Positivizing Operations

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During a recent lesson on absolute value equalities with my Mathematics 20 class, we were graphing the relation $|x| + |y| = 6$. The graph is a square, intersecting the axes at ± 6 as in Figure 1, below. One student asked how, when the intercepts were found, we knew it wasn't a circle. We had been graphing circles and other conics the few days previously, and she was surprised the graph was not a circle. So I sketched Figure 2 on the board, and asked her for the equation of the circle. She responded correctly with $x^2 + y^2 = 36$. I wrote the 36 as 6^2.

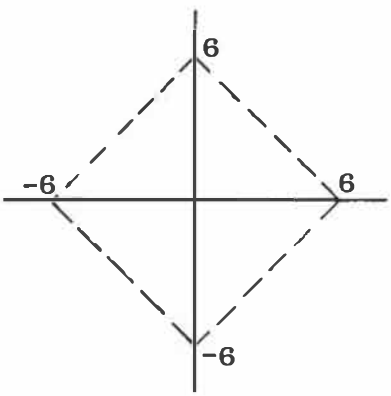

Figure 1: $|x| + |y| = 6$

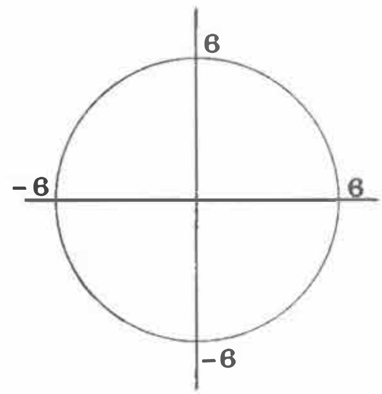

Figure 2: $x^2 + y^2 = 6^2$

Then the students were asked to look at the two relations carefully. Most of them were puzzled as to just what to look for, so I wrote $y = x^2$ and $y = |x|$ on the board. These were familiar to them, and they were asked to sketch the two side-by-side. Figures 3 and 4 were sketched by most of the students.

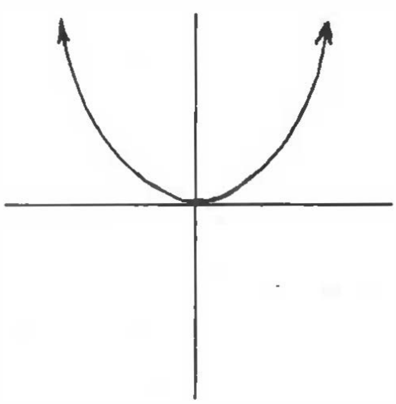

Figure 3: $y = x^2$

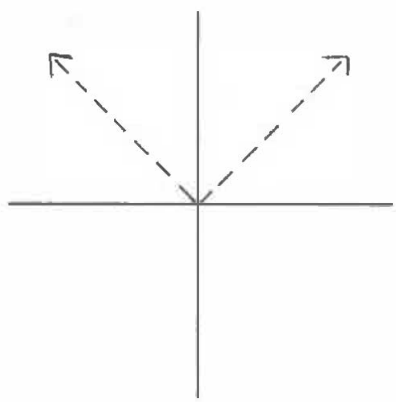

Figure 4: $y = |x|$

Someone suggested that "the curves are always the same, except one is straight". We tried the pair of equations $\frac{x^2}{4} + \frac{y^2}{1} = 1$, and $\left|\frac{x}{2}\right| + \left|\frac{y}{1}\right| = 1$, which are sketched below.

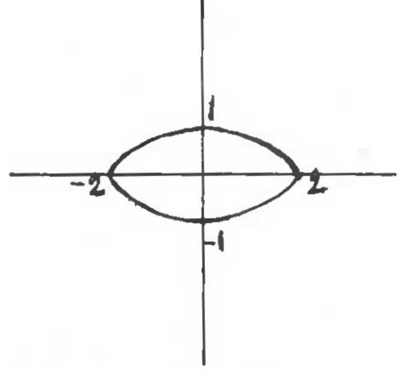

Figure 5: $\frac{x^2}{4} + \frac{y^2}{1} = 1$

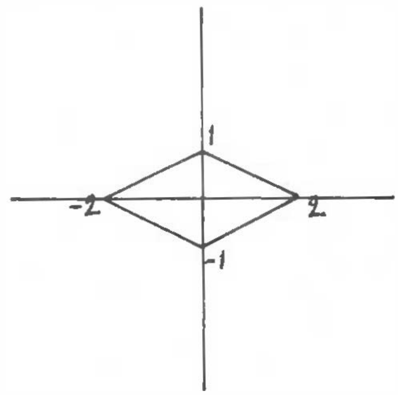

Figure 6: $\left|\frac{x}{2}\right| + \left|\frac{y}{1}\right| = 1$

They were familiar with hyperbolas too, so we tried $x^2 - y^2 = 1$ and $|x| - |y| = 1$, which yielded the graphs in Figures 7 and 8.

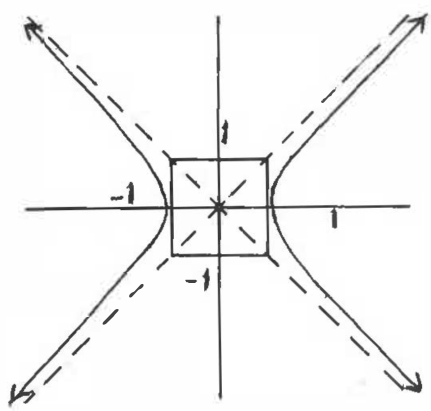

Figure 7: $x^2 - y^2 = 1$

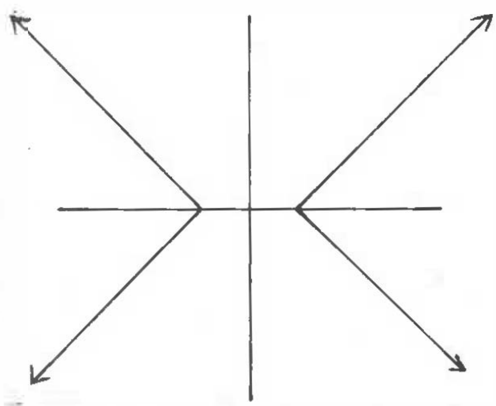

Figure 8: $|x| - |y| = 1$

By now, practically everyone had the idea, and many of the students were wondering "Why?". I heard, "Hey, that's neat!" from the back of the room.

There were only a very few minutes left in the period for an explanation, so it had to be brief. It centered around the fact that the operations of squaring and of taking the absolute value are "Positivizing" operations - that is, they always yield a positive number, whether the operand is positive or negative. Hence the relations behave similarly, but with the difference that one kind produces curves that are curved, the other produces curves which are straight.

For homework, I suggested that they test the hypothesis by comparing $y = (x - 1)^2$ with $y = |x - 1|$; $y = x^2 - 5$ with $y = |x| - 5$; $y = -2(x - 3)^2 + 4$ with $y = -2|x - 3| + 4$; and $x = y^2$ with $x = |y|$. Why don't you try these examples, too?

Also, can you come up with a better explanation than mine?