Finding the Non-real Roots of a Quadratic Equation by Graphing

Wayne Harvey
New Trier West High School
Illinois

NOTE: Most teachers and students of mathematics would agree that the quadratic function is most easily graphed when the equation is written in the form $y = a(x-h)^2 + k$. But to find the roots of a quadratic equation, the same teachers and students use the form $ax^2 + bx + c = 0$ and the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The following theorem shows that it is not necessary to change from the first form to the second to find the roots. In fact, this theorem provides a much easier way to find the roots of a quadratic equation than the more traditional use of the quadratic formula.

Theorem: $a(x-h)^2 + k = 0 \leftrightarrow x = h \pm \sqrt{-\frac{k}{a}}$

$$a(x-h)^2 \pm k = 0$$

$$(x-h)^2 + \frac{k}{a} = 0$$

$$(x-h)^2 - \left(-\frac{k}{a}\right) = 0$$

$$(x-h)^2 - \left(\sqrt{-\frac{k}{a}}\right)^2 = 0$$

$$\left(x-h - \sqrt{-\frac{k}{a}}\right)\left(x-h + \sqrt{-\frac{k}{a}}\right) = 0$$

$$x-h - \sqrt{-\frac{k}{a}} = 0 \quad \text{or} \quad x-h + \sqrt{-\frac{k}{a}} = 0$$

$$x = h + \sqrt{-\frac{k}{a}} \quad \text{or} \quad x = h - \sqrt{-\frac{k}{a}}$$

$$x = h \pm \sqrt{-\frac{k}{a}}$$

Since these steps are reversible:

$$a(x-h)^2 + k = 0 \leftrightarrow x = h \pm \sqrt{-\frac{k}{a}}$$

I presented this proof to my advanced junior math class and challenged each of them to come up with a clever, simple way to find the solution of a quadratic equation *by graphing*, when the roots were *non-real, complex numbers*, that is, when the parabola did not intersect the x-axis. (I didn't know the answer.) The accompanying article is a response to this challenge.

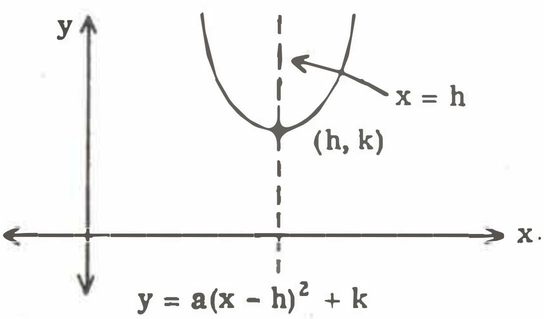
George Kelso

The graph of a quadratic equation is a parabola. If the equation is written in the form $y = a(x-h)^2 + k$, then:

1. The coordinates of the vertex are (h,k).
2. The minimum or maximum value of y is k.
3. i. If $a>0$, the parabola opens up.
   ii. If $a<0$, the parabola opens down.
4. i. If $|a| >1$, the parabola is "acute".
   ii. If $|a| <1$, the parabola is "obtuse".
5. i. The constants "a" and "k" are both positive or both negative, if and only if the parabola does not intersect the x-axis. In this case, the "discriminant" $-\frac{k}{a} <0$ and the roots are not real and can be written as
$$x = h \pm \sqrt{-\frac{k}{a}} \quad \text{or} \quad x = h \pm \sqrt{\frac{k}{a}}\,i$$
   ii. One of the constants "a" and "k" is positive and the other is negative, if and only if the parabola intersects the x-axis $(y=0)$. In this case, the "discriminant" $-\frac{k}{a} > 0$ and the roots are real and can be written as
$$x = h \pm \sqrt{-\frac{k}{a}}$$
6. The equation of the axis of symmetry is $x = h$.
7. i. If the parabola intersects the x-axis, then the real roots are the x-coordinates and both y-coordinates are 0. That is, the coordinates of the points of intersection are:
$$(x_1, y_1) = (h + \sqrt{-\frac{k}{a}}, 0) \qquad (x_2, y_2) = (h - \sqrt{-\frac{k}{a}}, 0)$$
   ii. One point is $\sqrt{-\frac{k}{a}}$ units to the left of the line of symmetry $(x=h)$, and the other is $\sqrt{-\frac{k}{a}}$ units to the right of this line.

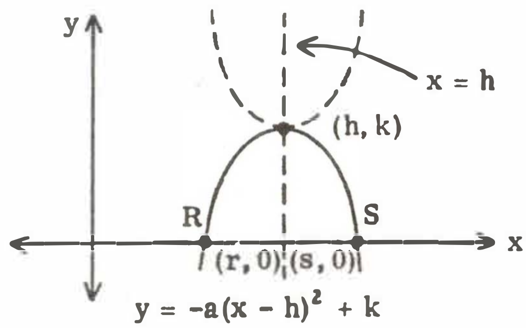
The above information enables us to approximate the roots of a quadratic equation when it is written in the form $0 = a(x-h)^2 + k$, by graphing the corresponding equation $y = a(x-h)^2 + k$, *even if the roots are non-real, complex numbers!*

Suppose we are given a parabola which does not intersect the x-axis. Then, according to 5.i. above, the roots are non-real numbers. Our problem is to find the approximate values of these roots by graphing. If the given parabola appears as it does in Figure 1, let us invert it to obtain the parabola in Figure 2.



