Contemporary Mathematics and its Mathematicians

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Volume 39, 1970, of *Mathematical Reviews*, published by The American Mathematical Society, indicates that some 95 general subject classifications are being used for the many areas of mathematics. In this volume, there are approximately 900 different subject listings. This relatively small sample hardly does justice to the vastness of the field of contemporary mathematics.

Back in the year 1941, Richard Courant and Herbert Robbins first published their book *What is Mathematics?** After some attempt in their introduction to answer this profound question, they finally concluded that "For scholars and laymen alike it is not philosophy but active experience in mathematics itself that alone can answer the question: what is mathematics?"

For this article we selected a very small subset of the many areas of mathematics and asked various members of the Department of Mathematics, University of Alberta, to participate. Specifically, each professor was asked to identify an area of interest and specialization by briefly indicating what it is all about. He was then asked to name some of the outstanding contemporary mathematicians in his area, to tell where they are located today, and to state what aspect of their speciality they are investigating at this time. The following contributions have been received with thanks.

THEORY OF NUMBERS

Harvey L. Abbott

Number theory is the study of the properties of the positive integers. Because of their importance in counting, the positive integers were among the first objects of a mathematical nature to be considered by ancient civilizations. Since questions about numbers appeal to natural human curiosity, it is not surprising that number theory has a long history. For example, early Chinese mathematicians considered various questions concerning divisibility properties of integers, some of them of a decidedly non-trivial nature. One of the questions which they raised was whether the condition that n divides $2^n - 2$ implies that n is a prime. While they were not able to answer this question, it should be pointed out that it was not until the 18th century that the great German mathematician C.F. Gauss proved such an integer n is not necessarily a prime, and that the problem of finding all composite integers n which divide $2^n - 2$ is still unsolved. The ancient Greeks, noted mainly for their contributions to geometry, also investigated properties of numbers. Euclid was aware of the fact that there exist infinitely many primes, and in his "Elements" he also discusses the so-called perfect numbers; a number n, such as 6, 28, or 496, is said to be

*R. Courant and H. Robbins, *What is Mathematics?* Oxford University Press, Seventh Printing, 1956.

perfect if the sum of its divisors is 2n . It is still an unsolved problem as to whether there are infinitely many such numbers or, for that matter, whether there are any at all which are odd.

Number theory advanced very little during the dark ages. It can be said that modern number theory had its beginning during the 17th century with the work of the French mathematician Pierre Fermat. Fermat proved a large number of very interesting and often surprising results about the positive integers. For example, every prime, which leaves a remainder of 1 on divisions by 4, can be written in a unique manner as a sum of two squares. When one considers that there is no simple way of deciding whether a number is a prime or not (excepting the obvious but impractical method of finding all of its divisors) it becomes clear that this theorem of Fermat is remarkable. Practically all famous mathematicians of the 17th, 18th, and 19th centuries contributed to the theory of numbers.

During the past 100 years, number theory has developed in many directions. Perhaps the major advance has been the development of the analytic theory of numbers. This is not so much an "area" of number theory but a method of attack. Analytic number theory is characterized by the use of the calculus of functions of a complex variable. The first advances were made by Riemann in Germany, and Hadamard and de la vallée Poussin in France. Other pioneering work was done by Hardy and Littlewood in England and by Vinogradov in Russia. These and others have developed analytic tools very systematically, and now one can give answers to questions which would have been hopelessly difficult a century ago.

The additive theory of numbers is concerned with questions of the following type:

Given two sequences of positive integers $a_1 < a_2 < \dots$ and $b_1 < b_2 < \dots$,

what can be said about the set of numbers which are of the form $a_i + b_j$?

While some questions in additive number theory have been in existence for a long time, most of the results have been obtained during the past 40 years. The most active and prolific mathematician in this area is P. Erdös of Hungary.

Finally, consider the geometry of numbers which is concerned with the following basic question: How many lattice points (points with integers for coordinates) are there in a given region in the plane or in a space of a higher number of dimensions? This is a surprisingly difficult question, and even for such simple regions as the interior of the circle with equation $x^2 + y^2 = n$ the complete answer is unknown.

