1976 Alberta High School Prize Exams - Winners and Solutions

RESULTS -- 1976 Alberta High School Prize Exam

Prize	Amt.	Student	School
Canadian Math Congress Scholarship	$400	ROTH, Dwayne C.	Medicine Hat High School Medicine Hat
Nickel Foundation Scholarship	$400	CHIPPERFIELD, Randall	Ernest Manning High School Calgary
Third Highest	$200	MATTHEWS, Paul	Dr. E.P. Scarlett High Calgary
Fourth Highest	$200	SAMPLE, Rick	Harry Ainlay Comp. High Edmonton

SPECIAL PROVINCIAL PRIZES

Highest Grade 12 student (below first 4):

	$100	HEWITT, Lawrence	William Aberhart High School Calgary

Highest Grade 10/11 student (below first 4):

	$100	FENSKE, Keith	Harry Ainlay Comp. High Edmonton

DISTRICT PRIZES

District No.		Amt.	Name	School
1			No prize awarded in District #1	
2		$50	HERCHEN, Harald	Grand Trunk High School Evansburg
3		$50	OSBORNE, Norman	Ardrossan Jr. Sr. High Ardrossan
4		$50	YOUNG, Bryan	Wm. E. Hay Comp. High Stettler
5		$50	HUANG, LingLing	Cochrane High School Cochrane
6		$50	NEUFELDT, Kevin	Kate Andrews High School Coaldale
7	(1)	$50	DOMIER, Calvin	Harry Ainlay Comp. High Edmonton
7	(2)	$25	GOON, Veiven	Harry Ainlay Comp. High Edmonton
		$25	FUJINAGA, Masao	Harry Ainlay Comp. High Edmonton
8	(1)	$50	PATTULLO, Andrew	William Aberhart High Calgary
8	(2)	$50	PATTERSON, Roger	Bishop Grandin High Calgary

CANADIAN MATHEMATICAL CONGRESS

1976 ALBERTA HIGH SCHOOL

PRIZE EXAMINATION IN MATHEMATICS

PART I ANSWER SHEET

To be filled in by the Candidate.

PRINT: ____

Last Name First Name Initial

____

Candidate's Address Town/City

____

Name of School

____

Grade

ANSWERS:

1	2	3	4	5	6	7	8	9	10

11	12	13	14	15	16	17	18	19	20

To be completed by the Department of Mathematics, University of Alberta:

	Points	Points Correct	Number Wrong
1 - 20	5	5 × =	1 × =
Totals		C = ______	W = ______

SCORE = C - W =

Do all problems. *<u>Each</u> problem is worth <u>five</u> points.* *<u>TIME</u>: 60 Minutes*

1. What is the value of $5^{\log_5 6}$?

(A) 1 (B) 5 (C) 6 (D) $\log_6 5$ (E) none of these

2. Let S be the set of points, (x,y), in the plane satisfying both $x^2 + y^2 \le 1$ and $x^2 + y^2 \ge r^2$. A value of r such that S is the empty set is:

(A) 1 (B) -1 (C) $\frac{1}{2}$ (D) $-\frac{1}{2}$ (E) none of these

3. If S, T, and V are any sets, then $[(S \cap T) \cup (S \cap V)]$ is the same set as

(A) S, (B) $T \cup V$, (C) $T \cap V$, (D) $S \cap (T \cup V)$,

(E) none of these

4. A metal disc has one face marked "1" and the other face marked "2" A second metal disc has one face marked "2" and the other marked "3" . Assume that, when tossed, the two faces of a disc are equally likely to turn up. If both discs are tossed, what is the probability that "4" is the sum of the numbers turning up?

(A) $\frac{1}{4}$ (B) $\frac{1}{2}$ (C) $\frac{3}{4}$ (D) $\frac{1}{3}$ (E) none of these

5. Which of the following statements about $\frac{1 + \sqrt{2}}{1 - \sqrt{2}}$ is true?

(A) It is irrational (B) It is rational (C) It is imaginary

(D) It is positive (E) None of these are true

6. If $i = \sqrt{-1}$, then i^6 is

(A) 1 (B) -1 (C) i (D) -i (E) none of these

7. The solution set of the inequality $x^2 - x - 2 < 0$ is the interval

(A) $-2 \le x \le 1$ (B) $-2 < x < 1$ (C) $-2 \le x < 1$

(D) $-2 < x \le 1$ (E) none of these

8. At the end of a party, everyone shakes hands with everyone else. Altogether there are 28 handshakes. How many people are there at the party?

