Contemporary Mathematics and its Mathematicians

Dr. William J. Bruce, Editor
Department of Mathematics
University of Alberta

Here follows the conclusion of the article started in the last issue of *Delta-K* (Vol.X, No.3, May, 1971).

PARTIAL DIFFERENTIAL EQUATIONS

Walter Allegretto

The branch of mathematics called Partial Differential Equations deals, as its title indicates, with the study of equations involving a function of more than one variable and several of its derivatives. The main problem usually considered is to find solutions of these equations which satisfy specific conditions which, in turn, depend on the special problem under consideration. Often, one also investigates the properties of the solutions of such equations, without actually finding them.

Partial differential equations arise in almost every branch of modern science. Fluid flow, heat flow, wave motion, atomic physics, relativity theory, electromagnetic waves, quantum mechanics, and reaction rates are but a few of the fields which extensively use partial differential equations.

Together with the above-mentioned applications, in recent years, elegant new theories have been developed. Now the subject also forms an abstract branch of mathematics somewhat removed, at the present time, from physical problems.

Since the subject is so vast, the names of only very few of the mathematicians working in this area can be mentioned. Many omissions are clearly unavoidable. From the list of notable names emerge Browder, Bers, Protter, Courant and John, all of whom work in North America; Sobolev and Miklin in the USSR; Lions, Stampacchia and Hörmander in Western Europe; and Agmon in Israel.

GENERAL TOPOLOGY

Richard L. McKinney

The word "topology" is derived from the Greek word "τοπος" meaning "place" or "space". Thus, it is a rudimentary form of geometry which analyzes the fundamental properties of very general spaces. Some isolated problems of a topological nature were considered long ago by such famous mathematicians as Descartes (1640), Euler (1736) and Gauss (1794). The proper birth of the subject grew out of the work of Weierstrass in the 1860s in which he investigated the foundations of the calculus. Cantor's development of the theory of point sets a decade later was instrumental in determining the direction of general topology during its rapid growth in the present century. The adjective "general" is used to distinguish this point set topology from combinatorial or algebraic topology which started in the 1890s with some remarkable work of Poincaré.

As a consequence of its general nature, topology is a unifying branch of mathematics which has applications in many seemingly strange quarters. A brief perusal of the current crop of mathematical research journals will dispel any doubt that topology is now one of the most active fields of mathematical activity A few of the important twentieth century contributors to topology are Brouwer of Holland, Banach of Poland, Alexandroff of Russia and Moore of the United States. An excellent introductory survey of topology is available in the paperback *First Concepts of Topology* by W. G. Chinn and N. E. Steenrod, published in the New Mathematical Library series of Random House.

APPROXIMATION THEORY

Amram Meir

The best-known classical result of approximation theory is Weierstrass' Theorem which states that any continuous function on a bounded closed interval can be uniformly approximated by polynomials.

Since this theorem was proved, approximation theory has developed in several directions, had interactions with different other fields of mathematics and become a very fruitful and wide area of research. Important new research is being carried out at present in a number of mathematical centers.

The general problems usually can be formulated as follows:

1. Given a normed linear vector space and a certain subspace of it, how well, by what method and under what conditions can an element in the space be approximated by an element from the subspace?

2. Given a functional on a function space, how well can it be approximated by other functionals of a certain type?

To the first type of results belong all those statements which claim the approximability (in some sense) of a function of polynomials, spline functions, trigonometric functions, solutions of differential equations, and so forth. To the second type belong the quadrature formulae (integral approximation).

Some of the outstanding men who are doing important and active research in approximation theory today are A. F. Timan of the USSR; G. Freud and P. Turán at the Academy of Science in Hungary; G. Birkhoff at Harvard University; W. Cheney and G. G. Lorentz in Texas; A. Sard and D. J. Newman in New York; T. J. Schoenberg in Wisconsin, and J. L. Walsh in Maryland.

FUNCTIONAL ANALYSIS

John L. B. Gamlen and Sherman D. Riemenschneider

Functional analysis is an abstract branch of analysis which began earlier in this century when a new and very powerful approach to analysis succeeded in solving some difficult long-standing problems. Present day functional analysis is, broadly speaking, an extension of the same methods into many different mathematical situations. The methods have obtained many striking new insights, and have also simplified much of the classical analysis.

We illustrate the method with two general examples. In electric field theory, the action of certain charges is described by a function (the potential function). It is known that this function satisfies a differential equation (Poisson's equation). Very frequently, the main concrete information one has about the potential function is that it solves such an equation (its form being unknown). Solution of the equation produces the function, but for a long time mathematicians were unable to show that the equation must always have a solution. The classical methods dealt with detailed properties of individual functions, whereas the new method considers a collection of functions, and attempts to reason from the properties of the collection as a whole. This is the so-called global approach. The use of the new methods enabled mathematicians to solve the problem.

To illustrate further the global point of view, we consider the property of continuity for a function. Many functions have this property and many do not. The collection of all functions that have the continuity property is a linear space (the sum of two continuous functions is a continuous function, and a number times a continuous function yields a continuous function). The global point of view studies the linear space as a whole, and asks: What properties do all the continuous functions have in common, and can these properties be expressed in useful terms? It turns out that two continuous functions have a "distance" between them (with the same properties as the distance between real numbers), and many of the common properties of continuous functions can be expressed in terms of this distance.

