common fractions, should we, as mathematics teachers, make proposals to the stock exchanges, the home economists, and other groups?

There is no longer any doubt *whether* Canada will move to the metric system—the only question is *when*. It is time for us to look, not merely at teaching the metric system, but at the effects this move will have on teaching *all* mathematics.

Stretch Your Sketching Skill

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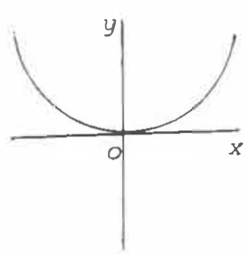
A number of elementary techniques are available for sketching graphs but, to our knowledge, they have not appeared in any definitive fashion in textbooks. This presentation is an attempt to ameliorate that situation.

The rules of thumb which follow are non-rigorous suggestions that usually assist the student in determining the nature of the graph.

1. DEGREE: The degree of the equation suggests the number of "bends" the resulting curve has. The degree of the equation is determined by the highest power of the independent variable (i.e., x). Therefore, in $y=x^2$ (degree two) there is one bend and in $y=x^3+3x$ (degree three) there are two bends, but in $y=2x - 3$ (first degree) there are no bends. It appears that the number of twists in the curve is one less than the highest power of the independent variable, that is, if $f(x)=ax^n + bx^{n-1} + ex^{n-2} + \ldots + wx$, then we should expect that there should be n-1 bends. Of course, if f(x) equals x to the three quarters, we cannot use the rule, so perhaps we should restrict the domain of n to the set of positive integers.

2. NUMERICAL COEFFICIENTS: The numerical coefficient of the highest power of the independent variable also helps determine the complexion of the graph. In the quadratic function we have a "smile" in the sketch of

