Construction of conic sections

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Certain difficulties are encountered in teaching the unit on Conic Sections in the Math 30 course. Students find it hard to understand the graphical implications of the constant b, the length of the shorter semi-axis of the hyperbola or ellipse. Rather than presenting this constant theoretically as part of the proof of a theorem, I have found it useful to give students the opportunity to construct the graph using the appropriate definition. This can be followed by a derivation of the equation in standard form.

Each of the four problems is presented as a lab exercise at the beginning of the study of each of the four conic sections. Poster paper is unrolled by the students and stapled to the bulletin boards. This serves as the material to draw diagrams. Other materials required are yardsticks, string, pins and thumbtacks. For construction of the parabola, a trapezoid made of stiff cardboard or plywood is required. This should be made by the students in advance. A ring is required to construct the hyperbola. It is preferable to use one without a stone, as this tends to tear the paper. Students seem to work best in groups of two or three. If there is not enough bulletin board space in your room, several pieces of "Ten-Test" can be cut from a 4' x 8' sheet and used as a drawing surface.

- *PROBLEM 1: To graph the locus of all points which are equidistant from a fixed point, and to find an equation for the locus.*

PART 1

1. Draw an x-y coordinate system with each axis about 24" long (including positive and negative portions).
2. Tie a cord around the pin of a thumbtack and push the thumbtack into the board at the origin. Tie the pencil to the other end of the cord, so that the distance between the tack and the pencil is about 5". (Leave an extra 1" to loop the string around the pencil and tie a knot.)
3. Keeping the cord tight, draw a continuous curve with the pencil.
4. Choose any point on the locus and call it (x_2, y_2). Then, using the distance formula, write an equation for the locus you have just drawn: The distance between any point and the fixed point - in this case the origin (0, 0) - is equal to a constant of 5".

   distance formula: $D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

   Simplify the equation by squaring to remove radicals.

PART 2

1. Using the same coordinate system, locate the point (2, 3) on the graph.
2. Using the same radius as in Part 1, and fixed point (2, 3), draw the locus using the same method.
3. Using the distance formula, write an equation for the locus with fixed point or center (2, 3) and constant distance or radius of 5".

   After squaring to remove radicals, leave the equation in the form

   $(x_2 - x_1)^2 + (y_2 - y_1)^2 = r^2$, where "r" is the constant distance.

   This results in an equation called "the equation of the circle in standard form".
4. By writing the equation of the circle in standard form, what information can be obtained directly from the constants in the equation?

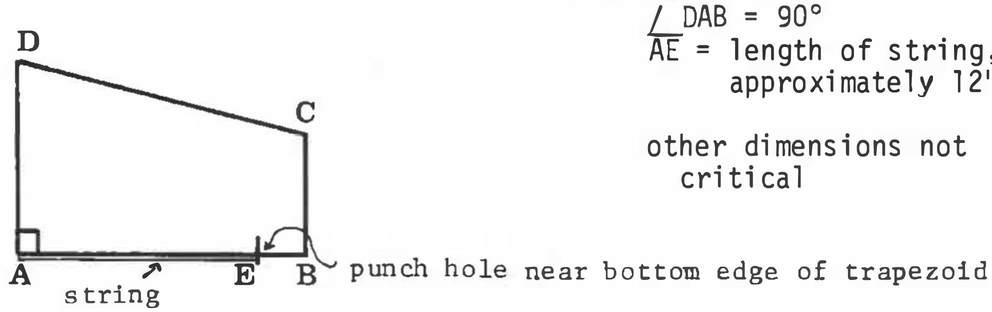
PART 3 - no construction required, but a sketch would be useful.

1. Let (h, k) be the center of a circle anywhere on the plane and let its radius be "r".
2. By methods outlined previously, derive a formula for this circle. Simplify by squaring to remove radicals. Leave the equation in standard form.
3. What are quantities required in order to write the equation of a circle in standard form?

