THE IMPACT OF MODERN MATHEMATICS ON THE DEVELOPING PROGRAM IN HIGH SCHOOL MATHEMATICS

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I - Pressures for Changes in the Mathematics Curriculum

Dyer, Kalin and Lord(1) state:

> "All this suggests that the mathematics curriculum must grow with the times. The fact is that it has changed very little during the past century. The elementary school has offered arithmetic and some intuitive geometry; the college preparatory curriculum in high school has generally consisted of algebra in Grade IX, demonstrative geometry in Grade X, more algebra in Grade XI, and trigonometry, solid geometry, and occasionally advanced algebra in Grade XII. There have, of course, been some variations on this pattern and changes in emphasis in some schools from time to time, but by and large the mathematics curriculum in general use today looks remarkably like the 1890 model."

That the 1890 model is not adequate for the twentieth century is obvious from the characteristics of this century. Moving from the machine age into the atomic age and into the space age, this century has produced pressures which can no longer be ignored with reference to the high school mathematics program. We now look at four main kinds of pressure.

The Impact of Mathematical Research

Beginning at the graduate level, an explosion in mathematical knowledge has taken place during the twentieth century. This growth of pure mathematics has been the handmaiden to the explosion in scientific knowledge. In general, with deference to a time lag, the universities affect the curricula of the high schools (and below) by pushing back the frontiers of knowledge. This new mathematical (and scientific) frontier has produced a wide gap between the high school courses and the university courses. A great pressure has been built up therefore to change the mathematics curriculum of the college-bound student.

J. Dieudonne(2) states:

> "In the last 50 years, mathematicians have been led to introduce, not only new concepts, but a new language, which grew empirically from the needs of mathematical research, and whose aptitude for expressing mathematical statements concisely and precisely has repeatedly been tested and has won universal approval. But up to now the introduction of this new terminology has been steadfastly resisted by the secondary schools, who desperately cling to an obsolete and inadequate language: so that when a student enters the university, he will most probably never have heard such common mathematical words as set, mapping, group, vector space, etc. No wonder if he is baffled and discouraged by his contact with higher mathematics."

Professor Dieudonne goes on to evaluate the situation and comes up with a startling remedy:*

> "I think the day of such patchwork is over, and we are now committed to a much deeper reform, unless we are willing to let the situation deteriorate to the point where it will seriously impede further scientific progress. And if the whole program I have in mind had to be summarized in one slogan it would be: <u>Euclid must go</u>!"

Realizing that this would create a great vacancy, he goes on to suggest what should be put in its place:

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*Dieudonne was speaking of the situation in France: there seems to be an internationality about mathematical problems - he could have been talking about the United States or Canada.

"In more detail, I would list the following:

(i) Matrices and determinants of order 2 and 3.
(ii) Elementary calculus (functions of one variable).
(iii) Construction of the graph of a function and of a curve given in parametric form.
(iv) Elementary properties of complex numbers.
(v) Polar coordinates."

Growth in Scientific Research

To understand, to participate in, to contribute to the new age of science, students must have a mathematics program geared to the scientific knowledge of the twentieth century.

Einstein's relativity theory and his equation of energy are mathematical concepts. Nuclear physics is grounded in mathematics, much of it of the "modern" variety such as matrices, topology, group theory. The science of cosmology has demanded the revival and creation of mathematical theories to supply the answers to new problems. In the world of new scientific frontiers, the mathematically uneducated become illiterate.

New Mathematical Applications in Other Fields

The National Council of Teachers of Mathematics in its 1959 report remarked on the "stochastic" nature of the world. This nature has asked mathematics to sharpen the theories of probability and statistics to give patterns and relationships to a multitude of factors relevant to a multitude of problems.

Many of these problems occur in the field of social sciences. Investigators are saying that factors which exist must exist in some amount; these amounts can be measured, providing mathematics can produce the relevant measures. For example, the theory of matrices enables the research workers in many fields to consider, simultaneously, a multitude of diverse factors that impinge upon any situation. Mathematical theories of patterns and structures are applied to patterns and structures in the social, or economic or industrial world. Logical reasoning and the very nature of proof is manifest in mathematical deduction and implication. The law student may carry a book of torts in one hand supported by a mathematical set of "truth tables" in the other.

Automation

The greatest image of twentieth century automation is the so-called electronic brain, the digital computer. Descriptions in the popular media of communication say that it can "think". Actually, the thinking is done by (a) the designer, (b) the programmer - both mathematicians. And this is an application of the most fundamental principle in mathematics - the principle of counting (base 2!) and of determining equality. Behind all the wonders of automation stand the mathematicians. An understanding of fundamental mathematical principles is necessary to appreciate these machines; their operators must be mathematical technicians; their designers must be creative mathematicians.