(A) 8 (B) 14 (C) 20 (D) 56 (E) none of these

9. Let $[x]$ denote the largest integer not exceeding x. Thus, e.g. $[2] = 2$, $[3.99] = 3$, $[-.5] = -1$. Which of the following statements are always true?

(i) $[x+y] = [x] + [y]$

(ii) $[2x] = 2[x]$

(iii) $[-x] = -[x]$

(A) (i) (B) (ii) (C) (iii) (D) all (E) none

10. If the sum of the first n positive integers is $n(n+1)/2$, the sum of the first n positive <u>odd</u> integers is

(A) $n(n+1)/4$ (B) $\frac{n(2n+1)}{2}$ (C) n^2 (D) $n^2 - 4$ (E) none of these.

11. Suppose that $d = x^2 - y^2$ where x and y are two odd integers. Which of the following statements are always true?

(i) d is odd

(ii) d is divisible by 4

(iii) d is a perfect square.

(A) (i) (B) (ii) (C) (iii) (D) (ii) and (iii)

(E) none of these

12. If each term of the sequence $a_1, a_2, \ldots, a_n$ is either $+1$ or -1 then $a_1 + a_2 + \ldots + a_n$ is always

(A) 0 (B) 1 (C) $\begin{cases} \text{odd if } n \text{ is odd} \\ \text{even if } n \text{ is even} \end{cases}$

(D) $\begin{cases} \text{odd if } n \text{ is even} \\ \text{even if } n \text{ is odd} \end{cases}$ (E) none of these

13. If x is a real number satisfying the equation

$$x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = 2$$

then x is equal to

(A) ∞ (B) 2 (C) $\sqrt[4]{2}$ (D) $\sqrt{2}$ (E) none of these

14. The number of pipes of inside diameter 1 unit that will carry the same amount of water as one pipe of inside diameter 6 units of the same length is:

(A) 6π (B) 6 (C) 12 (D) 36 (E) none of these

15. $2^{-(2k+1)} - 2^{-(2k-1)} + 2^{-2k}$ is equal to

(A) 2^{-2k} (B) $2^{-(2k-1)}$ (C) $-2^{-(2k+1)}$ (D) 0 (E) none of these

16. Let P be the product of any 3 consecutive positive odd integers. The largest integer dividing all such P is:

(A) 15 (B) 6 (C) 5 (D) 3 (E) none of these

17. If $|x - \log y| = x + \log y$ where x and $\log y$ are real, then

(A) $x = 0$ (B) $y = 1$ (C) $x = 0$ and $y = 1$ (D) $x(y-1) = 0$

(E) none of these

18. Each of a group of 50 girls is blonde or brunette and is blue or brown eyed. If 14 are blue-eyed blondes, 31 are brunettes and 18 are brown eyed, the number of brown-eyed brunettes is

(A) 7 (B) 9 (C) 11 (D) 13 (E) none of these

19. After finding the average of 35 scores, a student carelessly included the average with the 35 scores and found the average of these 36 numbers. The ratio of the second average to the true average was

(A) 1:1 (B) 35:36 (C) 36:35 (D) 2:1 (E) none of these

20. If the line $y = mx + 1$ intersects the ellipse $x^2 + 4y^2 = 1$ exactly once, then m^2 is equal to

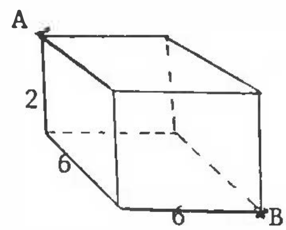
(A) 1/2 (B) 2/3 (C) 3/4 (D) 4/5 (E) none of these

<u>PART II</u> <u>TIME</u>: *110 Minutes*

1. Prove that it is impossible to find positive integers a,b satisfying

$$\frac{21}{23} = \frac{1}{a} + \frac{1}{b} \quad \text{and} \quad 0 < a < b \ .$$

2. A fly sitting in the upper corner A of a room measuring $6 \times 6 \times 2$ wishes to walk to the opposite lower corner B. What is the minimum distance the fly must walk and how many different paths achieve this distance?