● PROBLEM 2: *To graph the locus of all points which are equidistant from a fixed point and a fixed line, and to find an equation for the locus.*

PART 1

1. Draw an x-y coordinate system with each axis about 20" long.
2. Draw the line $x = -2$ (the fixed line) and mark the point F (2, 0) - the fixed point.
3. Cut a trapezoid out of thick cardboard to the dimensions shown and letter it as follows:



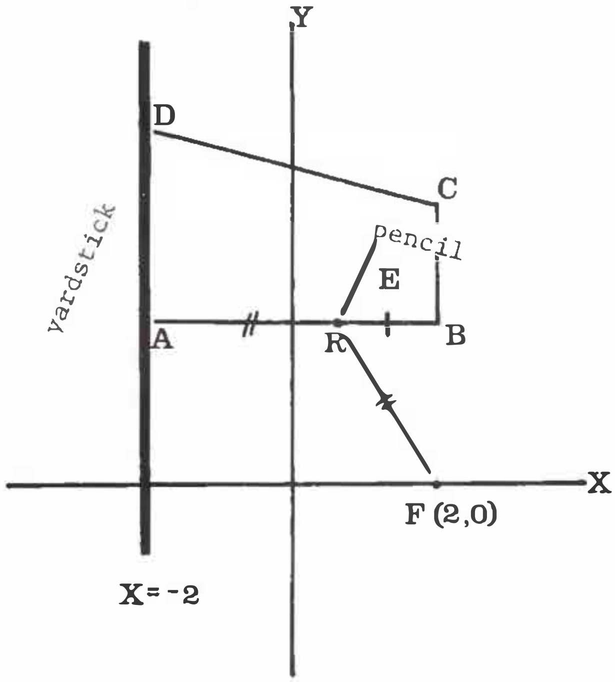
Note: AB = 15"
$\angle$DAB = 90°
$\overline{AE}$ = length of string, approximately 12"

other dimensions not critical

4. Place the trapezoid on the paper so that AD lies along the fixed line $x = -2$ and the end of the cord is fastened at point (2, 0) with a thumbtack. The cord should be left slack.

5. Using the point of your pencil, pull the cord tight along AB (do not press down on the paper during this process).

6. Keeping the cord tight, move the trapezoid downward, with AD moving along the fixed line $x = -2$. This is accomplished by pushing the pencil slightly upward and to the left and holding the straight edge along the fixed line.



Note: AE = length of cord
AE = AR + RE

Also: AE = RF + RE
AR = RF

∴ the pencil is equidistant from point F and fixed line x = -2.

7. Move the trapezoid down until AE is on the x-axis. Then start below the x-axis and move upward. *Do not remove the thumbtack.*

8. Repeat the whole procedure twice more, making the following changes:
   (a) let the fixed line be $x = -1$, and fixed point (1, 0)
   (b) let the fixed line be $x = 1$, and fixed point (-1, 0). (Flip the trapezoid over for this one.)

PART 2

You now will have drawn three different loci which obey the same kind of condition: The distances between a general point on the graph and a *fixed point* is the same as the distance between the general point and a *fixed line*– e.g., in the diagram for Part 1, 6 - RF = RA.

Note: Distance from a point to a line is always considered to be distance measured *perpendicular* to the line.

1. Choose any point other than the origin which is on the graph having fixed point (2, 0) and fixed line x = -2. Let this point be P (x, y).

2. Using the chosen point, write the equation of the locus: The point is equidistant from the fixed point and the fixed line.

3. Simplify the equation by removing the radicals. This results in the "Standard form of the equation".

4. Using a different diagram, let A be any point on the locus with coordinates (x, y). Let (p, o) be the fixed point and x = -p be the fixed line. Draw a sketch and using the method outlined above, write the standard form of the equation of the locus.

● PROBLEM 3: *To graph the locus of all points such that the sum of their distances from two fixed points is a constant, and to find an equation for the locus.*

PART 1

1. Draw a coordinate system making each axis about 24" long, including the positive and negative portions. Let 2" equal one unit.

2. Mark the points F_1 (3, 0) and F_2 (-3, 0).

3. Place a thumbtack at each point F_1 and F_2 and fix the string at these two points. The total length of the string should be 21" (20" plus 1" for the loop to hold the pencil).

4. Make a notch around the pencil about 1/2" from the point. This will be used to hold the cord in place without slipping.

5. Loop (do not tie a knot) the cord around the notch and, keeping the string tight, trace out a curve with the pencil. Start with the pencil on the positive x-axis, and move around toward the negative x-axis, finishing the graph by moving below the x-axis up to the starting point.