References

(1) Dyer, Henry S., Kalin, Robert, and Lord, Frederick M., "Problems in Mathematical Education," Educational Testing Service, Princeton, N. J., 1956, pp. 18-19.

(2) Dieudonne, J., "Paper on Teaching Geometry", Background paper, Seminar on New Thinking in School Mathematics, (unpublished), Paris, 1959.

II - Nature of the Changes in Secondary School Mathematics

Reselection of Ideas for the Details of the Courses

Some of the changes in the mathematics curriculum will be the normal, gradual changes in details which occur in any discipline as civilization marches onward and the environmental world changes about the student.

Part of the mathematics courses will become obsolete because of technological changes. Here is a specific example. In trigonometry, the solution of a triangle is important; if, say, three sides are known, it is important to know how to determine the measures of the angles. There are many relationships that could be used. Here are two:

The Cosine Law:

In A, B, C,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

The Half-Angle Tangent Law:

$$\tan \frac{A}{2} = \frac{s(s-b) \quad (s-c)}{(s-a)}$$

$$\text{Where } s = \frac{a+b+c}{2}$$

The development of the Cosine Law is elementary; its validity is readily proven and it is easily understood. However, because it involves the operations of addition and subtraction this relationship is not suited to the use of logarithms nor of a slide-rule. Twenty years ago, logarithms and slide-rules were the prize tools of the engineer. Therefore the more clumsy half-angle Tangent Law was emphasized in high school trigonometry because it involved multiplication, division and extracting square roots which could be done by logarithms or on a slide-rule. But the half-angle Tangent Law was difficult to develop and many high school pupils resorted to memorization in lieu of understanding. Today, fast and efficient computers are available which add, subtract, multiply, or divide with equal facility. Therefore the simpler Cosine Law can replace the more complicated half-angle Tangent Law.

Many other parts of mathematics become obsolete because of a change in treatment, in sequence or approach. It is particularly easy to spot the deadwood which persists from the days of mental discipline because of a traditional lag. B. E. Meserve(1) suggests that the following ideas are obsolete or at least should receive less emphasis: Horner's method for solving equations, formulas for solving general cubic and quartic equations, trigonometric identities for multiple angles, obscure types of factoring, use of logarithms.

Introduction of New Material

(a) Theory of Sets: B. E. Meserve states:

> "The theory of sets is of fundamental importance in all branches of mathematics. It should permeate the thinking of teachers at all levels - kindergarten through graduate school."

Most educators, working in the field of modern mathematics, feel that set theory is a great unifying idea for many parts of the discipline. The set idea carries with it new terms - elements, disjoint sets, subsets, null sets, complementary sets, finite and infinite sets, union and intersection. The theory of sets may be applied to such diverse "elements" as numbers, points, vectors, atoms, members of a woman's club, baseball leagues, electors, polynomials, switching circuits, mathematical operators.

(b) Mathematical Systems: In the introduction to the twenty-third yearbook of The National Council of Teachers of Mathematics, C. V. Newson says:

> "As already implied, the same system may occur frequently as the underlying pattern in many diverse real and ideal situations. Thus there is validity in the assertion that the mathematician is less concerned with development of general patterns that have widespread applicability in the study of particular situations."

For example, much of mathematics deals with the properties and operations of a set of numbers called the integers. We learn how to add, subtract, multiply, divide, factor, find powers, extract roots. But modern mathematics looks beyond these computational skills to discover underlying structures that may have wider applications. In this respect we find that the integers form a number system because they have defined for them two binary operations (addition and multiplication) which have closure, are commutative, associative, and one operation is distributive over the other.*

But integers are not the only number system. The structure of a number system is found with rational numbers and with real numbers. Indeed, there are number systems without "numbers" in them at all - for example, the collection of all subsets of any set (note that the elements of this set may be anything - you name it!) with the operations of "union" and "intersection" form a number system.

*This sentence summarizes the axiomatic definition of "a number system".

Getting back to integers, we might look at them as an example of a <u>group</u>. (A group is a mathematical system with clearly defined properties.) Group theory has many applications. For example, the whole area of <u>permutations</u> belongs to this mathematical structure.

Integers are also <u>rings.</u> This is another structure with certain properties, which contains many systems besides the integers (for example, polynomials). That is, if we study <u>rings</u>, rather than <u>integers,</u> we study a mathematical model with more generality and hence with greater application.