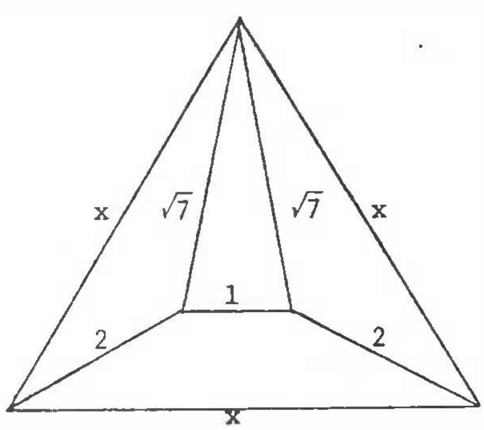


3. Find all real values of k so that the equation $x^3 + x^2 - 4kx - 4k = 0$ has two of its three roots equal.

4. In a round-robin tournament, each of n players $P_1, P_2, \ldots, P_n$ plays one game with each of the other players. The rules are such that no ties can occur. If w_i and ℓ_i denote respectively, the number of games won and lost by player P_i, show that

$$w_1^2 + \ldots + w_n^2 = \ell_1^2 + \ldots + \ell_n^2$$

5. Find the side x of the triangle shown in the diagram



6. Given real numbers a,b,c which satisfy $a^2 + b^2 + c^2 = 1$, find the smallest value of r so that $-\frac{r}{2} \le ab + bc + ca \le r$ is always true.

7. Find all possible integers a,b,c such that

$$a \log_{10} 2 + b \log_{10} 5 + c \log_{10} 7 = 1$$

8. Given that

$$\cos 3\theta = A \cos^3 \theta + B \cos \theta$$

holds for every real number θ , find A,B .

9. Nine lines are drawn parallel to the base of a triangle so that the other two sides are divided into 10 equal segments and the area into 10 portions, the largest of which is 38 square units. Find the area of the triangle.

10. Let a and b be odd integers and let n be a positive integer. If $a - b$ is divisible by 2^n prove that $a^3 - b^3$ is divisible by 2^n . Moreover, if $a^3 - b^3$ is divisible by 2^n prove that $a - b$ is divisible by 2^n .

SOLUTIONS

1976 Alberta High School

PRIZE EXAMINATION IN MATHEMATICS

PART I - KEY

C	E	D	B	A	B	E	A	E	C
1	2	3	4	5	6	7	8	9	10

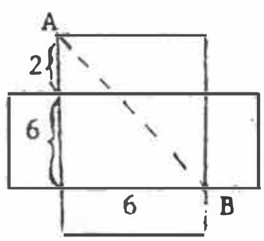
B	C	D	D	C	D	D	D	A	C
11	12	13	14	15	16	17	18	19	20

PART II - ANSWERS

1. If a and b are positive integers such that $0 < a < b$, then if $a = 1$, $\frac{1}{a} + \frac{1}{b} > 1$, and if $a \geq 2$, then

$$\frac{1}{a} + \frac{1}{b} \leq \frac{1}{2} + \frac{1}{3} = \frac{5}{6} < \frac{21}{23} .$$

2. Unfold the room to get:



The closest distance is the straight dotted line which is the hypotenuse of a right triangle with sides 6 and 8 . Thus

$d = \sqrt{6^2 + 8^2} = 10$. Using the floor and ceiling, there are 4 distinct paths achieving this distance.

3. Factoring, one gets $x^3 + x^2 - 4kx - 4k = x^2(x+1) - 4k(x+1) =$ $= (x+1)(x^2-4k)$. Hence the roots are given by $x = -1$ and $x = \pm 2\sqrt{k}$. Clearly then two of the roots are 0 if $k = 0$, and two of the roots are -1 if $k = 1/4$.

4. Taking the difference of the two sides:

$$(w_1^2 + \ldots + w_n^2) - (\ell_1^2 + \ldots + \ell_n^2) = (w_1^2 - \ell_1^2) + \ldots + (w_n^2 - \ell_n^2)$$

$$= (w_1 - \ell_1)(w_1 + \ell_1) + \ldots + (w_n - \ell_n)(w_n + \ell_n)$$

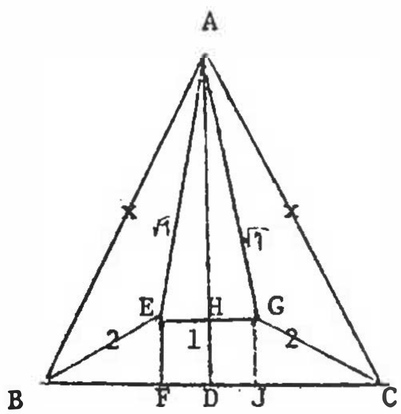
$$= (w_1 - \ell_1)(n - 1) + \ldots + (w_n - \ell_n)(n - 1)$$

$$= [(w_1 - \ell_1) + \ldots + (w_n - \ell_n)](n - 1)$$

$$= [(w_1 + \ldots + w_n) - (\ell_1 + \ldots + \ell_n)]\ (n - 1)$$

$$= 0 .$$

Since every game is either won or lost, the total of the wins is equal to the total of all the losses. We also used the fact that for each player, the total of his wins and losses is the number of games $(n - 1)$ that he played.

