Geometry and Axiomatics: An Historical Perspective

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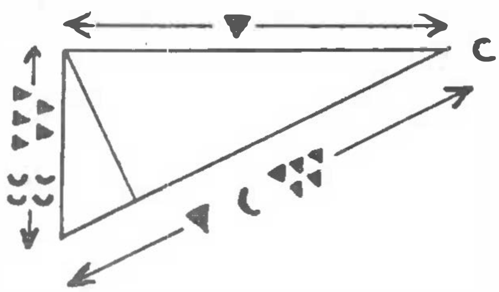
The evolution of axiomatics was predicated upon the evolution of geometrical thought. Since the concepts of what constitutes a geometry and what constitutes axiomatic development change as a function of the time period under consideration, it becomes essential that we examine the evolution of these ideas from antiquity to the present. The profound contributions made by geometrical thought to the genesis of axiomatics were principally derived through the critique of Euclid's fifth postulate, through the rise of non-Euclidean geometry, through the construction of abstract geometry, nay, through the development of logic. Hence, an investigation of this evolution, which has shaped our present perspective, would be most valuable.

The earliest extant records of geometrical activity are on baked-clay tablets from Mesopotamia believed to date from Sumerian times (3000 B.C.). From the first Babylonian dynasty of King Hammurabi's era, the new Babylonian empire of Nebuchadnezzar II, and the following Persian and Seleucid eras, there exists a superabundance of cuneiform tablets which suggest that Babylonian geometry involved practical mensuration. Between 2000 and 1600 B.C. the Babylonians had derived rules for computing the area of a rectangle and the areas of right and isosceles triangles. An example of the state of geometry in the Babylonian period of the seventeenth century B.C. derives from a 1958 excavation of a unique mathematical tablet containing the following right triangle problem:



Given ABC is a right triangle at A.
AD perpendicular to the hypothenuse BC.

AC = 60 (1), AB = 45, BC = 75 (1,15).

$BD = \sqrt{AB/AC \times 2 \times \text{area } ABD}$

$= \sqrt{45/60 \times 2 \times 486}$

27.

(The area 486 was written as 8,6 in Babylonian numeration, as shown on the diagram.)*

*Daniel B. Lloyd, "Recent Evidences of Primeval Mathematics," *The Mathematics Teacher* (Washington, D.C.: National Council of Teachers of Mathematics, 1965), p. 721.

Ancient Egyptian geometry is contained in the Moscow and Rhind papyri and of the 110 problems in these texts, 26 are devoted to geometry. Problem number 14 of the Moscow Mathematical Papyrus indicates that the Egyptian scribes were familiar with the formula $V = h/3\ (a^2 + ab + b^2)$ for the volume of a truncated pyramid, where h is the height and a and b are the edges of the square base and the square top, respectively.

Both the Babylonians and the Egyptians displayed mathematical initiatives. They used intuition, experiment, induction, and plain guessing to create some of their results, but irrespective of how well these rules agreed with experiences and no matter how exact the measurement, the rule was not deduced from explicit assumptions. The time when and the place where the distinction between inductive inference and deductive proof from a set of postulates became clear is not known, but it is known that the Greeks were the first to transform geometry from a set of empirical conclusions of the Egyptians and Babylonians to a deductively-based, systematic geometry. Perhaps the paradoxes of Zeno and the problem of incommensurables structured Greek thought to the direction of a logical base which ultimately led to an axiomatic treatment of geometry.

The Eudemian Summary of Proclus places the genesis of Greek geometry with Thales of Miletus in the first half of the sixth century B.C. He was the first among the Seven Wise Men of Greece. He is also the first known individual with whom the use of deductive methods in geometry is associated. Thales brought geometry from Egypt on his commercial ventures and then applied Greek procedures of deduction to his findings. He is credited with the following propositions relating to plane figures:

1. Any circle is bisected by its diameter.
2. The angles at the base of an isosceles triangle are equal.
3. When two lines intersect, the vertical angles are equal.
4. An angle in a semicircle is a right angle.
5. The sides of a similar triangle are proportional.
6. Two triangles are congruent if they have two angles and a side respectively equal.

