CANADIAN MATHEMATICAL CONGRESS
1974 Alberta High School Prize Examination

N O T E

We were asked by the Department of Mathematics, University of Alberta, to publish the following CORRECTION TO PART I of the Alberta High School Prize Examination in Mathematics:

The answer "Key" to Part I of the above exam is to be amended to read 1. = "C" and 5. = "E".

The Mathematics Council congratulates the following for outstanding achievement in the 1974 Alberta High School Prize Examination in Mathematics:

Place	Name	Grade	School
1.	Paranjape, Manu B.	12	Strathcona Composite High School, Edmonton, Alberta
2.	Whitney, Arthur T.	12	Strathcona Composite High School, Edmonton, Alberta
3.	Graham, William R.	11	Harry Ainlay Composite High School, Edmonton, Alberta
4.	Smith, Robert J.	12	Frank Maddock High School, Drayton Valley, Alberta
5.	Mallet-Paret, Louis C.	11	Archbishop MacDonald High School, Edmonton, Alberta
6.	Romanycia, Marc H.	12	Ecole J.H. Picard, Edmonton, Alberta
7.	Gretton, Jeremy S.	12	Western Canada High School, Calgary, Alberta
8.	Reid, Douglas R.	12	Bonnie Doon Composite High School, Edmonton, Alberta
9.	Davenport, Michael R.	12	Lord Beaverbrook High School, Calgary, Alberta
10.	Stochinsky, Tamin	12	Jasper Place Composite High School, Edmonton, Alberta
11.	Elliott, John F.	12	Camrose Composite High School, Camrose, Alberta
12.	DeGroot, E. Thomas	12	Edmonton Christian High School, Edmonton, Alberta

13.(A)	Sande, Gregory J.	12	St. Francis Xavier High School, Edmonton, Alberta
13.(B)	Hwang, Helena	12	Bishop Grandin High (see below)
14.	Ady, Michael S.	12	Bishop Grandin High School Calgary, Alberta
15.	Steffler, Peter	12	Louis St. Laurent High School, Edmonton, Alberta
16.	Grant, Donald B.	12	Ernest Manning High School, Calgary, Alberta

We wish to extend our best wishes to all who participated and our heartfelt desire that they will continue to achieve and to contribute to society the results of their achievements.

As a service to the teachers of participating students, and for the students and the coordinating personnel, we include the solutions to Part II of the examination which were not available for distribution with Part I at the time of mailing the results.

Solutions to Part II

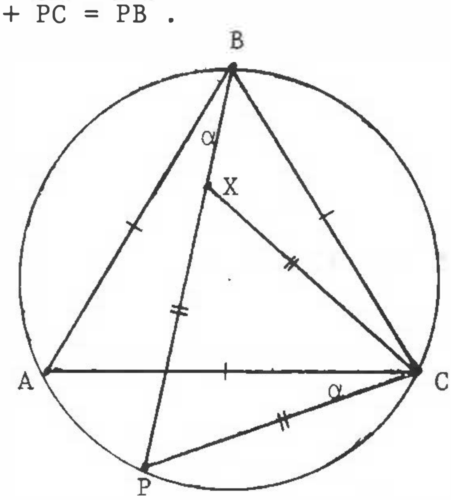
1. Suppose we divide the set of numbers $A = \{1,2,3,4,5\}$ into two sets C and D $(C \cup D = A)$. Prove that at least one of the sets C, D must contain two numbers and their difference.

   Solution: Let's try to show the conclusion need not be true. Put the number 1 into set C. Then the number 2 must go into D, since if $\{1,2\} \subset C$ then $1,2$ and $2-1$ would be in C. Then the number 4 must go into C, otherwise $2,4$ and $4-2$ would be in D. So far, $\{1,4\} \subset C$, $\{2\} \subset D$. The number 3 cannot go in C, since then, $4,3$ and $4-3$ would be in C. So $\{1,4\} \subset C$, $\{2,3\} \subset D$. No matter where you put 5, the conclusion follows.