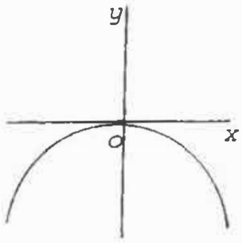
$y = x^2$



positive coefficient
concave upwards

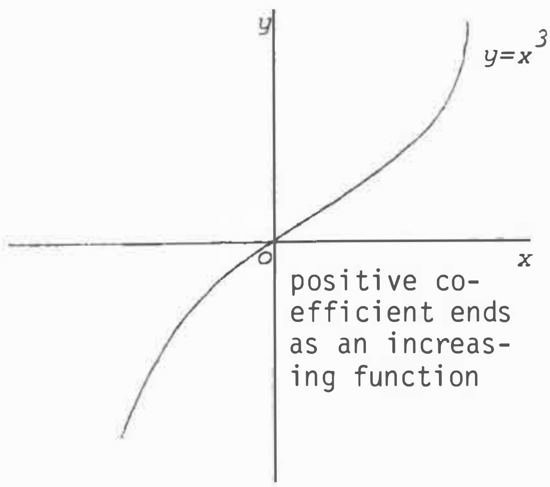
or a "frown" in

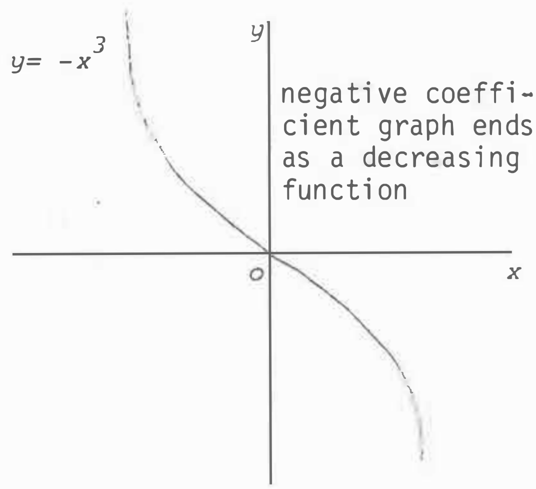
$y = -x^2$

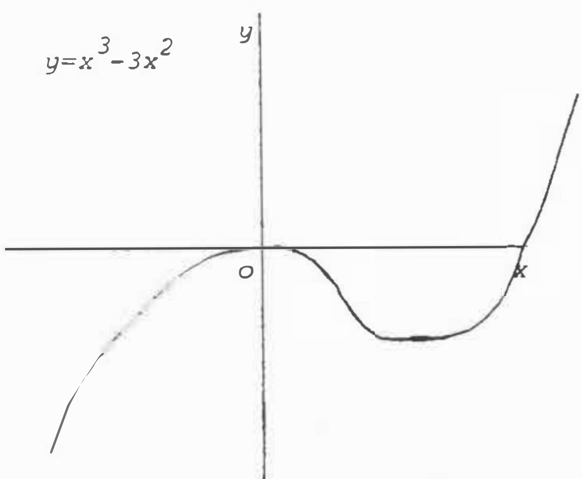


negative coefficient
concave downwards

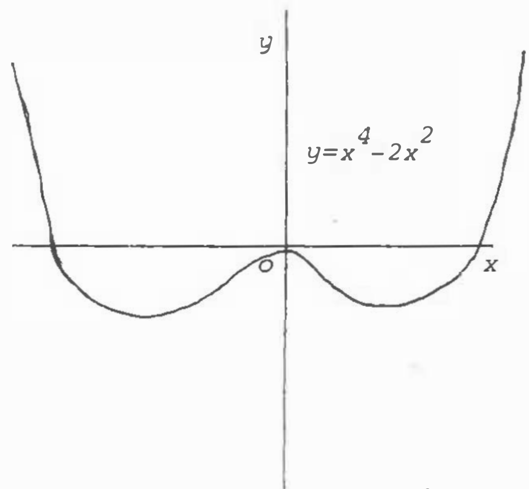
The use of 'smile' or 'frown' to indicate concave upwards and concave downwards is a useful memory device for the beginning student of the quadratic. But wait, there is more! When the coefficient of x is positive the graph ending is always increasing. We note that in $y = x^2$ the graph ends as an increasing function but in $y = -x^2$ the function ends as a decreasing function. The following examples are illustrative of this generalization.







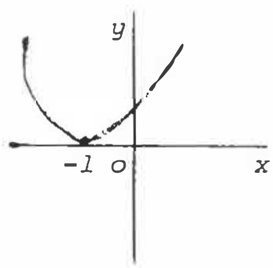
positive coefficient of x^3 means graph ends as increasing function



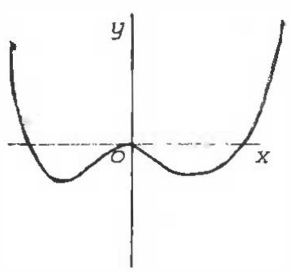
positive coefficient of x^4 means graph ends as increasing function

It is interesting and perhaps instructive to note also that, in $y=x^3$ we have an initially increasing graph, while in $y = -x^3$ we have an initially decreasing graph. The odd power and the positive coefficient combine to produce a graph whose genesis is always increasing, while the odd power and the negative coefficient produce an initially decreasing graph. By extrapolation, the even power and the positive coefficient initiate a decreasing graph while the increasing mode is born of an even power and a negative coefficient.

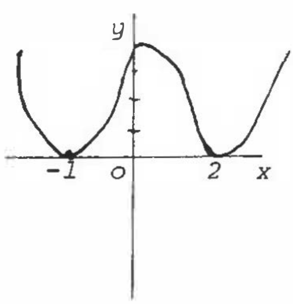
3. SQUARED FACTOR: If f(x) can be expressed as a factor squared, that is, if $f(x) = (x-a)^2$, the graph will be tangent to the x-axis at x=a. Consider the following examples.



$y=(x+1)^2$ the graph touches the x-axis at x= -1



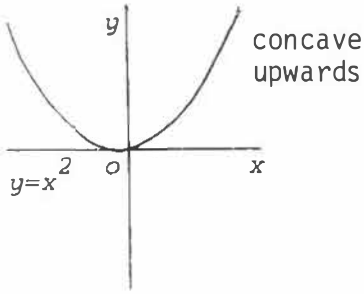
$y=x^4 -2x^2 = x^2(x - 2)$
since x is squared,
at x=0 the graph touches the x-axis

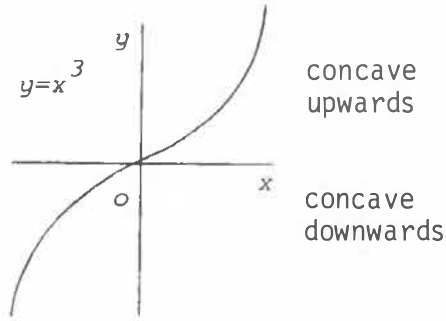


$y=(x-2)^2\ (x+1)^2$
graph is tangent to x-axis
at x=2
x= -1

4. INFLECTION POINTS: In order to properly sketch a function, it is essential to know the inflection points. An inflection point is a point on the graph

