Maximization of Areas of Circle-Inscribed Triangles by an Oscillating Algebraic Sequence

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If a scalene triangle and an isosceles triangle are inscribed in a circle on the same side of a common chord as base, the altitude of the isosceles triangle is greater than the altitude of the scalene triangle and, consequently, the area of the isosceles triangle is greater than the area of the scalene triangle.

In Figure 1, $AB = BC > AC$. If a second isosceles triangle is constructed on one of these equal sides, say on AB, we obtain ΔABC_1. Since C_1 is the mid-point of arc BC_1A, the altitude drawn from C_1 to the base AB of ΔABC passes through the center of the circle and is greater than the altitude from C to the base AB of ΔABC. Thus the area of ΔABC_1 is greater than the area of ΔABC, both on the same side of AB as common base. A third such isosceles triangle can be constructed on BC as base, and so on, to produce a sequence of isosceles triangles each of area greater than that of the preceding one.

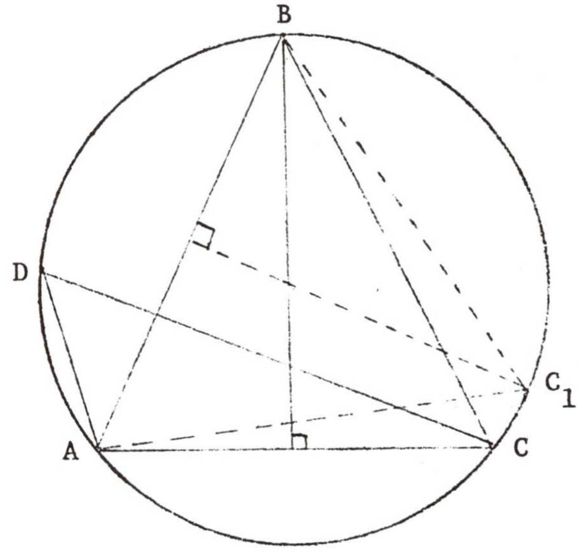

Fig. 1

Note that, should our first isosceles triangle not bound the center of the circle, the second and succeedings one will. Thus, we shall start always with an isosceles triangle that does bound the center, since no loss of generality occurs.

Various sequences are associated with the successive isosceles triangles formed as stated. These include the sequence of vertex angles, two equivalent sequences of lengths of bases, two equivalent sequences of altitudes, and two equivalent sequences of areas. The sequence of vertex angles is the easiest to set up and its limit is elementary. Consideration of the sequence of isosceles

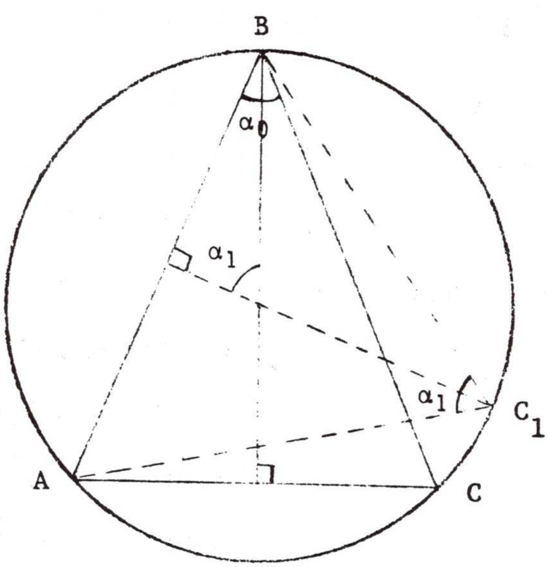

Fig. 2

triangles, shown in Figure 2, yields

$$\alpha_1 = 90° - \frac{\alpha_0}{2} = 60° + (-1/2)(\alpha_0 - 60°),$$

$$\alpha_2 = 90° - \frac{\alpha_1}{2} = 90° - \frac{1}{2}[60° + (-1/2)(\alpha_0 - 60°)]$$

$$= 60° + (-1/2)^2(\alpha_0 - 60°),$$

$$\alpha_3 = 90° - \frac{\alpha_2}{2} = 60° + (-1/2)^3(\alpha_0 - 60°),$$

$$\alpha_n = 90° - \frac{\alpha_{n-1}}{2} = 60° + (-1/2)^n(\alpha_0 - 60°).$$

The presence of the factor $(-1/2)^n$ indicates that this sequence oscillates. Since $\lim_{n\to\infty} (-1/2)^n = 0$, it follows that $\lim_{n\to\infty} \alpha_n = 60°$, which shows that the limiting isosceles triangle is an equilateral triangle, the triangle of maximum area.

That the formula

$$\alpha_n = 60° + (-1/2)^n(\alpha_0 - 60°)$$

is correct can be proved by mathematical induction as follows:

1. $\alpha_1 = 60° + (-1/2)(\alpha_0 - 60°) = 90° - \frac{\alpha_0}{2}$,

   which is true by construction.

2. Assume

   $$\alpha_k = 60° + (-1/2)^k(\alpha_0 - 60°)$$

   true. Then

   $$\begin{aligned} \alpha_{k+1} &= 90° - \frac{\alpha_k}{2}, \text{ by construction} \\ &= 90° - \frac{1}{2}[60° + (-1/2)^k(\alpha_0 - 60°)] \\ &= 60° - \frac{1}{2}(-1/2)^k(\alpha_0 - 60°) \\ &= 60° + (-1/2)^{k+1}(\alpha_0 - 60°). \end{aligned}$$

   Hence, if α_k is true, so is α_{k+1}.

3. Since α_1 is true, it follows by the principle of mathematical induction that

   $$\alpha_n = 60° + (-1/2)^n(\alpha_0 - 60°)$$

   is true for every natural number n.

For an equilateral triangle inscribed in a circle of radius r, it is easily shown that the length of its side is given by $\sqrt{3}\, r$, whereas its area is given by $\frac{3}{4}\sqrt{3}\, r^2$. This, then, gives the area of the triangle of maximum area inscribed in any circle.■