1975 Alberta High School Prize Exams - Winners and Solutions

Following is a list of the winners and their prizes in the 1975 Alberta High School Prize Examination in Mathematics. Congratulations to the finalists and our commendation to all who participated.

PROVINCIAL PRIZES

Student	Amount	School
[1]Terry Wu	\$400	Lindsay Thurber High School, Red Deer
[2]Norman C. Hutchinson	\$400	Brooks High School, Brooks
[3]Donald J. Reble	\$200	Concordia College, Edmonton
[4]William R. Graham	\$200	Harry Ainlay High School, Edmonton

SPECIAL PRIZES

[5]Malcolm W. Kern	\$100	W.R. Myers High School, Taber
[6]Keith Fenske	\$100	Harry Ainlay High School, Edmonton

DISTRICT PRIZES (\$50 EACH)

District Number	Student/School
1	(none awarded)
2	Shawn R. Golby, Lorne Jenken High School, Barrhead
3	Larry I. Tennis, Wetaskiwin High School, Wetaskiwin
4	Leo B. Hartman, Camrose Composite High School, Camrose
5	Daniel A. Boulet, Olds High School, Olds
6	Emil L. Hallin, Crescent Heights High School, Medicine Hat
7 (1)	Raymond Kwan, Bonnie Doon High School, Edmonton
(2)	Masao Fujinaga, Harry Ainlay High School, Edmonton
8 (1)	Herman J. Ruitenbeek, Lord Beaverbrook High School, Calgary
(2)	John Soong, St. Mary's Community High School, Calgary

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[1]Canadian Mathematical Congress Scholarship

[2]The Nickle Foundation Scholarship

[3]Third place winner

[4]Fourth place winner

[5]Grade 12 student placing highest (below first 4)

[6]Grade 10/11 student placing highest (below first 4)

NATIONAL OLYMPIAD NOMINEES

	Name (grade)	School
1.	WU, Terry C.-Y. (12)	Lindsay Thurber High School, Red Deer
2.	REBLE, Donald J. (12)	Concordia College, Edmonton
3.	GRAHAM, William R. (12)	Harry Ainlay High School, Edmonton
4.	HUTCHINSON, Norman C. (12)	Brooks Composite High School, Brooks
5.	KERN, Malcolm W. (12)	W.R. Myers High School, Taber
6.	HARTMAN, Leo R. (12)	Camrose Composite High School, Camrose
7.	FENSKE, Keith (10)	Harry Ainlay High School, Edmonton
8.	KWAN, Raymond R.K. (12)	Bonnie Doon High School, Edmonton
9.	RUITENBEEK, Herman J. (12)	Lord Beaverbrook High School, Calgary
10.	GOLBY, Shawn R. (12)	Lorne Jenken High School, Barrhead
11.	SOONG, John H.-C. (12)	St. Mary's Community High, Calgary
12.	FUJINAGA, Masao (11)	Harry Ainlay High School, Edmonton
13.	LAKE, Robert M. (12)	Ross Sheppard High School, Edmonton
14.	CAMPBELL, Murray S. (12)	Ross Sheppard High School, Edmonton
15.	ROGERSON, Richard D. (12)	McNally Composite High School, Edmonton
16.	HOUSTON, Gary M. (12)	Lindsay Thurber High School, Red Deer

CANADIAN MATHEMATICAL CONGRESS

1975 Alberta High School Examination

PART I - Answer Sheet

To be filled in by the candidate.

PRINT: ____

Last Name First Name Initial

____

Candidate's Address Town/City

____

Name of School

ANSWERS:

1	2	3	4	5	6	7	8	9	10

11	12	13	14	15	16	17	18	19	20

To be completed by the Deparmtent of Mathematics, University of Alberta.