As propositions in geometry, they may appear to be trivial since they are intuitive but we must be reminded that prior to this time geometry was confined almost exclusively to the measurement of surfaces and solids. The fundamental contributions of Thales consisted of his geometry of lines, and he is also credited with the idea of a logical proof to substantiate his geometrical results.

Pythagoras of Samos (572 B.C.) continued the systematization of geometry. Particularly important in the deductive aspects of geometry was the founding of the Pythagorean school. Members of the Pythagorean society developed the properties of parallel lines and used them to prove that the sum of the angles of any triangle is equal to two right angles. In the Eudemian Summary we are lead to believe that a Pythagorean, Hippocrates of Chios, was the first to attempt a logical presentation of geometry as a sequence of propositions which were ultimately based on some initial definitions and assumptions. Leon, Theudius and others developed this concept further until approximately 300 B.C. Euclid produced his treatise, the Elements, which consisted of an elegant chain of some 465 propositions on plane geometry (Books I to IV), the theory of proportions (Books V and VI), the theory of numbers (Books VII to IX), the theory of incommensurables (Book X) and solid geometry (Books XI to XIII).

It is generally accepted, however, that Aristotle's work as a systematizer of logic really prepared the way for Euclid's organization of the geometry of his time. Sir Thomas Heath, in the introductory chapters of his definitive English translation of the Elements, quotes a long passage from Aristotle's Posterior Analytics, containing a very careful analysis of the idea of a demonstrative science:

> Every demonstrative science, says Aristotle, must start from indemonstrable principles: otherwise, the steps of demonstration would be endless. Of these indemonstrable principles some are (a) common to all science, others are (b) particular, or peculiar to the particular science; (a) the common principles are the axioms, most commonly illustrated by the axiom that, if equals be subtracted from equals, the remainders are equal. Coming now to (b) the principles peculiar to the particular science which must be assumed, we have first the genus or subject-matter, the existence of which must be assumed, viz. magnitude in the case of geometry, the unit in the case of arithmetic. Under this we must assume definitions of manifestations or attributes of the genus, e.g. straight lines, triangles, deflection etc. The definition in itself says nothing as to the existence of the thing defined: it only requires to be understood. But in geometry, in addition to the genus and the definitions, we have to assume the existence of a few primary things which are defined, viz. points and lines only: the existence to everything else, e.g. the various figures made up of these, as triangles, squares, tangents, and their properties, e.g. incommensurability etc., has to be proved (as it is proved by construction and demonstration). In arithmetic we assume the existence of the unit: but, as regards the rest, only the definitions, e.g. those of odd, even, square, cube, are assumed, and existence has to be proved. We have then clearly distinguished, among the indemonstrable principles, axioms and definitions. A postulate is also distinguished from a hypothesis, the latter being made with the assent of the learner, the former without such assent or even in opposition to his opinion.

So the conception of a demonstrative science as a deductive sequence from an accepted set of initial statements was developed during the first 300 years B.C. by Greek mathematicians and philosophers. Certainly one of the greatest achievements of the Greeks was the creation of the postulational form of thinking (now called "material axiomatics") and the geometry they structured according to this posture.

Since changes and additions have been made in what now appears as Euclid's Elements, it is not certain precisely what statements Euclid assumed for his postulates and common notions nor what definitions he made, but the available evidence suggested that there were five postulates:

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any centre and distance.
4. That all right angles are equal to one another.

5. That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles.

There were five common notions (axioms):

1. Things which are equal to the same thing are also equal to one another.
2. If equals be added to equals, the wholes are equal.
3. If equals be subtracted from equals, the remainders are equal.
4. Things which coincide with one another are equal to one another.
5. The whole is greater than the part.

There were twenty-three definitions, some of which appear below:

1. A point is that which has no part.
2. A line is breadthless length.
3. A straight line is a line which lies evenly with the points on itself.
4. A place angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.
5. Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

It appears then that Euclid adopted Aristotle's distinction between postulates and common notions. He could also have adopted Plato's assumptions for classical constructions in the first three postulates. However, the remarkable insights of Euclid are contained in his ideal to establish geometry on any unimpeachably-logical foundation. Though his attempt failed, he nevertheless conceived it.

