreement on the mathematics that should be taught in twelfth-grade courses for college-bound students. Many different courses are being taught, and still others are in the planning stage. No particular program seems to be the most appropriate one at the present time. Much more content has been proposed than can be taught in any given program.

3. Acceleration and enrichment have generally accompanied the development of strong mathematics programs. Larger schools can be extremely selective in grouping talented students for honors courses. These gifted students are capable of learning more advanced mathematics than is usually offered in high school. Even very small schools are attempting to offer ambitious programs, in which as many as one-fourth of all the students are accelerated. In some schools acceleration appears to be overdone. Questions of "why accelerate?" and "how much acceleration?" have not been adequately studied.

4. Concern for calculus prevails. Most of the advanced courses are either calculus and analytic geometry, or algebra and analysis courses intended to prepare students for calculus. Although many high schools are teaching calculus courses of good quality, there is a trend toward the teaching of nonrigorous courses in calculus that emphasize only the mechanics of differentiation and integration. In effect, these are warm-up courses to give students a "running start" in the beginning course in college.

5. There is little acceptance of a course in probability and statistics as the fourth or fifth-year mathematics offering in the college preparatory program. There is even less acceptance of courses in linear algebra, matrices, and computer mathematics.

This situation is probably due to the lack of knowledge of these subjects by the great majority of high school teachers.

6. <u>Analytic geometry as a separate course has achieved only slight popularity</u>.

7. <u>It has been demonstrated that college-level calculus courses can be successfully taught at the high school level provided the following conditions are met</u>: (a) That there are enough capable students who, at the beginning of the twelfth grade, are prepared to study calculus. (b) That the teacher is adequately prepared to teach calculus.

(Reprinted from Math and Science Newsletter for Secondary Schools, State Department of Education, Charleston, West Virginia, October, 1965.)

SURVEY ON HIGH SCHOOL MATH

Newberry, Holland and Reed

<u>Editor's Note</u>: I publish in its entirety the following report of a survey and ask the reader: (1) Are the questions loaded? (2) Is the sampling limited; if so, to what extent do the limitations affect the conclusions? (3) Do the implications, as contained in the last two paragraphs, lack validity?

A survey has just been conducted at the University of Alberta, Calgary, with a view to determining the effectiveness, or otherwise, of the high school program in mathematics. The figures given below are percentage replies to the indicated questions; they incorporate the replies of 114 (virtually all) UAC students whose major subject is mathematics and who took Grades XI and XII in Alberta. Replies from other students are excluded from the figures. Among the questions and answers were:

<u>Question One: Do you feel that the study habits developed in high school form an adequate basis for the study habits necessary at university</u>?

<u>Answer</u>: Inadequate - 75%, adequate - 19%, good - 6%.

<u>Question Two: Do you consider that a potential university student could profitably cover more material in Grades XI and XII in your field of specialization? If so, how much more?</u>

<u>Answer</u>: No more - 18%, zero to twenty per cent more - 54%, more than twenty per cent more - 28%.

<u>Question Three: On the average, in Grades XI and XII</u>, <u>( ) written exercises per year were in my present major field and were corrected and returned to me</u>.

<u>Answer</u>: Disregarding the non-numerical answers like "several", "too few", etc., there were 82 numerical answers of which 21 per cent indicated that the number of written exercises per year was zero. There were also a few replies in the neighborhood of 100. However, the allegation that there are gross inequalities in the Alberta school system has - the Minister assures us - "no basis in fact". (<u>Calgary Herald</u> 26/2/1965).

There were other questions and answers on the survey, but these did not appear to exhibit any very conspicuous trend. On the whole, the members of the teaching profession did not incur serious criticism, although the adequacy of their mathematical training was occasionally questioned. Questions One and Two appear to be an indictment of our administrators and syllabus-makers, Question Three reflects upon the inspectors (if any).

