# A Computer Application in Trigonometry

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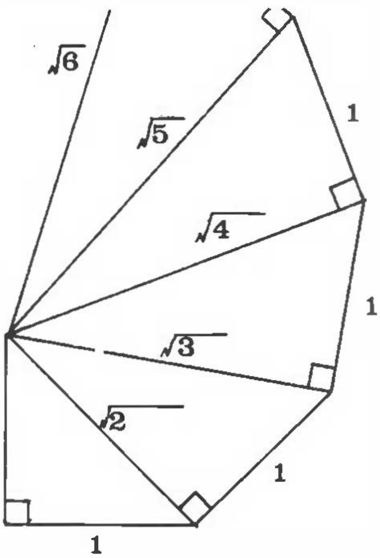
Editor's note: The following article was submitted in response to "A Problem", which was published in *Delta-K*, Volume XI, Number 4, August 1972, p.8. Dr. Bruce is a member of the Department of Mathematics at the University of Alberta.

We consider the compass and straight-edge construction of the square roots of the natural numbers, a result that leads to a spiral effect as shown.




Starting with a right triangle with unit legs, we obtain a measure of $\sqrt{2}$ units for the hypotenuse. With this hypotenuse as base, another right triangle with one unit leg can be constructed. Its hypotenuse measures $\sqrt{3}$ units. The process can be repeated to produce segments that measure $\sqrt{n}$, where n is any natural number. Incidentally, the area of each triangular region so formed is given by $\frac{1}{2}\sqrt{n}$.

It is intuitively obvious that eventually the triangular regions will overlap, but when does this first occur? Careful drawing makes it apparent that the segment representing $\sqrt{18}$ is the first one to intersect the original triangle. However, it is rather close, so we need to examine the problem more closely to be sure.

When the sum of the central angles of the triangles is first greater than 360° (or $2\pi$) we shall know that the overlap has occurred. The measures of these angles are given by Arccot $\sqrt{n}$, where n is a natural number and principal values are indicated. Clearly, we need to compute

$$\sum_{n=1}^{k} \text{Arccot } \sqrt{n}$$

to find the necessary integral value of "k" for the first overlap.

Whether we use one of the new pocket computers, or otherwise, we find that Arccot $\sqrt{n}$ is not programmed. However, Arctan $\sqrt{n}$ is programmed. Now

$$\sum_{n=1}^{k} \text{Arccot } \sqrt{n} = k(\pi/2) - \sum_{n=1}^{k} \text{Arctan } \sqrt{n}.$$

The computer print-out gives us the following results in radian measure for n=1 to n=17:

| | | | | | |
|---|---|---|---|---|---|
| n=1 | 0.7853981634 | n=7 | 3.557622403 | n=13 | 5.369914672 |
| n=2 | 1.400877872 | n=8 | 3.897459312 | n=14 | 5.631072083 |
| n=3 | 1.924476648 | n=9 | 4.219209866 | n=15 | 5.883752338 |
| n=4 | 2.388124257 | n=10 | 4.525487236 | n=16 | 6.128731001 |
| n=5 | 2.808658592 | n=11 | 4.818330007 | n=17 | 6.366672126 |
| n=6 | 3.196255279 | n=12 | 5.099364909 | | |

The last entry is the first one to exceed $2\pi$ (approx. 6.2832). Hence the triangle with base $\sqrt{17}$ is the first one to overlap the first triangle. The line segment of measure $\sqrt{18}$ intersects the first triangle. In degree measure, the computer yields

n=16 351.15°
n=17 364.78°.

The pocket computer yields $\sum_{n=1}^{k}$ Arctan $\sqrt{n}$ in degree measure. This has to be subtracted from k(90°) to get the above results.

Since the central angles are continually getting smaller, it is obvious that considerably more terms of the series will need to be summed to find when the second overlap will occur. Try to make a conjecture and test it on a computer.

A diller, a dollar,
A witless trig scholar
On a ladder against a wall.
If length over height
Gives an angle too slight,
The cosecant may prove his downfall.

*L.A. Graham*

"Math teachers never die; they just reduce to lowest terms."