Like all other branches of mathematics, number theory is growing at a very fast rate. This will continue, because while there is a great deal now known about positive integers, there are still many things which we do not know.

SET THEORY

Henry F. J. Lowig

Set theory is a mathematical discipline which is due to George Cantor, a German mathematician (1845 - 1918). Cantor's most important achievement is the discovery that the number of elements of an infinite set can be defined in a similar way as the number of elements of a finite set. Two finite sets have the same number of elements if, and only if, there is a one-to-one correspondence between them. Cantor extended this criterion of "having the same number of elements" to infinite sets. According to Cantor, any two sets have, by definition, the same number of elements if, and only if, there is a one-to-one correspondence between them. The number of elements of a set is called its "cardinal number" or "cardinality". A set which is in one-to-one correspondence with the set {1,2,3, ...} of positive integers is called <u>countable.</u> This implies that any two countable sets are of the same cardinality. The set of odd positive integers and the set of all rational numbers are examples of countable sets.

On the other hand, Cantor proved that the set of all real numbers is uncountable. His argument runs essentially as follows. Assume that we have an enumeration of all real numbers x such that $0 < x \leq 1$, namely,

$$
\begin{array}{l}
0.\ a_{11}\ a_{12}\ a_{13} \cdots \\
0.\ a_{21}\ a_{22}\ a_{23} \cdots \\
0.\ a_{31}\ a_{32}\ a_{33} \cdots \\
\quad \cdots
\end{array}
$$

Here it is supposed that each of these numbers is written as a non-terminating decimal. For example, 1/4 would be written as 0.24999 ... , not as 0.25 or 0.25000 ... . Let, for n = 1,2,3,...,

$$
a_n = \begin{cases} 9 & \text{if } a_{nn} \neq 9 \\ 1 & \text{if } a_{nn} = 9 . \end{cases}
$$

Then $0.\ a_1 a_2 a_3 \ldots$ is a real number > 0 and ≤ 1 which does not occur in the above enumeration, contrary to our assumption. Hence such an enumeration cannot exist, or the set of all real numbers x such that $0 < x \leq 1$ is uncountable. Now it is easily shown that the set of all real numbers (without restriction) is also uncountable.

Therefore, the cardinality of the set of all real numbers is different from the cardinality of the set of all positive integers; so there exist different infinite cardinal numbers just as there exist different finite cardinal numbers.

Another part of set theory which is very important for its applications is the theory of <u>well-ordered</u> sets. An ordering of a set is called a well-ordering if, under this ordering, every non-empty subset of the given set has a smallest (or first) element. For example, the set of positive integers is well-ordered by its natural ordering. The same set can be well-ordered as follows:

2, 4, 6, 8, ...; 1, 3, 5, 7, ...

(that is, all even positive integers precede all odd positive integers, while the even positive integers as well as the odd positive integers are left in their natural order). On the other hand, the usual ordering of the set of all non-negative real numbers is not a well-ordering because, for example, there is no smallest positive real number.

Nowadays, set theory is used in practically all branches of mathematics. A few mathematicians who are currently working in abstract (or pure) set theory are J. W. Addison, University of California at Berkeley; G. Fodor, Szeged, Hungary; K. Hrbáček, Charles University of Prague, Czechoslovakia; K. Kuratowski, Warsaw, Poland; E. C. Milner, University of Calgary, Alberta; J. Mycielski, University of Colorado, U.S.A.

GROUP THEORY

Ronald D. Bercov

The rotations about a point in the plane can be regarded as a mathematical system in which we obtain a third rotation, $R_1 o R_2$, from two given ones, R_1 and R_2, by following the first by the second. For instance, if R_1 and R_2 are 15° and 30° counterclockwise rotations, then $R_1 o R_2$ is a 45° counterclockwise rotation. In this mathematical system we have for any three rotations that $R_1 o (R_2 o R_3)$ is the same as $(R_1 o R_2) o R_3$ (associative law); the 0° rotation E plays a role analogous to that of the number zero for addition, namely R o E = R for all rotations R (identity element); and for each rotation R, the rotation -R, which has the same magnitude and is opposite in sense, satisfies R o (-R) = E (inverse). Such an associative system with an identity and inverses is called a group.