2. Is the product of any n consecutive natural numbers divisible by $n!$ (with remainder 0)? Prove or give a counterexample.

Solution: Yes $(k+1)(k+2)\ldots(k+n) = \binom{n+k}{k} n!$, where $\binom{n+k}{k}$ is the binomial coefficient, which is always a natural number for n and k natural numbers.

3. Given an equilateral triangle ABC inscribed in a circle, let P be an arbitrary point on the arc AC . Show that $\overline{PA} + \overline{PC} = \overline{PB}$.



Solution: ABC is equilateral and P is an arbitrary point on AC . Prove $\overline{PA} + \overline{PC} = \overline{PB}$. Choose a point X on $\overline{PB}$ such that $|\overline{PX}| = |\overline{PC}|$. We will show that ΔCXB is congruent to ΔCPA , so that $|\overline{AP}| = |\overline{BX}|$ which in turn implies $|\overline{PB}| = |\overline{PX}| + |\overline{BX}| = |\overline{PC}| + |\overline{PA}|$. First $m(\angle BAC) = m(\angle BPC) = 60°$. Since ΔPXC is isosceles, $m(\angle PXC) = m(\angle XCP) = 60°$ and thus $|\overline{XC}| = |\overline{PC}|$. However $|\overline{BC}| = |\overline{AC}|$. Also $m(\angle XCA) = 60 - m(\angle ACP)$ so that $m(\angle BCP) = 60 - m(\angle XCA) = m(\angle ACP)$. Thus ΔACP is congruent to ΔBCX , and $|\overline{PB}| = |\overline{PC}| + |\overline{PA}|$.

4. Prove that $\sum_{k=2}^{n} \frac{1}{k^2-1} = \frac{(3n+2)(n-1)}{4n(n+1)}$ for $n \geq 2$.

Solution: The proposition can be shown by induction. If $n = 2$, $\frac{1}{2^2-1} = \frac{1}{3}$ and $\frac{(3\cdot 2+2)(2-1)}{4\ 2(2+1)} = \frac{1}{3}$. Suppose now that $\sum_{k=2}^{n} \frac{1}{k^2-1} = \frac{(3n+2)(n-1)}{4n(n+1)}$ where n is an arbitrary integer ≥ 2 . Then

$$\sum_{k=2}^{n+1} \frac{1}{k^2-1} = \sum_{k=2}^{n} \frac{1}{k^2-1} + \frac{1}{(n+1)^2-1} = \frac{(3n+2)(n-1)}{4n(n+1)} + \frac{1}{(n+2)n}$$

$$= \frac{1}{n} \cdot \frac{(3n+2)(n-1)(n+2) + 4(n+1)}{4(n+1)(n+2)}$$

$$= \frac{1}{n} \cdot \frac{(3n+5)\ n^2}{4(n+1)(n+2)} = \frac{(3(n+1)+2)((n+1)-1)}{4(n+1)((n+1)+1)}$$

which proves the formula.

Alternate Proof:

$$\sum_{k=2}^{n} \frac{1}{k^2-1} = \sum_{k=2}^{n} \left(\frac{1/2}{k-1} - \frac{1/2}{k+1}\right)$$

$$= \frac{1}{2}\left(\sum_{k=2}^{n} \left(\frac{1}{k-1}\right) - \sum_{k=2}^{n} \left(\frac{1}{k+1}\right)\right) = \frac{1}{2}\left(\sum_{k=2}^{n} \left(\frac{1}{k-1}\right) - \sum_{k=4}^{n+2} \left(\frac{1}{k-1}\right)\right)$$

$$= \frac{1}{2}\left(\frac{1}{2-1} + \frac{1}{3-1} - \frac{1}{(n+1)-1} - \frac{1}{(n+2)-1}\right) = \frac{(3n+2)(n-1)}{4n(n+1)} .$$