Integers are <u>not</u> "fields". (A field is yet another mathematical system). But the <u>rational</u> numbers (fractions) are <u>fields</u>. In other words, as we expand our number domains through the common stages - natural to integer to rational to real - we find new patterns and structures - groups, rings, and fields - that have different and wider applications. We have begun to nibble on a great rug that has a beautiful but intricate design woven into it.

<u>(c) Probability and Statistical Thinking</u>: The demand from the social sciences for mathematical models to deal with many factors that impinge simultaneously upon some situation, has brought forth the importance of the study of probability, variability, sampling, normal and hypergeometric distributions, etc. As the general standard of education is raised with the impact of a complex society, new ideas such as "norms", "means", "standard deviation", "statistic", "correlation", find their way into the literature of the day. Mathematical statistical models are a part of "modern mathematics". Indeed, this is a real frontier area with considerable growing pains due to its rapid crash development. Some of these ideas will undoubtedly be added to the secondary school mathematics program.

<u>(d) Other Specific Examples of New Material</u>:
(i) <u>Numeration Systems with Bases Other than 10</u>. - It is important to differentiate between the "mechanics" of mathematics and the "ideas" of mathematics. Symbolization, notation, numeration systems, algorisms all belong to the mechanics - they are used for the manipulation of symbols. If some of these "mechanics" are involved, then it is important to know and to understand the

rationale that has been used to design the symbolization. This is the case with the numeration system that is used to symbolize our numbers. One way to sharpen the understanding of the Hindu-Arabic numeration system is to consider a different system with a base other than ten. This immediately focuses attention on the essential characteristics of the system, such as its base, the use of zero, place value, grouping and regrouping.

For example, let us devise the numberal system that would be used to "count" the elements of this set,

X X X X X X X X X X X X X X X X X X X X X X X X X X X

first in base 10 (with usual word symbols as well), then in base 4 (with semi-creative word symbols with a canine slant!), and finally in base 2.

Base 10		Base 4		Base 2
Numerals:	Word Symbols:	Numerals:	Word Symbols:	Numerals
1	one	1	one	1
2	two	2	two	10
3	three	3	three	11
4	four	10	doggy	100
5	five	11	doggy-one	101
6	six	12	doggy-two	110
7	seven	13	doggy-three	111
8	eight	20	twoggy	1000
9	nine	21	twoggy-one	1001
10	ten	22	twoggy-two	1010
11	eleven	23	twoggy-three	1011
12	twelve	30	throggy	1100
13	thirteen	31	throggy-one	1101
14	fourteen	32	throggy-two	1110
15	fifteen	33	throggy-three	1111
16	sixteen	100	a houndred	10000
17	seventeen	101	houndred & one	10001
18	eighteen	102	houndred & two	10010

What is the sum of throggy-three and twoggy-two? Using our common algorism we compute:

$$\begin{array}{r} 33 \\ \underline{22} \\ 121 \end{array}$$

The answer is one houndred and twoggy-one!

What is the product of these two numbers? Obviously, it is

$$\begin{array}{r} 33 \\ \underline{22} \\ 132 \\ \underline{132} \\ 2112 \end{array}$$

or twoggy-one houndred and doggy-two!

Note that it is necessary to know the "basic facts" for addition and multiplication in base 4 in order to do this computation. Also note that the task of remembering these facts has been reduced (compared to base 10) to 16 from 100. In the above computations attention has been given to place value and to grouping by fours.

Of course, this study of other bases is more than a novel, intellectual work-out. There <u>are</u> practical applications. The digital computer, for example, is designed to "count" with a base 2 numeration system, the 2 digits (1 and 0) being an open and a closed circuit respectively. This means that the programmer must convert all numerals into a base 2 system.

(ii) <u>Modular Systems</u>--Infinite sets, for example, natural numbers, 1, 2, 3, 4, 5, 6, ... may be "mapped" into finite sets (sets with a definite number of elements) by means of a modular system. For example, all the hours of time, beginning at some arbitrary point of time could be represented by the infinite set 0, 1, 2, 3, 4, ... These "hours" could be "mapped" (and indeed, are) into the 12 hours on the face of a clock as follows:

Infinite Set: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, \ldots\}$

Finite Set:
Modular 12: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$

We have used this finite set, modular 12, for centuries to tell the "time of day". We can "add" with this finite set, mod. 12, for example, $11 + 3 = 2$; $5 + 7 = 0$; $6 + 9 = 3$. We can also multiply: $3 \times 7 = 9$; $2 \times 6 = 0$. Indeed, these two operations have closure, are commutative, associative, and multiplication is distributive over addition. This means that this finite set, of 12 elements only, is a <u>number system</u>. (Refer to

The idea of a modular number system carries us amazingly far into abstract algebra and has many used and applications both in the mathematical world and in the physical world of real events.