What Euclid did not seem to realize is that in a sense his postulates impose constraints on points and lines which therefore are defined. Hence any further attempt to define these terms is redundant mathematically. The question now arises as to why Euclid insisted on defining what should have remained undefined. The reason, in historical perspective, appears to be that Euclid thought his geometry constituted a description of the physical universe. It follows that his geometry should refer to external realities and therefore be definable.

Another remarkable insight of Euclid was his recognition of the importance of the parallel postulate and the necessity of assuming it. Given that he avoided the use of the fifth postulate until Proposition 29 of Book I, we may infer that Euclid himself may even have questioned its inclusion. From the beginning this postulate was criticized. It lacked the brevity of the other four postulates; and its converse, viz. - "The sum of two angles of a triangle is less than two right angles" - was proved as a theorem. Consequently, it was thought to be capable of proof. Posisonius, in the first century B.C., who defined parallel lines as lines that are coplanar and equidistant, attempted to prove the equivalent of Euclid's parallel postulate. In the second century, Claudius Ptolemy of Alexandria also worked on a proof of this postulate. Even

Proclus in the fifth century considered it alien to the special character of postulates. He was successful in reducing the proof to one that depended on the establishment of the following: Given any two parallel lines and a third distinct line which intersects one of the given lines, then it also intersects the remaining given line. Proclus' argument was based on the assumption that parallel lines everywhere are equidistant and this is tantamount to the fifth postulate. The logical anathema was that Proclus assumed what he was trying to prove.

Apollonius of Perga (225 B.C.) studied under the successors of Euclid. Their influence was manifest in his eight books on the Conics. Only seven books have survived, four in Greek and three in Arabic which contain 387 propositions. While Apollonius used a systematic and deductive process of construction and demonstration in his treatise, the form of the propositions was horrendous by virtue of the subject he was treating - a distinct contrast to the elementary conceptions of the line and circle of Euclid. Though Apollonius marks the termination of the golden age of Greek geometry, such geometers as Heron of Alexandria (A.D. 75), Menelaus (100), Ptolemy (85 - 165) and Pappus (320) did make some contributions to geometry. Of these, perhaps the greatest work was that of Pappus. His "Collection" contained original propositions and improvements.

A gradual decline in original thinking typified the period of the Roman Empire which devoured Greece in 146 B.C. Then, from the fall of the Roman Empire in the middle of the fifth century, the Dark Ages gave rise to little new mathematical thinking in Western Europe. It was during this time that mathematics was influenced by the Hindu and Arabian people. But the idea of deductive proof was alien to the thinking of the Hindus as exemplified in Aryabhata's book called Aryabhatiya which was written in 499. The Hindus' chief interest was in numbers and there was little influence of Greek geometry. However, the Arabian scholars of geometry were attracted to the proof of Euclid's fifth postulate. Alhazen (ibn-al-Haitham) (965-1039) "proved" that the fourth angle in a trirectangular quadrilateral must also be a right angle. (Actually, the fifth postulate follows from the assumption that Alhazen made.)

Omar Khayyam (1044-1123) also attempted the proof of Euclid's fifth. Khayyam's second book, Commentaries on the Difficulties in the Postulates of Euclid's Elements, was in part an attempt to connect the fifth and fourth postulates by means of five Aristotelian principles. For instance, Khayyam used the principle that quantities can be divided without end, *id est*, there are no indivisibles. He used the principle that a straight line can be indefinitely produced; two principles of intersecting lines and the axiom of Archimedes.

In order to prove one of his propositions it was necessary for him to conceive of three situations which in later history became known as (a) the acute angle case, (b) the obtuse angle case, and (c) the right angle case. This trichotomy which ultimately became known as non-Euclidean Bolyai-Lobachevskii geometry, non-Euclidean Riemann geometry, and Euclidean geometry, was also quoted by the Persian mathematician Nasir ed-din (1201-1274) who tried to prove the parallel postulate from the hypothesis:

> If a line u is perpendicular to a line w at A, and if line v is oblique to w at B, then the perpendiculars drawn from u upon v are less than AB on the side on which v makes an acute angle with w and greater on the side on which v makes an obtuse angle with w.