5. Which is bigger, $(\sqrt{5})^{\sqrt{3}}$, $(\sqrt{3})^{\sqrt{5}}$? Give a complete proof.

Solution: If we raise $(\sqrt{5})^{\sqrt{3}}$ to the $2\sqrt{3}$ power we get

$$[(\sqrt{5})^{\sqrt{3}}]^{2\sqrt{3}} = (\sqrt{5})^{6} = 5^{3} = 125 .$$

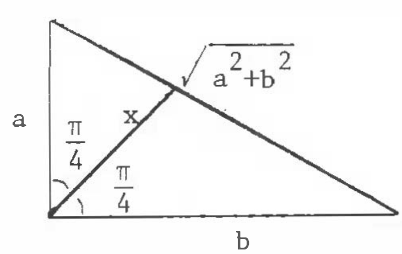
If we raise $(\sqrt{3})^{\sqrt{5}}$ to the $2\sqrt{3}$ power we get

$$[(\sqrt{3})^{\sqrt{5}}]^{2\sqrt{3}} = 3^{\sqrt{15}} < 3^{\sqrt{16}} = 3^{4} = 81 .$$

$$\therefore \quad (\sqrt{5})^{\sqrt{3}} > (\sqrt{3})^{\sqrt{5}}$$

6. A right triangle has legs of length a,b . Prove that the bisector of the right angle has length $\frac{ab\sqrt{2}}{a+b}$.

Solution:



$$\sqrt{a^2+b^2} = \{x^2 + a^2 - 2ax\cos\frac{\pi}{4}\}^{1/2} + \{x^2 + b^2 - 2bx\cos\frac{\pi}{4}\}^{1/2}$$

(law of cosines twice)

$$a^2 + b^2 = 2x^2 + a^2 + b^2 - \sqrt{2}\,x(a+b)$$

$$+ 2\{x^2 + a^2 - \sqrt{2}\,ax\}^{1/2}\{x^2 + b^2 - \sqrt{2}\,bx\}^{1/2} - 2x^2 + \sqrt{2}\,x(a+b)$$

$$= 2\{x^2 + a^2 - \sqrt{2}\,ax\}^{1/2}\{x^2 + b^2 - \sqrt{2}\,bx\}^{1/2}$$

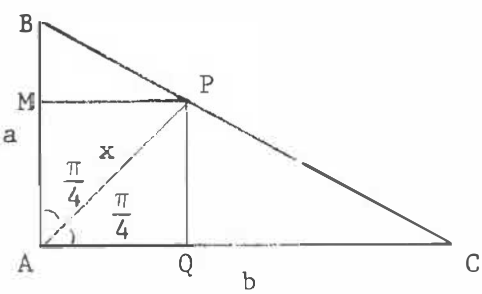
$$4x^4 - 4\sqrt{2}\,x^3(a+b) + 2x^2(a+b)^2 = 4\{x^2 + a^2 - \sqrt{2}\,ax\}\{x^2 + b^2 - \sqrt{2}\,bx\}$$

$$0 = 2x^2(a^2+b^2) + 4a^2b^2 - 4\sqrt{2}\,a^2bx - 4\sqrt{2}\,b^2\,ax + 4ab\,x^2$$

$$0 = (a+b)^2x^2 - 2\sqrt{2}\,ab(a+b)x + 2a^2b^2$$

$$x = \frac{2\sqrt{2}\,ab(a+b) \pm \{8a^2b^2(a+b)^2 - 8a^2b^2(a+b)^2\}^{1/2}}{2(a+b)^2} = \frac{\sqrt{2}\,ab}{a+b} .$$

Alternate Proof:



Draw line MP parallel to AC .

Draw line PQ parallel to BA .