Points	Points Correct	Number Wrong
1 - 20 5	5 x =	1 x =
Totals	C =	W =

SCORE = C - W =

(DO ALL PROBLEMS. EACH PROBLEM IS WORTH FIVE POINTS.) *TIME: 60 Minutes*

1. The sum of three consecutive positive integers is always

(a) odd (b) even (c) a perfect square

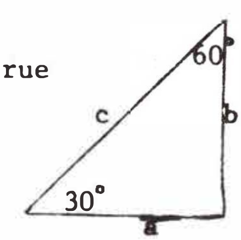
(d) divisible by 3 (e) none of these

2. Which of the following holds true?

(a) $\log_3 2 < \log_2 3$ (b) $\log_3 2 = \log_2 3$ (c) $\log_3 2 > \log_2 3$

(d) $\log_3 2 = 1$ (e) $\log_2 3 = 1$

3. For the triangle as shown, which of the following is true



(a) $a = b$ (b) $b = 2a$ (c) $c = 2a$

(d) $c = 2b$ (e) none of the previous are true

4. "The operation $\circ$ is commutative" means

(a) $x \circ 1 = 1$ (b) $x \circ x = x$ (c) $x \circ y = y \circ x$

(d) $x \circ (y \circ z) = (x \circ y) \circ z$ (e) none of the previous

5. If $x = .1102$ (base 3), then x^2 is (base 3)

(a) .11021102 (b) .10102 (c) .01222111

(d) .010211 (e) none of the previous

6. Given a square inscribed in a circle inscribed in an equilateral triangle, if each side of the triangle has length 6, what is the length of each side of the square?

(a) $\frac{1}{2}\sqrt{6}$ (b) $\sqrt{3}$ (c) $\sqrt{6}$

(d) $2\sqrt{3}$ (e) none of these

7. Which is larger, the volume of a sphere of radius 1 or the volume of a right circular cone of height 1 and base radius 2?

(a) these volumes do not exist (b) they are equal

(c) the sphere (d) the cone

(e) none of the above are true

8. $\dfrac{a^4 + a^2b^2 + b^4}{a^2 + ab + b^2} =$

(a) $a^2 + ab + b^2$ (b) $a^2 + ab - b^2$, (c) $a^2 - ab - b^2$

(d) $a^2 - ab + b^2$ (e) none of the previous

9. Five years from now Bill will be twice as old as he was two years after he was half as old as he will be in one year from now. His age is

(a) 16 (b) 13 (c) 8

(d) 41 (e) cannot be determined

10. The number $1.1313\overset{\frown}{13}\ldots$ (the pair 13 is repeated ad infinitum) is the same as

(a) $\frac{112}{99}$ (b) $\frac{113}{99}$ (c) $\frac{100}{99}$

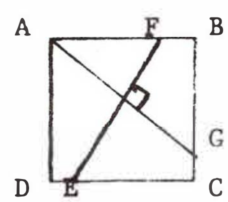
(d) $\frac{1131313}{1000000}$ (e) none of these

11. A jar contains 15 balls, of which 10 are red and 5 are black. If 3 balls are chosen at random the probability that all three will be red is

(a) 0 (b) $\frac{2}{3}$ (c) $\frac{4}{9}$

(d) $\frac{8}{27}$ (e) none of these

12. The square ABCD has side length 1. Given that $AG \perp EF$ and that $\overline{EF} = \sqrt{\frac{7}{6}}$ $\overline{BF} = \sqrt{\frac{1}{7}}$ then $\overline{AG}$ is



(a) $\sqrt{\frac{7}{6}}$ (b) $\sqrt{7}$ (c) $\sqrt{\frac{6}{7}}$

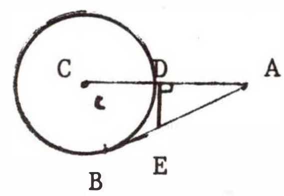
(d) $\sqrt{\frac{8}{7}}$ (e) none of these.

13. Which is the largest of 2^{4^3}, 2^{3^4}, 4^{2^3}, 3^{2^4}, 3^{4^2}?

(a) 2^{4^3} (b) 2^{3^4} (c) 4^{2^3}

(d) 3^{2^4} (e) 3^{4^2}

14. A circle of radius 3 has centre C. Let A be at a distance 5 from C and AB be a tangent to the circle. Let AC meet the circle at D and let E lie on AB with ED $\perp$ AC. Then the length $\overline{ED}$ is



(a) 1 (b) 2

(c) $\sqrt{2}$ (d) $\sqrt{3}$

(e) none of these

15. The system of equations $2x - 3y = 4$, $2y - 4x = 8$ has

(a) ten solutions (b) two solutions, (c) one solution

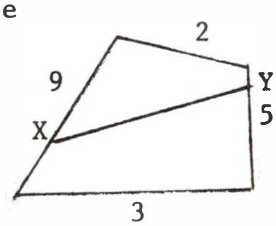
(d) no solutions (e) none of the previous

16. Suppose that a_1 is an integer not divisible by 3 and that $a_1^2 + a_2^2 + \dots + a_n^2$ is divisible by 3, where $a_2, \dots, a_n$ are integers. Then n is

(a) arbitrary (b) at least 3 (c) at most 2

(d) Always odd (e) none of these

17. According to the diagram, $\overline{XY}$ cannot equal



(a) 2 (b) 4

(c) 7 (d) 10 (e) any of these

18. Assume the earth is a perfect sphere and a wire is stretched tightly around the equator. The wire is lengthened one meter and then expanded uniformly so as to form a somewhat larger circle. The new radius will be approximately how many meters larger than the old one?