In considering these results it must be borne in mind that the survey was confined to a restricted class of individuals - university students of mathematics. These people are of well above average intelligence and it may be that some of their views (e.g., concerning enrichment of the syllabus, Question Two) are not directly applicable on a universal basis. Nevertheless, the survey does indicate that the more intelligent students are not learning as much as they could and should learn in high school. This is improvident. The situation may soon arise in which Alberta needs its best brains far more than the best brains need Alberta. The time will come when Albertans must live by their talents, not by their oil revenues. Although the survey was for convenience, confined to mathematicians, the problem is undoubtedly much broader. There is evidence to believe that a similar survey conducted in the physics or English departments would yield a similar result. There is no reason why this province should not have the best educational system on the continent. We have the money and the facilities, and the target is not all that high.

But we can't be bothered.

A BEGINNING APPROACH TO VECTORS

Merrill B. Hill

Editor's Note: The following is from a presentation to the California Council of Teachers of Mathematics (Southern Section, Long Beach, California, December 12, 1964). Some thought might well be given to including some material of this kind in our high school mathematics program.

The study of vectors is an important part of both pure and applied mathematics. Consequently, it is receiving greater attention as an integral part of up to date school mathematics programs. A beginning unit on vectors could be taught at about the ninth grade level from an intuitive point of view before the student becomes involved in the formality of logic and proof. By this time the student should have the necessary background for such a beginning unit. This would include concepts such as directed segments, parallel lines, rays, similar triangles, the Pythagorean formula, ordered pairs, number systems, and others usually included in carefully planned seventh and eighth grade programs today.

How can we profitably involve the student in these interesting ideas that differ from the usual algebra and geometry? How can we emphasize applications related to the problems studied in science at this level? How can we use ideas about vectors to extend experiences with mathematical structures and provide practice in a new framework for ideas and skills previously developed? How can we provide pre-requisite experiences for more formal study of vector algebra and geometry?

The following is a brief description of one sequence of basic ideas about vectors which answers the above questions.

There are many examples of vectors. Here we will be concerned with only two kinds of vectors, a directed segment and an ordered pair of real numbers. First we will work with vectors that are directed segments since they can be generated from concrete experiences in a very natural way. Applications to simple physics problems involving displacements, velocity, and force will be suggested at appropriate places.

Intuitive notions of directed segments can be generated from experiences with displacements or trips in the physical world. A displacement from one place to another can be described in terms of a certain distance in a certain direction. Vectors which have both length (magnitude) and direction, a starting point and a terminal point, can be used to represent displacements.

Two successive trips can be represented by two successive vectors in a plane where the terminal point of the first vector is the starting point of the second vector.

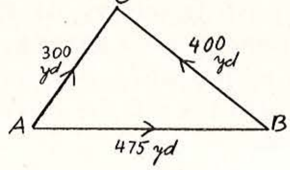

Figure 1

In Figure 1, vector $\overrightarrow{AB}$ (denoted $\overrightarrow{AB}$) represents a trip from the first tee to the second tee and $\overrightarrow{BC}$ represents a trip from the second tee to the third tee on a golf course. The result of the two successive trips is the same as a single trip from the first tee to the third tee. Likewise, the two successive vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ used to represent the trips have the same starting point, A, and the same terminal point, C, as the single vector $\overrightarrow{AC}$. Therefore, we say that AC is the sum of $\overrightarrow{AB}$ and $\overrightarrow{BC}$, and we write

$$\overrightarrow{AB} \oplus \overrightarrow{BC} = \overrightarrow{AC}.$$

In finding a vector sum, we must find both its length and direction. The sum of the measures of the two noncollinear vectors is not the same as the measure of the vector sum. Finding a vector sum is not the same as finding the sum of the measures of the vectors.

Next, the sum of two vectors on a line where the terminal point of the first is the starting point of the second could be developed. If two successive vectors in a line have the same direction, then the measure of the vector sum will be the sum of the measure of the two vectors. If they have opposite directions, the measure of the sum vector is the absolute value of the difference of the measures of the two vectors.

It can be finally concluded that if two vectors are such that the terminal point of the first vector is the starting point of the second, then the vector sum, both for vectors in a plane and vectors in a line, is the vector that has the same starting point as the first vector and the same terminal point as the second vector.