The history of the development of axiomatics is replete with attempts to prove the fifth postulate. These so-called proofs may be catalogistically trichotomized according to the following types: direct proof from the other four postulates; substitution for the fifth postulate some other ostensibly simpler one, of which Alhazen, Omar Khayyam and Nasir ed-din are typical examples, and indirect proof. While we know now that these approaches were illogical, they were instrumental in the evolution of synthetic methods.

The latter half of the eleventh century saw the infiltration of Greek learning into Europe. Significant though in terms of the history of mathematics was the twelfth century. The Elements appeared in Latin from the Arabic and were translated in 1142 by Adelard of Bath (1075-1160). It also appeared as a revised Latin translation from the Arabic work of Thabit ibn Qurra.

The rise of the universities in the thirteenth century at Paris, Oxford, Cambridge, Padua, and Naples contributed to the development of mathematics. It was during this century in approximately 1260 that Johannes Campanus of Novara made a commanding Latin translation of Euclid's Elements, which later, in 1482, became the first printed version. The hundred-year hiatus of the fourteenth century abounded in unproductiveness by virtue of the Black Death and the Hundred Years' War but was followed by the invention of the printing press in the fifteenth century which revolutionized the dissemination of knowledge.

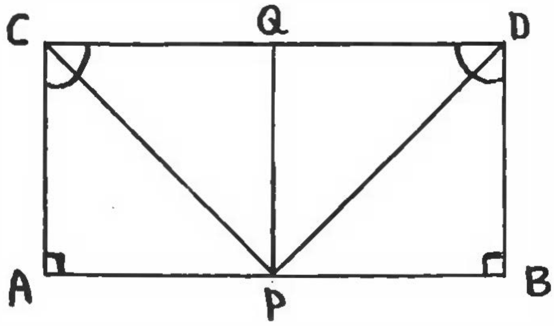
The mathematical achievements of the sixteenth century were more algebraic than geometric, although the symbolization of algebra was to have a pervasive effect on the development of geometry. More importantly though were the 1533 translation of Proclus' Commentary on Euclid, Book I, the 1566 Latin translation of Books I-IV of Apollonius' Conic Sections, and the 1572 Commandino translation of the Elements of Euclid. With an increasing number of the great Greek works in geometry readily accessible it would only be a question of time before the attention of scholars would again be focused on the development of geometry.

From the Greek era until the seventeenth century there was an enormous gap in the axiomatization of geometry. But it appears that the impetus of symbolized algebra and the general climate in the arts and sciences affected the formation of different conceptual patterns in geometry. One such development was that of non-Euclidean geometry.

Non-Euclidean geometry was created as a direct consequence of a critique of Euclid's parallel postulate. It became a compulsive challenge to prove this postulate from the others. It was hereditary stress. It appeared that there was a cultural "intuition" not unlike that of the Pythagoreans who, when presented by Theodorus of Cyrene with the proof of the irrationality of $\sqrt{2}$, refusing to scrap their philosophy, and, unable to ferret out the "faulty" step in the proof, arrived at a solution by labelling the discovery "alagon" (unutterable) and swore never to tell of this new number. This cultural bias then was probably responsible for the prevailing opinion in the Middle Ages that the fifth postulate could not conceivably be "independent" of the other postulates.

So it became almost a lifetime pursuit of Girolamo Giovanni Saccheri (1667-1733) to demonstrate once and for all that Euclid's system of geometry with its postulate of parallels was the only one possible. The first work, *Logica demonstrativa,* appeared in 1697 but was published under the name of Count Gravere, one of Saccheri's students. In this text, Saccheri makes a clear distinction between *definitiones quid nominis* and *definitiones quid rei* (nominal and real definitions). The nominal definition refers to a specific term while the real definition refers to the existence of a thing or its constructibility. Hence the nominal becomes real with a postulate.

His most definitive attempt to prove the parallel postulate was published in 1733 under the title of *Euclides ab omni naevo vindicatus sive conatus geometricus quo stabiliuntur prima ipsa geometriae principia.* In this masterpiece, Saccheri, using the method of *reductio ad absurdum,* tried to clear Euclid of all blemishes, including the supposed error of assuming the fifth postulate. His technique was to deny Euclid's parallel postulate, retain the other postulates and consequently derive a self-contradictory geometry. To arrive at a contradiction, he used a figure contained in the Clavius edition of Euclid's Elements, the birectangular quadrilateral (the Saccheri quadrilateral)



which consisted of two equal perpendiculars AC and BD to segment AB. Saccheri called ∠ACD and ∠BDC the summit angles of the quadrilateral. He noted three possibilities:

1. The summit angles are right (the right-angle hypothesis).
2. The summit angles are obtuse (the obtuse-angle hypothesis).
3. The summit angles are acute (the acute-angle hypothesis).