5. 

Construct the altitude AD, thus bisecting EG at H. Then $\overline{HG} = 1/2$, $\overline{AG} = \sqrt{7}$ and by Pythagoras' Theorem applied to $\Delta\ AHG$ we have

$$\overline{AH} = \sqrt{(\sqrt{7})^2 - (1/2)^2} = 3\sqrt{3}/2$$

Now if the perpendicular EF is dropped to BC, then

$\overline{EF} = \overline{HD} = \overline{AD} - \overline{AH}$ and by Pythagoras' Theorem again, applied to $\Delta\ ABD$,

$$\overline{AD} = \sqrt{x^2 - (x/2)^2} = x\sqrt{3}/2$$

and hence

$$\overline{EF} = \sqrt{3}/2(x-3) .$$

Finally, since $\overline{FJ} = \overline{EG} = 1$, we must have $BF = (x-1)/2$ and applying Pythagoras' Theorem to $\Delta\ BEF$ we obtain

$$\left(\frac{x-1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}(x-3)\right)^2 = 2^2$$

and upon simplifying,

$$x^2 - 5x + 3 = 0$$

so that $x = (5 \pm \sqrt{13})/2$. Now since $(5-\sqrt{13})/2 < 1$ and we clearly require $x > 1$, we must have $x = (5+\sqrt{13})/2$.

6. $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$. Since $a^2 + b^2 + c^2 = 1$, $ab + bc + ac = \frac{1}{2}[(a+b+c)^2 - 1]$. The right side will be smallest if $a + b + c = 0$, and largest if $a = b = c = 1/\sqrt{3}$. Hence

$-\frac{1}{2} \leq ab + ac + bc \leq 1$ and $r = 1$.

7. Note that

$$\begin{aligned} 1 &= a \log_{10} 2 + b \log_{10} 5 + c \log_{10} 7 \\ &= \log_{10} 2^a + \log_{10} 5^b + \log_{10} 7^c \\ &= \log_{10} (2^a 5^b 7^c) \end{aligned}$$

Taking antilogs,

$$10 = 2^a 5^b 7^c$$

Since $10 = 2^1 5^1 7^0$ uniquely, the only solution in integers is $a = 1, b = 1, c = 0$.

8. Since $\cos 0^\circ = 1$, $\cos 60^\circ = 1/2$, and $\cos 180^\circ = -1$, if

$$\cos 3\theta = A \cos^3\theta + B \cos\theta$$

we must have

$$-1 = A\cdot(1/2)^3 + B\cdot(1/2)$$

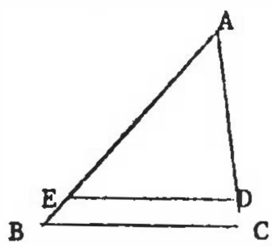
$$1 = A\cdot 1^3 + B\cdot 1$$

i.e. $$\frac{1}{8}A + \frac{1}{2}B = -1$$

$$A + B = 1$$

and solving we obtain $A = 4$, $B = -3$. (There are other methods of solution.)

9.



In Δ ABC , let ED denote the first of the 9 lines inserted above and parallel to the base BC . If h denotes the altitude of the quadrilateral BCDE , then the altitude of Δ ABC is $10h$ (by construction) and since Δ ABC is similar to Δ ADE , we must have $\overline{DE} = \frac{9}{10}\overline{BC}$. Now

$$\text{Area BCDE} = \frac{1}{2}(\overline{BC}+\overline{DE})\cdot h = \frac{1}{2}(1.9)\overline{BC}\cdot h$$

is given to be 38 . Hence

$$\text{Area ABC} = \frac{1}{2}\overline{BC}\cdot 10h = \frac{10}{1.9}\left[\frac{1}{2}(1.9)\overline{BC}\cdot h\right] = \frac{10}{1.9}\cdot 38 = 200$$

square units. (There are several other methods of solution.)

10. $a^3 - b^3 = (a-b)(a^2+ab+b^2)$. Clearly, if 2^n divides $a - b$, it must divide $a^3 - b^3$. Suppose 2^n divides $a^3 - b^3$, then since $a^3 + ab + b^2$ is odd, it must divide $a - b$.

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