Groups first became important as a means of describing geometric systems. For instance, any rotation about its center takes a circle onto itself, but only 0°, 90°, 180° and 270° rotations take a square onto itself, and only 0°, 120°, and 240° rotations take an equilateral triangle onto itself. Thus we can identify these geometric objects by knowing which rotations take them onto themselves. We say that a geometric object is described by its "symmetry group".

Similarly, physical systems can be described by symmetry groups. Those transformations which leave the system unchanged give a description of the system. In this way group theory has become important for modern theoretical physics. Indeed, the group concept was used recently by the Nobel prize-winning American physicist Murray Gell-Mann, whose work has contributed to the classification of the subatomic particles, of which there are a large number. The group-theoretic approach even indicated the existence of an at that time unknown particle which was then found experimentally.

The mathematician studies groups by breaking them up into "simple" factors much as large whole numbers are broken up into their prime factors. New such simple groups have been found in the last few years by, among others, a Czech, Janko, then in Australia, now in the United States; a Canadian, Higman, also in the United States; an American, Hall; and an Englishman, Conway. Major discoveries had been made in the five to ten previous years by a Frenchman, Chevalley; a Japanese, Suzuki, and a Canadian, Steinberg, both in the United States; and a Korean, Rhee, in Canada. There is now hope that almost all such finite simple groups have been found. This hope has been bolstered by the outstanding work of two young Americans, Feit, now of Yale, and Thompson, now at Cambridge in England, who have proved that there are no new such groups with an odd number of elements. The international character of modern mathematics is well illustrated by the above partial list of distinguished group theorists. Another name prominently associated with these discoveries is that of Richard Brauer, who came to the University of Toronto as a refugee from Nazi Germany and who is now at Harvard University. Much recent progress in the theory of groups has been based on methods developed by Brauer.

ORDINARY DIFFERENTIAL EQUATIONS

Jack W. Macki

Ordinary differential equations were of great interest in the 18th and 19th centuries because the many laws of physics are easily written in terms of these equations.

For example, Newton's law, $f = ma$, becomes $f(y(t),t) = m\frac{d^2y}{dt}$ if $y(t)$ describes the position of a particle in a force field, f.

Interest in such equations declined in the early part of this century, but since about 1940 there has been a tremendous increase in interest in ordinary differential equations. At the University of Alberta, work in ordinary differential equations centers on several different areas as follows:

(1) Suppose that a particle is subjected to a periodic (repeating) force. Will its motion be periodic? In more mathematical terms, if $q(t + 2\pi) = q(t)$ for all t, when are solutions of $y''(t) + q(t)y(t) = 0$ periodic?

(2) How can one determine the behavior for large time of solutions of differential equations? A typical result is the following: If $q(t)$ is an increasing function of t, then solutions $y(t)$ of the equation $y''(t) + q(t)y(t) = 0$ oscillate (that is, have infinitely many zeros) and are decreasing in amplitude. Questions like this are important because most of the interesting differential equations that occur in applied areas cannot be solved exactly. Our task is to get information about the solutions, even though we cannot determine the solutions exactly.

Many talented people are working on ordinary differential equations today. Probably one of the best in the world is F. V. Atkinson, University of Toronto. In the United States, there are very strong groups located in the universities of Maryland, Minnesota, and Wisconsin. Among the most eminent men in the field in the United States are Philip Hartman at Johns Hopkins University, Einar Hille at the University of New Mexico, W. T. Reid at the University of Oklahoma, and Lamberto Cesari at the University of Michigan. European schools are generally weak in ordinary differential equations. The notable exception is the Italian school centered in Florence. This school is led by Robert Conti. There are many strong groups researching differential equations in the Soviet Union. Notable among the workers there are the famous blind mathematician L. S. Pontryagin, in Moscow, and Bogolinboff, Krylov, and Mitropolsky of the outstanding Ukrainian school.