Saccheri showed that if any of these hypotheses is valid for one Saccheri quadrilateral, it is valid for every quadrilateral of the same type. He also showed that the parallel postulate is a logical consequence of the right-angle hypothesis. He showed further that by assuming a straight line is infinitely long, the obtuse-angle hypothesis is self-contradictory. To dispose of the actue-angle hypothesis was another matter, however. He obtained many results different from those that had been established by use of the fifth postulate, but he never did find a contradiction. Consequently, he concluded on the basis of intuition that the "hypothesis of the acute angle is absolutely false, because it is repugnant to the nature of a straight line." We now know that it would have been impossible for Saccheri to ever deduce a contradiciton from the acute-angle hypothesis.

In his attempt, though, he succeeded in creating a geometry independent of the parallel postulate. Perhaps the title of his treatise suggests that he expected to find no contradictions in Euclid, his idol. Certainly, the able logician Saccheri was cognizant that the system of fundamental propositions in every demonstrative science is precisely their indemonstrability. Perhaps Saccheri just could not conceive of this because of the extremely strong tradition that the only conceivable mathematics of space was Euclidean. He did not recognize his creation, yet he proved several theorems in two new geometries which were as sound logically as Euclid's.

The geometers of the eighteenth century were influenced by the work of Saccheri, especially Johann Heinrich Lambert (1728-1855), a Swiss mathematician, who, in 1766, analyzed the work of Saccheri and concluded that the obtuse-angle hypothesis is consistent with spherical geometry. Another geometer influenced by Saccheri, Adrien Marie Legendre (1752-1853) wrote *Eléments de geometrie* which clearly resembles Saccheri's work, except that Legendre proposed three hypotheses in which the sum of the angles in a triangle is equal to, greater than, and less than two right angles. He succeeded in producing a proof that the angle sum of a triangle cannot be greater than two right angles but failed to show that the sum cannot be less than two right angles. In any case, his "proof" was based on assumptions equivalent to what he was trying to prove. In 1809, Bernhard Friedrich Thibaut tried to prove Legendre's first hypothesis basing his argument on the assumption that every rigid motion can be resolved into a rotation and a translation, and assumption equivalent to that of the fifth postulate. John Playfair in 1813 tried to tidy up the errors in Thibaut's argument but with no success.

It remained for Gauss to bring the expression of the concept of non-Euclidean geometry to our attention. Carl Friedrich Gauss (1777-1855) also attempted to prove the parallel postulate by assuming its falsity. It is not known when Gauss recognized the existence of a logically-sound geometry without Euclid's fifth postulate but it is certain that he spent some thirty years in pursuit of such an aim given the cultural prejudice associated with it. In a letter to Franz Adolf Taurinus on November 8, 1824, he wrote:

> The assumption that the angle sum (of a triangle) is less than 180° leads to a curious geometry, quite different from ours but thoroughly consistent, which I have developed to my entire satisfaction. The theorems of this geometry appear to be paradoxical, and, to the uninitiated, absurd, but calm, steady reflection reveals that they contain nothing at all impossible.

However, the discovery of non-Euclidean geometry was not made by one person but by three almost simultaneously and independently. Gauss did not complete his discoveries but Janos Bolyai did. Bolyai (1775-1856) replaced the parallel postulate with: "In a plane two lines can be drawn through a point parallel to a given line and through this point an infinite number of lines may be drawn lying in the angle between the first two and having the property that they will not intersect the given line." Bolyai's work was published in 1832. Nikolai Ivanovich Lobachevskii (1793-1856) also invented a new geometry, published in 1829. Lobachevskii's replacement of Euclid's parallel postulate was: "Through a point P not on a line there is more than one line which is parallel to the given line." Consider the apocalyptical though logical consequences of this postulate: (1) No quadrilateral is a rectangle; if a quadrilateral has three right angles, the fourth angle is acute, (2) The sum of the measures of the angles of a triangle is always less than 180°, and (3) If two triangles are similar, they are congruent.