By making scale drawings of successive vectors and their sum, the approximate length and direction of the vector sum of two vectors can be obtained to solve problems involving displacements or trips in the physical world. When the vectors for a problem situation form a right triangle, the Pythagorean formula and the sine, cosine, or tangent formula can be utilized in determining the length and direction of the vector sum of the two vectors.

So far we have considered the sum of two vectors when they are directed segments and when the terminal point of the first vector is the starting point of the second. Many more problems could be solved if we knew how to find the sum of two vectors when they have the same starting point or the same terminal point.

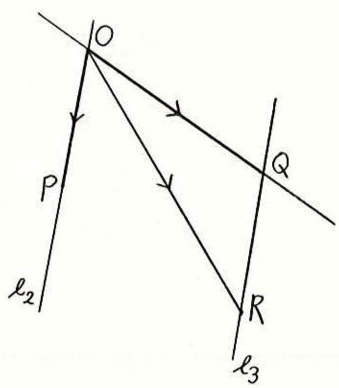

Figure 2

Vectors $\overrightarrow{OQ}$ and $\overrightarrow{OP}$ in Figure 2 represent two displacements. They have the same starting point and they are noncollinear. To find the vector sum of $\overrightarrow{OQ}$ and $\overrightarrow{OP}$, let us first find a vector with the same length and direction as $\overrightarrow{OP}$ and whose starting point is the same as the terminal point of $\overrightarrow{OQ}$. To do this, consider line l_1, which includes $\overrightarrow{OQ}$ and line l_2, which includes $\overrightarrow{OP}$. Line l_3 is parallel to l_2; it includes $\overrightarrow{QR}$, and $\overrightarrow{QR}$ has the same measure as $\overrightarrow{OP}$. Line l_1 is determined by the starting points of vectors $\overrightarrow{OP}$ and $\overrightarrow{QR}$, and l_1 separates the plane of the two given vectors into two half planes. Since $\overrightarrow{OP}$ and $\overrightarrow{QR}$ are as parallel and the terminal points of both $\overrightarrow{OP}$ and $\overrightarrow{QR}$ are in the same half plane which means that $\overrightarrow{OP}$ and $\overrightarrow{QR}$ have the same direction. Two vectors are equivalent if they have the same measure and the same direction. Thus $\overrightarrow{OP}$ and $\overrightarrow{QR}$ are equivalent vectors.

Now that we have the idea of equivalent vectors, we can develop a triangle method of finding the sum of two vectors that have the same starting point. Since $\overrightarrow{OQ}$ and $\overrightarrow{QR}$ are located in such a way that the terminal point of $\overrightarrow{OQ}$ is the starting point of $\overrightarrow{QR}$, we know that the vector sum of these two vectors is OR, and we can write

$$\overrightarrow{OQ} \oplus \overrightarrow{QR} = \overrightarrow{OR}$$

The student also knows that in the sum of two numbers, one or both of the numbers can be replaced with equal numbers, and the sum will be the same. Likewise, in the sum of two vectors, one or both of the vectors can be replaced by equivalent vectors, and the sum will be the same vector or a vector equivalent to it. Therefore, if we replace $\overrightarrow{QR}$ in the sum above by the equivalent vector $\overrightarrow{OP}$, we have

$$\overrightarrow{OQ} \oplus \overrightarrow{OP} = \overrightarrow{OR}$$

This method for finding the vector sum of two vectors in a plane that have the same starting point could be done by first finding a vector equivalent to either of the given vectors. In a similar way, equivalent vectors could be used to find the vector sum of two vectors that have the same terminal point.

The parallelogram method for finding the sum of two non-collinear vectors that have a common starting or a common terminal point could be developed next.