<u>Some Changes in Emphasis</u>

(a) <u>Logical Reasoning</u>: Euclidean geometry has long had a reputation as a discipline based upon deductive reasoning. Often the reasoning has been overlooked and students have merely memorized the "proofs". However, the development of reasoning ability is claimed to be an objective of high school mathematics, and certainly "modern" mathematics will emphasize this objective much more than it has been.

B. E. Meserve(2) states:

> "We lose many excellent opportunities for emphasizing deductive thinking when we restrict that topic to geometry. Often it is easier in algebra. We should make deductive thinking a part of all formal work in mathematics."

Mathematical systems are axiomatic systems; students become familiar with "undefined terms", "definitions", "axioms or postulates", "implications or theorems", "converses", "inverses", and "contrapositives", and so understand better the nature of truth.

(b) <u>The Language of Mathematics</u>: Modern mathematics suggests not only careful attention to a precise terminology, but presents the thesis that mathematics <u>is</u> a language.

E. A. Krug, in his book, <u>The Secondary School Curriculum</u>, begins a chapter on mathematics by saying: (3)

> "Mathematics is language, a more specialized, exact, precise, and refined form of language than any of those to which the term 'language' is ordinarily applied. It has become a powerful and indispensable tool in the description, analysis, and control of phenomena that can neither be accurately expressed nor understood in ordinary language alone. It is the most truly international of all languages. It is one of the most remarkable of all human achievements and as such forms part of the cultural heritage that schools seek to transmit and on which they build for the future.",

and he ends the same chapter:

> "And this is another way of saying that the mathematics curriculum is moving toward a fuller realization of mathematics not as a body of mechanical manipulations and routines, but as a language of unparalleled beauty, precision, and power."

Modern mathematics is concerned with ideas, with concepts. These are symbolized by many kinds of notation, which carry with them the sense of their referents. Some of the symbols are "nouns" - 5, x, AB; some of the symbols are "transitive verbs" - +, x, /; an important one is the verb "to be" - =. The mathematical sentence - the equation or inequality - expresses an idea.

This implies that modern mathematics is concerned with <u>thinking</u>, with <u>comprehension</u>, rather than just with manipulation. It is more than a tool or a skill, it is a structured body of related ideas extending the frontiers of knowledge.

<u>References</u>

(1) Meserve, Bruce E.; "Implications for the Mathematics Curriculum,"

Insights into Modern Mathematics, Twenty-Third Yearbook, NCTM, Washington, D.C., 1957.

(2) Ibid: page 415.

(3) Krug, E. A.; The Secondary School Curriculum, Harper and Brothers, New York: 1960, page 278.

III. Implications for Curriculum Planning

Aims and Objectives

H. L. Stein(1) states:

> "In summary, then, a search of the literature on objectives of mathematics in the secondary school reveals mild swings from one point of view to another. Roughly, until the early twenties, the emphasis was upon mathematics as a discipline. From the twenties to the early fifties there appeared to be a swing to utilitarian emphasis. Presently, the shift is towards a thoroughly modernized program which emphasizes, in the main, mathematical concepts - at least for the college-capable students."

When educators consider the objectives of high school mathematics, they often mention a dualism: (1) those objectives for college-bound students, and (2) those objectives so far as a "general education" is concerned. The Commission on Mathematics of the College Entrance Examination Board in its report, "Program for College Preparatory Mathematics" (2) have, in spite of their main interest, recognized the obligation of the schools to serve "all the children of all the people" and suggest the following specific objectives for "general" students:

1. An understanding of and competence in, the processes of arithmetic and the use of formulas in elementary algebra. A basic knowledge of graphical methods and simple statistics is also important.
2. An understanding of the general properties of geometrical figures and the relationships among them.

3. An understanding of the deductive method as a method of thought. This includes the ideas of axioms, rules of inference, and methods of proof.
4. An understanding of mathematics as a continuing creative endeavor with aesthetic values similar to those found in art and music,...