(a) .016 (b) .032 (c) .16

(d) .32 (e) 1

19. School X has 100 students and school Y has 50 students. These schools are to be replaced by a single school Z. If the students live in the immediate vicinities of their respective schools (X or Y), where should Z be placed so as to minimize the total distance travelled by all the students.

(a) at X (b) at Y (c) half way in between

(d) one third of the way from X to Y (e) at none of these

20. An urn contains 100 balls of different colours, 40 red, 27 green, 26 blue, and 7 white. What is the smallest number of balls that must be drawn without looking to guarantee that at least 15 balls have the same colour?

(a) 86 (b) 50 (c) 43

(d) 39 (e) none of these

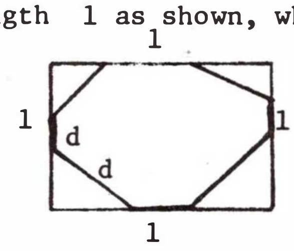
PART II

TIME: 110 Minutes

<u>Instructions to Candidates</u>

Attempt the problems in any order you wish. Partial credit is given for significant progress. Use any methods you like. Each problem is worth 20 points.

1. Given a regular octagon of side length d inscribed in a square of side length 1 as shown, what is d?



2. A prime number is a positive integer bigger than one, which is evenly divisible only by one and itself. Show that if a prime number is divided by 30, the remainder is prime.

3. Find all integers k so that

$$x^2 + k(x+1) + 7 = 0$$

has only integer solutions.

4. Given triangle ABC with M the midpoint of BC. Prove that $\overline{AB} + \overline{AC} > 2\,\overline{AM}$.

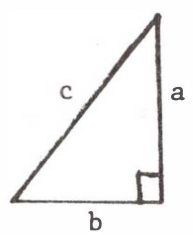
5. Find a positive number n, which is smaller than 25 such that the expression

$$(n - 1)^3 + n^3 + (n + 1)^3$$

is evenly divisible by 102.

6. Is the inequality $(99)^n + (100)^n > (101)^n$ always true, where n is a positive integer? Justify your answer.

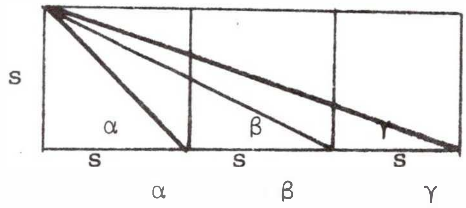
7. Given the right triangle as shown; and that a, b, c is an arithmetic progression. Show that $a : b : c = 3 : 4 : 5$.



8. Let a, b, c, d be four consecutive integers. Prove that $a \cdot b \cdot c \cdot d + 1$ is a perfect square.

9. A group of students write a set of k exams. Suppose that a_1 of the students failed at least 1 exam, a_2 failed at least 2 exams, · · ·, a_k failed exactly k exams. What was the total number of failed exams?

10. Given the three squares as shown, along with the angles α, β, γ. Show that $\alpha + \beta + \gamma = 90°$.



SOLUTIONS

1975 ALBERTA HIGH SCHOOL

PRIZE EXAMINATION IN MATHEMATICS

PART I: - KEY

1	2	3	4	5	6	7	8	9	10
D	A	D	C	C	C	B	D	E	A

11	12	13	14	15	16	17	18	19	20
E	A	B	E	D	B	D	C	A	B

PART II: - ANSWERS

1. By symmetry, the octagon divides the square into the octagon itself and four right triangles, each of side lengths x and hypoteneuse length d, such that $2x + d = 1$. By the Pythagorean theorem, $2x^2 = d^2$, or $x = \frac{\sqrt{2}}{2} d$, so that $\sqrt{2}d + d = 1$ or $d = \frac{1}{\sqrt{2} + 1} = \sqrt{2} - 1$.

2. Let p denote the prime and r the remainder. Then for some integer d we can write $p = 30d + r$. Note that $1 \leq r \leq 29$. All non-prime integers in this range are divisible by either 2 , 3 , or 5 . Thus, if r is not prime, then r is divisible by 2 , 3 or 5 (which also divides 30). The same number would then have to divide p . Since p is a prime, this cannot occur. Therefore, r is prime.