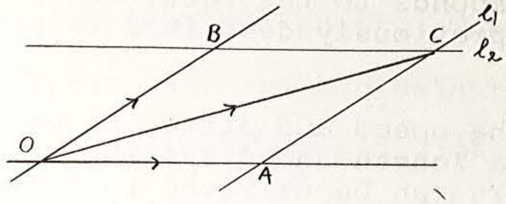

Figure 3

In Figure 3, $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are noncollinear vectors with a common starting point. Point C is determined by the intersection of l_1 and l_2, and OC is the vector sum of $\overrightarrow{OA}$ and $\overrightarrow{OB}$. The angle determined by the two given vectors, along with the idea of corresponding angles, can be used to determine l_1, containing point A, and l_2, containing point B, so that l_1 is parallel to the line that contains $\overrightarrow{OB}$, and l_2 is parallel to the line that contains $\overrightarrow{OA}$. In this way, the intersection of l_1 and l_2 is the terminal point of the vector sum $\overrightarrow{OC}$. Quadrilateral OBCA is a parallelogram since the opposite sides are contained in parallel lines. The three end-points A, O, and B of the two given vectors determine three vertices of the parallelogram. The fourth vertex is the terminal point of the vector sum. The sum vector $\overrightarrow{OC}$ determines a diagonal of parallelogram OBCA.

To summarize this method, the starting point of the vector sum of two noncollinear vectors that have a common starting point is the common starting point of the two given vectors. The terminal point of the vector sum is a vertex of a parallelogram. Two intersecting sides of this parallelogram are determined by the two given vectors.

The magnitude of a force (generally measured in pounds) corresponds to the length or magnitude of a vector. The line of action of a force corresponds to the direction of a vector. Two forces that are applied at the same point have the same effect as a single force called the resultant of the two forces. The vector sum of two vectors that have the same starting point or the same terminal point corresponds to the resultant force. Hence, we can now use the methods previously described to solve problems involving forces.

Similarly, the speed and direction characterizing velocity can be related to the length and direction of a vector. Thus, the sum of two vectors can be utilized to solve many kinds of problems involving such things as airplanes or boats moving through wind or current. Simple arithmetic can be used to find resultant forces or velocities acting in a line. Scale drawings using ruler and protractor can be used to solve problems requiring the notion of equivalent vectors and the triangle method. Scale drawings can also be used to solve problems involving the parallelogram method. If two forces or velocities are acting at right angles, the formulas for solving right triangles can be used to find the resultant force or velocity.

In some problems, the resultant force or velocity is given, and it is required to find two or more vectors called component vectors whose sum is the given vector. For the simple problems at this level, the component forces or velocities required act at right angles to each other, and the vector problem involves the solution of a right triangle. Component forces may be the vertical and horizontal components of a given force; component velocities may involve the east and south, west and north, etc., components of a given velocity.

Our second example of a vector is an ordered pair of real numbers. Each ordered pair of real numbers corresponds to a point in the plane, the correspondence determined by a pair of axes.

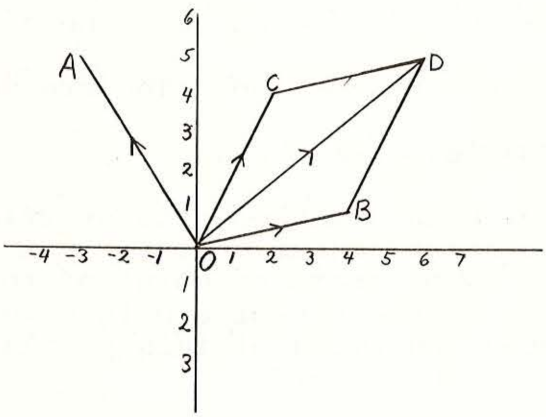

Figure 4

Vector $\overrightarrow{OA}$ pictured in Figure 4 has point (0, 0) as its starting point and (-3, 5) as its terminal point. There are infinitely many vectors in the plane that are equivalent to $\overrightarrow{OA}$, i.e., have the same length and the same direction as $\overrightarrow{OA}$. The vector with starting point (1, 0) and terminal point (-2, 5) is equivalent to $\overrightarrow{OA}$. However, $\overrightarrow{OA}$ is the only vector of an infinite set of equivalent vectors that has the origin as its starting point. Vector OA can therefore be associated with (-3, 5). For convenience, we will say "vector (-3, 5)" or just "(-3, 5)". Each vector in the plane that has the origin as its starting point can be thought of as the ordered pair of real numbers associated with its terminal point. Hence, we will agree that a vector in the plane that has (0, 0) as its starting point is the ordered pair of real numbers associated with its terminal point. If we agree that (0, 0) is a vector, called the zero vector, then each ordered pair of real numbers is a vector in the plane.