Their objectives for the college-bound student are more challenging "to know something of the intellectual excitement and deep satisfaction that mathematics can offer", and at the same time help to prepare him for the tasks demanded by a technological age. As the report points out,

> "Mathematics is no longer reserved for the use of engineers and physical scientists, even though a great many of its applications are still in their hands. ... Today, mathematics exerts actual leadership in natural science, social science, business, industry, and other fields.
>
> ... the work in the classroom must be constantly related to other fields of knowledge; to the pupil's immediate experience, and to other activities (e.g., science and geography) in which he is engaged. The ultimate aim is the recognition on the part of the pupil that mathematics is a dynamic system, constantly developing, and that creative self-activity, is a fascinating realm of endeavor."

A close examination of the objectives for "college-bound" and for "general" students as stated by different groups, produces not a dualism, but, in fact, a synthesis of objectives. Statements of objectives for both groups of students include (a) the utilitarian value of mathematics and (b) the cultural value of mathematics. The difference between the statements is not one of kind but of degree. This idea is elaborated further in the section below on individual differences.

The Need for Unification

First, there is a need to find some unifying ideas that will synthesize and fuse the various threads of mathematical concepts into some definite, organized and sequential pattern. We do not naively imply that the field can be simplified into an easy sequence from

A to Z, and that our only problem would then be to put our mathematics students through the alphabet. But the warp and the woof of the mathematics tapestry should all contribute to a unified pattern held together by relationships and rational associations. Especially should "new" topics be so interwoven. The Secondary School Curriculum Committee of the National Council of Teachers of Mathematics list certain new topics, "not to be isolated, but woven into the curriculum".

One such "unifying idea" is the Theory of Sets. H. F. Fehr, in a talk at a mathematics institute in Madison, July, 1960 emphasized that the greatest hope for finding a synthesizer of the various new ideas of modern mathematics was the "set idea". Most experimental programs(3) at the elementary level include a foundational consideration of sets. At the high school level, the Theory of Sets is a popular and promising addition and approach and is included in most of the modern mathematics textbooks written for secondary students.(4)

The Commission on Mathematics lists the following as "unifying ideas": sets, variables, functions, and relations. These will undoubtedly form an important part of the outline of any new high school mathematics program.

A Team Approach to Curriculum Planning

Van Engen(5) says:

> "The reform of the mathematics program rests ultimately on three pillars - a tripod of sound principles. The first leg of the tripod rests upon sound mathematics... The second leg of the tripod rests upon sound philosophical principles... The third leg of the tripod rests upon sound psychological principles."

We have suggested here, a team of three. The mathematician will produce an organized, sequential body of mathematical content that will rigorously develop this discipline as a science; the educational philosopher will relate this content to the aims and objectives of twentieth century education, and the school psychologist will suggest methods and approaches consistent with known theories of learning and of child development. Such a team, working together, may produce a dynamic and vital mathematics program to replace our "1890 model".

The Provision for Individual Differences

The Secondary School Curriculum Committee of the National Council of Teachers of Mathematics appointed ten subcommittees to study various phases of the "modern" program in mathematics; one of these prepared a report on the provision for individual differences. They suggested:

(a) Teach the same mathematical structures and concepts to all, but vary the amount, complexity, depth, and manner of organization and presentation.
(b) For pupils of below average ability, have a program of minimum essentials. Research is needed to provide teachers with a detailed outline for below average pupils.
(c) Program of enrichment for the gifted should provide real challenge and basic information.
(d) Ability grouping is a desired practice.

Here is a recommendation that realistically recognizes that individual differences do exist and that ability grouping is "a desired practice". However, the provision is not one of dualism, but rather to "teach the same mathematical structures and concepts to all". The courses will vary from "minimum essentials" to enrichment with a "real challenge".

Traditional mathematics could not be organized in this way. Based primarily upon the memorization of facts, and the development of computational skills, the "1890 model" does not possess enough versatility to make provision for wide individual differences. That is why "dualism" has been persistently advocated.

Modern mathematics, however, based upon <u>ideas</u>, and organized with unifying structures may be flexible enough to meet the demands of both a college-preparatory course and a general course. We may find that this program will point to the possibility of a high school marriage of the two objectives, to prepare for life and to prepare for higher education.

References

(1) Stein, H. L.; "Recent Views of Mathematics of the Secondary School," Canadian Research Digest, No. 6, C.E.A., Spring, 1960.

(2) CEEB, Report of the Commission on Mathematics, Program for College Prep. Math., New York, 1958.

(3) e.g. Hawley and Suppes of Stanford University at Grade I level; the Cleveland experiment; outlines of the SMSG. .

(4) e.g. Freund, J. E. A Modern Introduction to Mathematics, Prentice-Hall, 1956.

(5) Van Engen, Henry, The Theory of Arithmetic and Its Teaching, Mimeographed manuscript, 1960.