$y = a(x-h)^2 + k$

Figure 1



$y = -a(x-h)^2 + k$

Figure 2

From the figures, it is obvious that both parabolas have the same vertex (h,k). Therefore, the values for "h" and "k" in the equations of the given parabola and the inverted parabola will be the same. And inverting a parabola merely changes the sign of the constant "a" in the corresponding equation. Therefore, if the equation of the given parabola is $y = a(x-h)^2 + k$, then the equation of the inverted parabola is $y = -a(x-h)^2 + k$.

But the inverted parabola in Figure 2 intersects the x-axis. According to 7.i. above, the real roots are:

$$x = h \pm \sqrt{\frac{k}{-a}} \qquad \text{or} \qquad x = h \pm \sqrt{\frac{k}{a}} \quad \left(\frac{k}{a} > 0\right)$$

That is, the *real* roots of the inverted parabola in Figure 2 are

$$x = h \pm \sqrt{\frac{k}{a}}$$

and the *non-real* roots of the given parabola in Figure 1 are

$$x = h \pm \sqrt{\frac{k}{a}}\,i$$

In other words, the real roots of the equation, $0 = -a(x-h)^2 + k$, which corresponds to the inverted parabola in Figure 2, provide us with the real numbers "p" and "q" of the non-real numbers "p + qi" and "p - qi", where p = h and $q = \sqrt{\frac{k}{a}}$.

How can "h" and "$\sqrt{\frac{k}{a}}$" be determined from the graph? Suppose that the inverted parabola in Figure 2 intersects the x-axis at points R and S, where the coordinates of R are (r, 0) and the coordinates of S are (s, 0),

then $s + r = \left(h + \sqrt{\frac{k}{a}}\right) + \left(h - \sqrt{\frac{k}{a}}\right) = 2h$

or $h = \frac{s + r}{2}$

and $s - r = \left(h + \sqrt{\frac{k}{a}}\right) - \left(h - \sqrt{\frac{k}{a}}\right) = 2\sqrt{\frac{k}{a}}$

or $\sqrt{\frac{k}{a}} = \frac{s - r}{2}$.

The non-real, complex roots $p \pm qi$ can, therefore, be found from the graph of the inverted parabola and expressed in the form

$$\frac{s + r}{2} \pm \frac{s - r}{2}\,i, \text{ where } p = \frac{r + s}{2} \text{ and } q = \frac{r - s}{2}.$$

To illustrate this method, supposing a student is given the graph in Figure 3 and asked to approximate the roots of the corresponding quadratic equation (where y =0). To do so, he merely does three things:

1. invert the parabola
2. determine the x-coordinates of the points where the inverted parabola intersects the x-axis
3. substitute for r and s in the expression $\frac{s + r}{2} \pm \frac{s - r}{2}\,i$

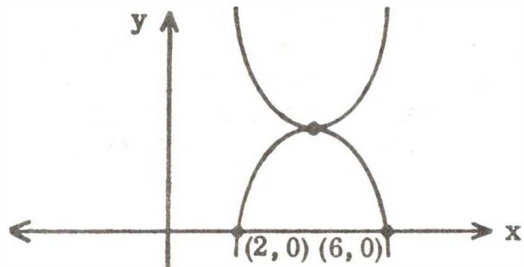


Figure 3

Solution: From the graph, $r = 2$ and $s = 6$.

Therefore, the non-real roots of the equation that correspond to the given parabola (where $y = 0$) are:

$$x = \frac{6 + 2}{2} \pm \frac{6 - 2}{2} i \qquad \text{or} \qquad x = 4 \pm 2i.$$

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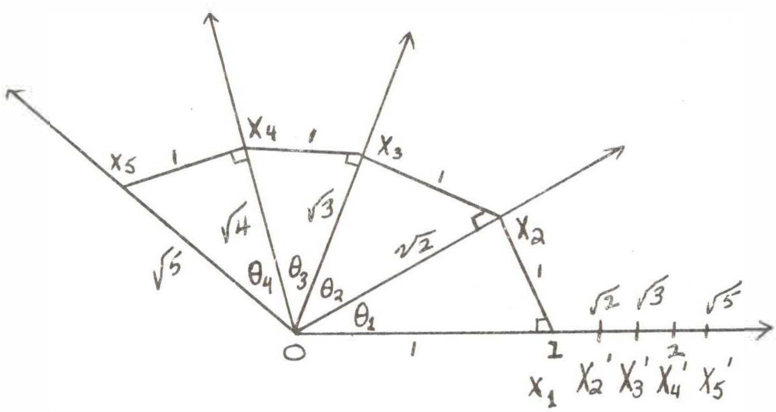
A Problem



Most teachers will be familiar with this diagram in which segments OX_i are constructed with lengths equal to $\sqrt{2}$, $\sqrt{3}$, and so on. The problem is to determine how many repetitions of the construction are necessary so that

$$\theta_1 + \theta_2 + \theta_3 + \ldots + \theta_i \geq 360°$$