This global approach to continuous functions leads to the study of other linear spaces of functions supplied with a distance or norm (a special distance). Among the linear spaces of functions are those composed of functions (called operators) that map one linear space into another. The study of these more general linear spaces and the operators defined on them yields practical and useful results including the problem that we posed earlier.

Thus, the functional analyst begins with "concrete" examples, performs an abstraction process with a global and linear point of view, to obtain a general picture of things that aid in the study of particular examples. Functional analysis has progressed rapidly and the abstraction process has been reiterated so that portions of the field no longer resemble concrete problems at all, but the usefulness is still there.

As one may suspect, the repeated abstractions lead to an immensely diverse field with many specializations and, consequently, to many outstanding individuals. As a sampling of these people and where they are located, we present the following list (which is by no means complete): Granirer, University of British Columbia; Gelfand, Naimark and M. Krein, USSR; L. Schwartz, J. Dieudonné and G. Choquel, France; L. Hörmander, Sweden; M. H. Stone, University of Massachusetts; J. T. Schwartz, New York University; S. Kahutani, Yale; and F. Browder, University of Chicago.

GENERAL RELATIVITY

Werner Israel

Einstein's theory of gravitation, after the initial burst of excitement around 1920, lay dormant for a long time because of a paucity of contacts with observation and with other branches of science. All this has changed in the last decade, with the discovery of quasars, radio galaxies, X-ray sources and pulsars, Weber's report of gravitational waves coming from the center of our galaxy, and direct observational evidence that our universe began as a highly compressed "primeval fireball". We now realize that regions of very intense gravity play a fundamental role in the universe, and that only by applying Einstein's theory in consort with nearly every other branch of theoretical physics can we hope to unravel their mysteries.

Probably the foremost relativist in the world today is Roger Penrose in London. He has made profound and highly original contributions to the mathematics of the theory. In the United States, the three leading groups are at Maryland under Charles Misner, studying the early history of the universe; the California Institute of Technology under Kip Thorne, studying gravitational fields of highly compressed bodies; and Princeton under John Wheeler, studying "black holes" and the formulation of a quantum theory of gravity. Mention must also be made of Zeldovich in Moscow, whose deep physical intuition has illuminated virtually every aspect of relativistic astrophysics.

PROBABILITY AND STATISTICS

E. S. Keeping

The mathematical theory of probability builds on intuitive notions of "chance" and "odds", derived from experience with horseracing, coin-tossing, playing with dice or cards, and so on, and formalizes these notions into a logical abstract theory. This theory in turn has many useful applications in such fields as insurance, engineering and quality control.

Mathematical statistics is based on probability and deals with the kinds of numerical inference that can be drawn from observations of the real world. Such inferences, or estimates, are never absolutely certain, but often decisions have to be made on this basis. Statistics, apart from the purely routine manipulation of data, is largely concerned with estimating the risks involved in making various types of decisions.

Among the great names in probability are A. Kolmogorov in Russia and W. Feller in the United States. The leading statistician, until his death in 1962, was Sir Ronald Fisher in England. Prominent still are H. Cramer in Sweden and J. Neyman in the USA.

THEORETICAL MECHANICS

Robert J. Tait

Mechanics seeks to describe and account for the motion of material bodies. In this discipline, one of the oldest branches of applied mathematics, the mathematician is interested in the formulation of its general principles, the development of its logical structure, and the efficiency and accuracy of its particular predictions. Along with philosophers, physicists and engineers, he is interested in deepening its foundations and widening its boundaries.

Classical mechanics, based on the "laws" of Newton, forms the foundation of much of physics, and has been the starting point of many subjects in pure mathematics. Its uses today range from designing bridges to arranging return trips to the moon.

If one applies the principles of classical mechanics to particles moving with very high speed or of very small size, one encounters conceptual and experimental difficulties. From these have stemmed the modern theories of relativity and quantum mechanics which are mathematically and physically more formal. In principle, these subjects, to which classical mechanics can be thought of as an approximation, are of universal application, and have raised questions of both a logical and computational nature.

On the other hand, if we deal with "ordinary" solids or fluids, Newtonian mechanics has been extended to formulate equations describing the behavior of such substances. We are then led to the study of continuum mechanics which includes elasticity, plasticity, fluid flow, aerodynamics, and related topics. A topic of present interest in this regard is rheology, the study of flowing materials such as lava or "Silly Putty". Among the large body of research workers in this area should be mentioned Truesdell and coworkers in North America and A. E. Green in Europe.

To indicate the mathematical subjects involved in discussions of these topics, and which have to some extent been influenced by them, one might list ordinary and partial differential equations, integral equations, calculus of variations, differential geometry, tensor calculus, probability and statistics, group theory, and topology.

New areas connected with mechanics are still opening up. To illustrate this, one can refer to S. Smale, working in differential topology, who has been examining the restricted three-body problem. As an example, consider the earth, moon, and a small satellite, and assume that all three travel in the same plane, and that the center of mass remains fixed. It is desired to find a mathematical model for this situation, one which is stable in the sense that small changes in the variables do not affect the essential character of the motion. Not all stability questions of this nature have been answered for the collection of models proposed.