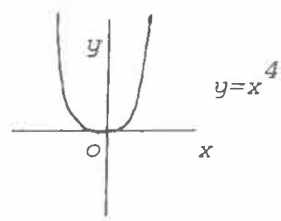
at which the concavity of the graph changes. Since $y=x^2$ is concave upward for $x \in R$ but $y=x^3$ has one inflection point at $x=0$,





seemingly the number of inflection points in $f(x) = ax^n + bx^{n-1} + cx^{n-3} + \ldots + wx$ would be $(n - 2)$ (two less than the highest power of the independent variable). If this is so, then let us consider $f(x) = x^4$. We would predict that it should have two inflection points. Before we propose a solution to this problem, let us consider an analogous but simpler problem, vis., the evaluation of the x-intercepts. To find these we solve $x^4 = 0$. Gauss showed that there are four roots to this equation, albeit they are all the same but there are still FOUR roots. This means that there are four x-intercepts all at the same point $x = 0$. Hence, when we solve $f''(x) = 0$ to find the inflection points we get $f'(x) = 4x^3$ and $f''(x) = 12x^2$. Therefore, if $f''(x) = 0$

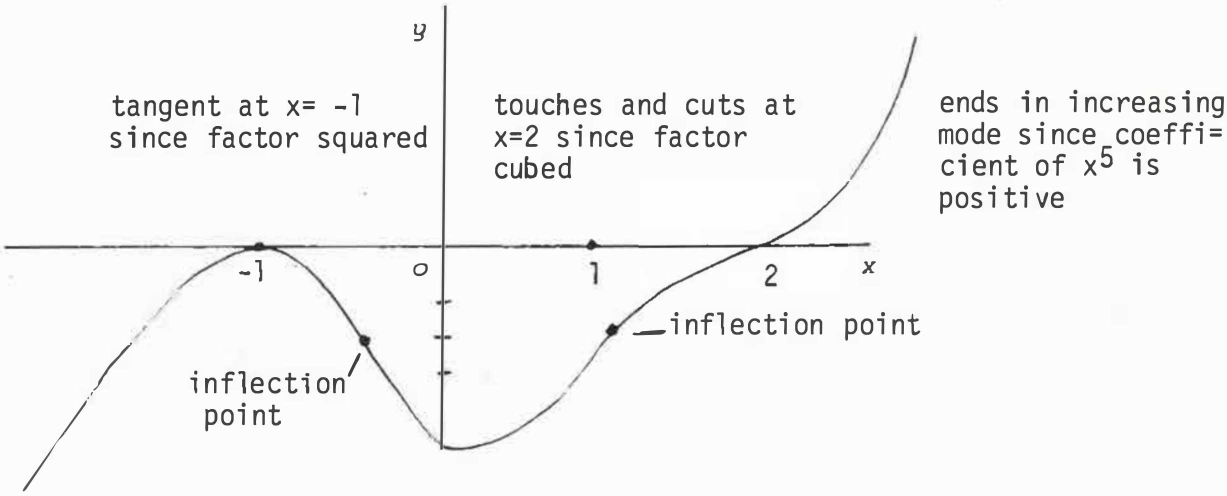
$$12x^2 = 0$$
$$\text{and } x^2 = 0$$
$$\text{so } x = 0 \text{ or } x = 0$$



Hence we have two inflection points at the origin. The inflection points are coincidental just as the roots are. This explanation for the phenomenon displayed by $f(x) = x^4$ is intuitively more satisfying to students than the explanation that this function just breaks the rules thus established. Further, this explanation offers a consistent theoretical position not heretofore deductible.

5. CUBED FACTOR: If $f(x)$ can be expressed as a factor cubed, that is, if $f(x) = (x-a)^3$, the graph will be tangent to the x-axis at $x=a$ AND the graph will cut the x-axis at $x=a$. For example, consider the curve of $y=x^3$. Here we have a touch and a cut at $x=0$.

In summary we should like to consider the complex function, $f(x) = (x - 2)^3 (x + 1)^2$ and observe how these generalizations are useful in determining its graph.



In graphing this function we can make use of more advanced techniques of calculus to determine the critical values of x. Setting the differential coefficient equal to zero, that is $f'(x)=0$ gives us $2(x-2)^3\,(x-1) + 3(x-1)^2\,(x-2)^2 = 0$, which yields $x = 2$ or $x = -1$ or $x = 1/5$, and these are the x values for which the slope of the tangent line is zero. This gives us a relative maximum at $x=-1$, a relative minimum at $x=1/5$ but neither of these at $x=2$. The geometrical interpretation of the function at $x=2$ is that the function is horizontal at $x=2$. This means that, if we were to describe the x values for which the function were increasing, they would be $\{x/\ x<-1,\ x > 1/5,\ x \neq 2,\ x \in R\}$ since at $x=2$ the function is neither increasing nor decreasing.

degree	number of bends	number of inf.pts if $n\geq 2$	POSITIVE COEFFICIENT begins	ends	NEGATIVE COEFFICIENT begins	ends
1	0					
2	1	0				
3	2	1				
4	3	2				
5	4	3				
⋮	⋮	⋮				