Next we need to know what is meant by equal vectors. Two vectors are equal if and only if they are the same ordered pair, i.e., they have the same first components and the same second components. For each ordered pair of real numbers (x, y) and (u, v), (x, y) = (u, v) if and only if, x = u and y = v. It can easily be shown that equality for vectors is reflexive, symmetric, and transitive since we already have these properties of equality for real numbers.

Graphs in the coordinate plane can be used to generate the definition of the sum of two vectors that are ordered pairs.

This can be done in such a way that it will be consistent with the sum of two vectors that are directed segments. In Figure 4, consider $\overrightarrow{OC}$, (2, 4), and $\overrightarrow{OB}$, (4, 1). Quadrilaterial OADB is a parallelogram which means according to the development of the parallelogram method, that

$$\overrightarrow{OC} \oplus \overrightarrow{OB} = OD$$

Stated in terms of ordered pairs this becomes

$$(2,4) \oplus (4,1) = (6,5)$$

We notice that the first component of the sum vector is the sum of the first components of the two given vectors, and the second component of the sum vector is the sum of the second components of the given vectors. Examples such as this, along with comparing horizontal and vertical displacements, will help us accept the following definition of the sum of two vectors:

$$(x,y) \oplus (u,v) = (x + u,\ y + v)$$

for each ordered pair of real numbers (x,y) and (u,v).

If we accept the parallelogram definition, then the ordered pair definition could be developed deductively but not as part of this intuitive development. Also, if we accept the ordered pair definition, then the parallelogram definition could be developed deductively.

Using the properties of addition for real numbers, it can be shown that addition of vectors has the corresponding properties shown below.

RXR (the set of vectors in the real plane)
Binary operation - Vector addition ($\oplus$)

Properties

1. Closure $(x,y) \oplus (u,v) \in RXR$
2. Commutativity $(x,y) \oplus (u,v) = (u,v) \oplus (x,y)$
3. Associativity $\left[(x,y) \oplus (u,v)\right] \oplus (s,t) = (x,y) \oplus \left[(u,v) \oplus (s,t)\right]$
4. Identity $(0,0) \oplus (x,y) = (x,y)$

5. Inverse — For each vector (x,y), there exists a vector (-x, -y) such that (x,y) ⊕ (-x, -y) = (0, 0)

The set of vectors together with the binary operation of vector addition and the five properties constitutes an example of a commutative group.

While referring to ordered pairs of real numbers as vectors, let us refer to real numbers as scalars. Scalar multiplication is the product of a scalar and a vector. As pictured in Figure 5, $\overrightarrow{OH}$ is twice as long as $\overrightarrow{OG}$.

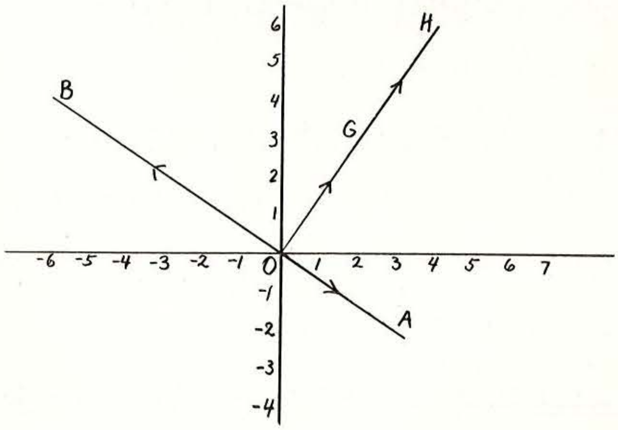

Figure 5

Both OG and OH have the same direction. We can think of OH as being two times OG and write