6. List the x and y intercepts of the graph you have drawn.

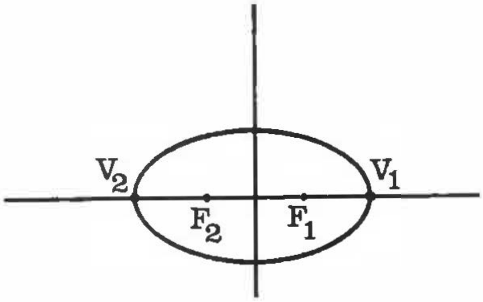
7. Repeat the above procedure, using the same length of string and

   (a) F_1 (2, 0), F_2 (-2, 0)

   (b) F_1 (0, 3), F_2 (0, -3)

PART 2

Several important line segments and points associated with the ellipse are illustrated below.



V_1 and V_2 are *vertices* of the ellipse.

F_1 and F_2 are *foci*.

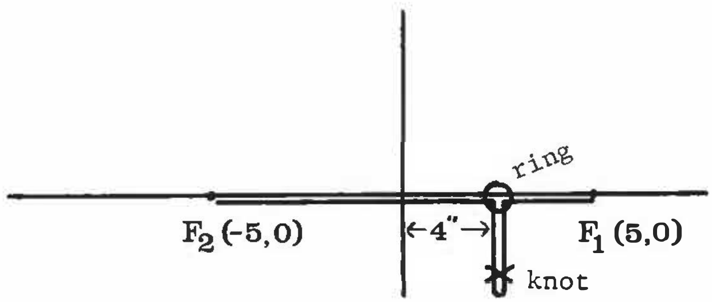
a = distance from origin to either vertex (x-intercepts in this case)
b = distance from origin to either of the remaining intercepts
c = distance from origin to either focus

1. Take any point on the ellipse P(x, y). Measure PF_1 and PF_2.
2. Measure V_2V_1. How are PF_1, PF_2 and V_2V_1 related?
3. Using the idea of locus, write a definition for the ellipse.
4. Find by measurement the relation between a, b and c.

● PROBLEM 4: *To graph the locus of all points such that the difference of their distances from two fixed points is a constant, and to find the equation of the locus.*

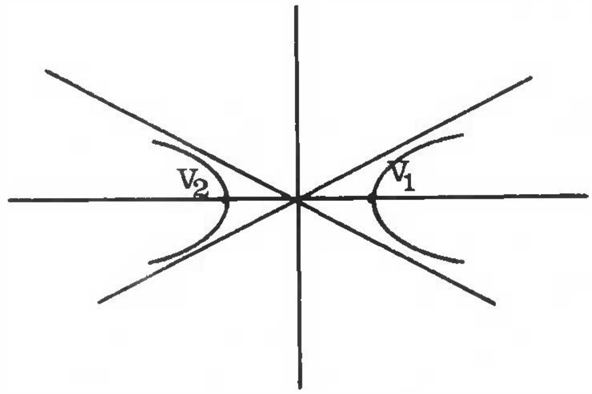
PART 1

1. Draw a coordinate system with axes about 18" long, letting 1" equal one unit. Mark the points F_1 (5, 0) and F_2 (-5, 0).
2. Using a string exactly 23" long, fasten its ends at F_1 and F_2.
3. Slip a ring over the cord and move it along the doubled portion until the string is tight along the x-axis. Tie a knot near the bottom of the doubled end.



4. Put the end of a pencil in the ring and, pressing the pencil against the upper portion of the ring, trace out a curve by moving the pencil upward while keeping the cord tight, but allowing it to slip through the ring.
5. To obtain the portion of the graph below the x-axis, swing the doubled part of the cord above the x-axis, and repeat the procedure, moving the ring downward.
6. Remove the tacks, *but keep the length of the string between the tacks unchanged.* The tack which was at F_2 is now moved to F_1, and vice versa.
7. Repeat parts 4 and 5 to obtain the second branch of the graph.

PART 2



1. Using a straight edge, draw two lines which pass through origin and approach each part of the graph *but do not touch it.* These lines are called *ASYMPTOTES* of the hyperbola.

2. At V_1 and V_2, construct perpendicular segments to cut the asymptotes in four places.

3. Join these four points to form a rectangle.

4. Find the points at which the horizontal lines of the rectangle cross the y-axis. The distance between these points of intersection and the origin is the quantity b for the hyperbola.

5. Having obtained b, find the relation between a, b, and c for the hyperbola.

   a is the distance from the origin to either vertex of the hyperbola, and c is the distance from the origin to either focus (F_1 and F_2).

6. Take any point P (x, y) on either branch of the hyperbola. Measure PF_1 and PF_2. Measure V_1V_2. How are these three quantities related?