Triangle MBP is similar to triangle QPC (all angles equal). Therefore,

$$\frac{\overline{MP}}{\overline{MB}} = \frac{QC}{QP} \quad , \quad \frac{x \sin \frac{\pi}{4}}{a - x \cos \frac{\pi}{4}} = \frac{b - x \cos \frac{\pi}{4}}{x \sin \frac{\pi}{4}}$$

$$\frac{x}{a - \frac{x}{\sqrt{2}}} = \frac{b - \frac{x}{\sqrt{2}}}{\frac{x}{\sqrt{2}}} \quad , \quad \frac{x^2}{2} = (b - \frac{x}{\sqrt{2}})\ (a - \frac{x}{\sqrt{2}})$$

$$0 = ba - \frac{(a+b)}{\sqrt{2}} x \quad , \quad x = \frac{\sqrt{2}\ ab}{a+b} \ .$$

7. At a party some people shake hands and some do not. Show that there must exist at least two people who shake hands with the same number of people.

Solution: If there are n people who shake hands, then the maximum number of possible handshakes for anyone is $n-1$ and the minimum number is 1 (since one does not shake hands with oneself). It follows therefore that there are only $n-1$ choices for the number of possible handshakes for anyone and hence two people (at least) must shake hands the same number of times.

8. Let $n-1$, n , $n+1$ be three consecutive natural numbers. Prove that the cube of the largest cannot equal the sum of the cubes of the other two.

Solution: Suppose, on the contrary, that we have $(n-1)^3 + n^3 = (n+1)^3$ for some natural number n . Then simplifying we get the equation $n^3 - 6n^2 - 2 = 0$, which implies $n^2(n-6) = 2$. Therefore the left hand side must be positive so $n > 6$ but also n^2 must divide 2, a contradiction. This proves the result.

9. How many natural numbers of n digits exist such that each digit is 1, 2 or 3 ? How many of these numbers use all three of the digits 1, 2 and 3 ?

Solution: 3^n .

$3^n - K$, where K is the number of such numbers that lack at least one of 1, 2, 3 . $K = M + N$

where M = numbers using only a single digit from $\{1,2,3\}$,

N = numbers using exactly two digits from $\{1,2,3\}$.

$M = 3$. $N = \binom{3}{2} \{2^n-2\}$. The $\binom{3}{2}$ represents choosing the two numbers to be used, the 2^n-2 represents all possible numbers of n digits formed from the two, less the pair of numbers that use exactly one digit. Thus the desired number is $3^n - 3 - \binom{3}{2} [2^n-2]$.

10. Let $x_1, x_2, \ldots, x_n$ be positive angles, with $\sum_{j=1}^{n} x_j \leq \frac{\pi}{2}$ (in degrees, $\sum_{j=1}^{n} x_j \leq 90°$). Show that

$$\sin (x_1+x_2+\ldots+x_n) \leq \sin x_1 + \sin x_2 + \ldots + \sin x_n .$$

Solution: The proof will be by induction on n . It is clear for $n = 1$. Suppose $\sin (x_1 + \ldots + x_n) \leq \sin x_1 + \ldots + \sin x_n$ for any positive angles $x_1, \ldots, x_n$ satisfying $\sum_{i=1}^{n} x_i \leq \frac{\pi}{2}$. If $x_1, \ldots, x_{n+1}$ are positive angles with $\sum_{i=1}^{n+1} x_i \leq \frac{\pi}{2}$ then $\sin ((x_1 + \ldots + x_n) + x_{n+1}) = \sin (x_1 + \ldots + x_n) \cdot \cos x_{n+1} + \sin (x_{n+1}) \cos(x_1 + \ldots + x_n)$. However $0 < \cos \alpha < 1$ for $0 < \alpha < \frac{\pi}{2}$ so that $\sin(x_1 + \ldots + x_{n+1}) \leq \sin (x_1 + \ldots + x_n) + \sin x_{n+1} \leq \sin x_1 + \ldots + \sin x_n + \sin x_{n+1}$.