3. If r , s are the roots, then $r \cdot s = 7 + k$, $r + s = -k$. From this second equation it is clear that if k and r are integers, so is s. If r is a root of $r^2 + k(r+1) + 7 = 0$, then

$$k = -\frac{r^2 + 7}{r^2 + 1} = -(r-1) - \frac{8}{r+1} .$$

Therefore <u>both</u> k and r are integers if and only if $8 \div (r+1)$ is an integer,

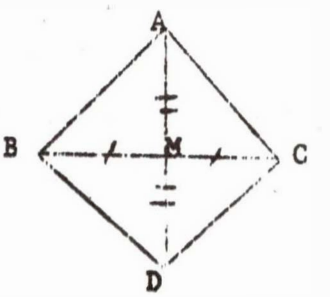
$$\frac{8}{r+1} = j , \quad \text{or} \quad r = -1 + \frac{8}{j} \quad \text{for}$$

some integer j . For r to be an integer, $j = \pm 1, \pm 2, \pm 4, \pm 8$. We can construct the following table.

j	1	-1	2	-2	4	-4	8	-8
$r = -1 + \frac{8}{j}$	7	-9	3	-5	1	-3	0	-2
$k = 1-r-j$	-7	11	-4	8	-4	8	-7	11

Therefore $k = -7, 11, -4, 8$ with corresponding root pairs (7,0), (-9,-2), (3,1), (-5,-3).

4.



In ΔABC , produce AM to D so that $\overline{AM} = \overline{MD}$. Then since $\overline{BM} = \overline{MC}$ and $\angle AMC = \angle BMD$, it must be that $\overline{AC} = \overline{BD}$.

Hence, $\overline{AB} + \overline{AC} = \overline{AB} + \overline{BD} > \overline{AM} + \overline{MD} = 2\overline{AM}$, since the shortest distance between two points is a straight line.

5. $n^3 - 3n^2 + 3n - 1 + n^3 + 3n^2 + 1 = 3n^3 + 6n = 3n(n^2 + 2)$

We require $3n(n^2 + 2) = 102j$ for some integer j, i.e.

$$\frac{n(n^2 + 2)}{2.17} = j$$

If n were odd, so would be $n^2 + 2$, and then so would be $n(n^2 + 2)$. Therefore n is <u>even</u>.

Also, since $0 < n < 25$, n cannot be an even multiple of 17, therefore $n^2 + 2$ is a multiple of 17, $n^2 + 2 = 17k$, or $n = \sqrt{17k-2}$.

We have:

k	1	2	3	4	5	6
n	$\sqrt{15}$	$\sqrt{32}$	$\sqrt{49}=7$	$\sqrt{66}$	$\sqrt{83}$	$\sqrt{100}=10$

So $n = 10$ will do ($n = 7$ is not even). If one checks all the cases $k = 1, 2, .., 37$ one finds that this is the only solution with $n \leq 25$. (The checking was done with the help of a digital computer).

6. Divide the inequality on both sides by $(100)^n$ to get $(.99)^n + 1 > (1.01)^n$. The left hand side is always less than or equal to 2 for every positive n. However, the right hand side increases without bound as n gets larger, so that for large enough n, the inequality must be false. (The smallest integer for which the inequality is not true is $n = 49$)..

7. Let $a = b - d$ and $c = b + d$. Then $(b + d)^2 = b^2 + (b - d)^2$, or upon simplifying, $b = 4d$, $a = 3d$, $c = 5d$. Hence $a: b: c = 3d: 4d: 5d = 3: 4: 5$.

8. $n(n+1)(n+2)(n+3) = n^4 + 6n^3 + 11n^2 + 6n + 1 = (n^2 + 3n + 1)^2$.

9. We first count how many <u>people</u> failed exactly z exams for $z = 1, 2, \ldots, k$:

$a_1 - a_2$	failed	exactly	1 exam
$a_2 - a_3$	"	"	2 exams
$\vdots$			
$a_{k-1} - a_k$	"	"	k - 1 exams
a_k	"	"	k exams

The total number of failed exams is then $1(a_1 - a_2) + 2(a_2 - a_3) + \cdots + (k-1)(a_{k-1} - a_k) + ka_k = a_1 + a_2 + \cdots a_k$.

10. $\tan \alpha = s/s = 1$, $\tan \beta = s/2s = 1/2$, $\tan \gamma = s/3s = 1/3$.
Hence $\alpha = 45^\circ$. $\tan(\beta+\gamma) = \dfrac{\tan\beta + \tan\gamma}{1 - \tan\beta\tan\gamma} = \dfrac{1/2 + 1/3}{1 - 1/6} = 1$, giving $\beta + \gamma = 45^\circ$. Hence $\alpha + \beta + \gamma = 90^\circ$.