$$2 \odot (2,3) = (4,6)$$

Examples such as these show that multiplying a vector by a number greater than one has the effect of "stretching" the vector. These examples suggest the following definition of the product of a scalar and a vector:

$$a \odot (x,y) = (ax, ay)$$

for each real number a and for each vector (x,y). From this definition we get

$$-2 \odot (3, -2) = (-6,4)$$

and as shown in Figure 5, we see that $\overrightarrow{OB}$ which is negative two times OA is twice as long as $\overrightarrow{OA}$ and has the opposite direction. Hence, we observe that multiplying a vector by a negative number has the effect of reversing the direction of the vector.

With the definitions of vector addition and scalar multiplication together with the properties of addition and multiplication of real numbers we can derive the following properties of scalar multiplication of vectors.

Closure property of scalar multiplication

$a \odot (x,y) \in RXR.$

Associative property of scalar multiplication

$[a \cdot b] \odot (x,y) = a \odot [b \cdot (x,y)]$.

Distributive property of scalar multiplication over addition of vectors

$a \odot [(x,y) \oplus (u,v)] = [a \odot (x,y)] \oplus [a \odot (u,v)].$

Distributive property to scalar multiplication over addition of scalars

$[a + b] \odot (x,y) = [a \odot (x,y)] \oplus [b \odot (x,y)].$

Identity-element property of scalar multiplication

$1 \odot (x,y) = (x,y).$

Scalar multiplication combines elements from the field of real numbers with vectors and has the five properties as listed above. The commutative group which involves vectors and vector addition together with scalar multiplication and those five properties forms another mathematical system which is an example of a two-dimensional vector space.

In such a brief treatment, many important features of a beginning unit on vectors had to be sacrificed. No sample problems could be given and these are necessary for increasing the student's problem-solving ability. We have stressed the importance of presenting each new idea so that it builds on preceding concepts and feeds into developments that follow. These kinds of experiences will increase the student's mathematical understanding and prepare him for more formal work at high school and college level.

(Reprinted from New York State Mathematics Teachers' Journal, April, 1965.)

TRIGONOMETRY OF THE REAL NUMBER BY WAY OF THE WRAPPING FUNCTION

J. D. Bristol

Editor's Note: J.D. Bristol is the newsletter editor for the Ohio Council of Teachers of Mathematics. In this article he condenses Roy Dubisch's idea which is advocated as a technique for teaching trigonometry.

The circular functions can be nicely developed in terms of the unit circle, $x^2 + y^2 = 1$ and the real number line $x = 1$.

On this line $x = 1$ there are such values as $\pi/6$, $\pi/4$, $\pi/3$, $\pi/2$, ... as well as 0.5, 1, 1.5, $1\frac{3}{4}$, ...

As we "wrap" the line about the circle, the real number 0 maps into a unique point on the circle. For example, $\pi/2$ maps into (0,1), n maps into (-1,0).

The x- and y-coordinates of such points are "defined" as cos θ and sin θ respectively.

Because of the unit circle, $x^2 + y^2 = 1$, the first important identity, $\sin^2 \theta + \cos^2 \theta = 1$ (and the related identities: $\sec^2 \theta - \tan^2 \theta = 1$ and $\csc^2 \theta - \cot^2 \theta = 1$) is quickly established.

Two important ordered pairs are studied: (θ, cos θ) and (θ, sin θ).

With the distance formula and two real numbers θ and ϕ on the number line x = 1, the formula for the cosine of the difference of two numbers is developed.

All of the remaining important sum, difference, double-number, half-number formulas are developed sequentially.

The complete study of the Circular Functions follows easily and naturally. The real number concept (as against "angle") is emphasized so that the use of the Circular Functions in the Calculus will follow.

A suggested reference is: Trigonometry by Roy Dubisch, Ronald Press, N.Y., 1955, $5.50.