The obtuse-angle hypothesis was not a consideration of Bolyai or Lobachevskii but of Georg Friedrich Bernhard Riemann (1826-1866), a student of Gauss. Replacing the parallel postulate of Euclid with: "Through a point in a plane there can be drawn in the plane no line which does not intersect a given line not passing through the given point," Riemannian geometry was born. It gave rise to other curious results:

1. Two perpendiculars to the same line intersect.
2. Two lines enclose an area.
3. The sum of the measures of the angles of a triangle is greater than 180°.
4. If two sides of a quadrilateral are congruent and perpendicular to a third side, the figure is not a rectangle, since two of the angles are obtuse.

But, we may add, it was Eugenio Beltrami in 1868 who finally established the relative consistency of the new geometries by interpreting plane non-Euclidean geometry as the geometry of geodesics on a certain class of surfaces in Euclidean space.

The solution of the parallel postulate was finally found. It is rather curious that its initial solution should occur almost simultaneously and independently through the work of Gauss, Bolyai and Lobachevskii unless we invoke a cultural explanation referred to at an earlier time in this paper. The equipment needed, the ideas prerequisite to appropriate analogies are additive within the mathematical community, being ubiquitous and yet accumulative until sufficient stress is created that the problem commences to be solved by several investigators in the same temporal domain. (There are many such examples of this phenomenon in the history of mathematics. Witness, for example, the development of calculus.) The solution of the parallel postulate problem was long in coming and yet its discovery occurred because the concepts and ideas that were prevalent just prior to its solution, such as the advent of axiomatic systems in algebra, initiated new insights unique to a solution.

The enigma of the fifth postulate was also slow in coming not only because of the inherited tradition that surrounded it but also because of the prevailing philosophy of Kant (1724-1804) who treated space not as empirical but as something existing in the mind and hence non-experiential. The obstacle to overcome then was to view geometry as an experimental science complete with postulates as a function of convenience but correlated with the data in the physical world. Certainly Kantian philosophy was responsible for the lack of true regard for the discoveries of Gauss, Bolyai and Lobachevskii. But there is no question that the role of the non-Euclidean geometries did have a pervasive effect on mathematical and philosophical thought especially in the nineteenth century. It lent credence to the idea that mathematics ought not to be bound to specific patterns à la Kant or even patterns displayed in the physical universe, but rather that mathematics ought to create its own patterns predicated on contemporary thought.

The stage was set for the abstract conception that geometric theories are true only in the sense that they are logical consequences of the axioms that constitute their bases. Mathematicians were almost ready to accept the notion that Euclidean geometry is no more true than non-Euclidean geometry.

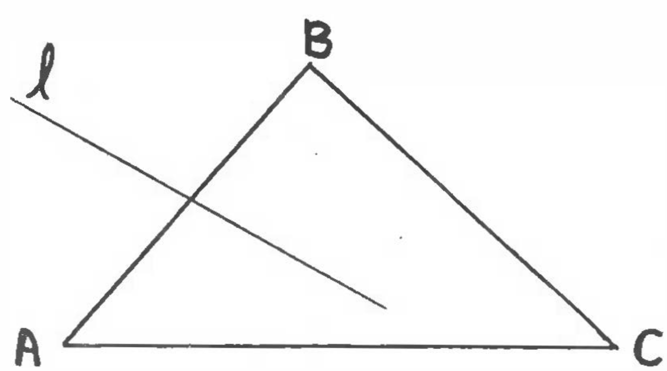
Euclid defined point and line as an approximation of the physical universe because he thought that they represented something extant. For centuries, this conception of the universe was not questioned. Even today the mathematician hesitates to define which geometry is more representative of his world. The truth of the matter is that he may never know if he agrees with the Henri

Poincaré model: If our universe were enclosed in a sphere of finite radius and our planet were close to its centre; if, when entities approach the boundary of the sphere they become increasingly but uniformly smaller, then they can never reach the boundary and we would be unaware of this shrinking since our measuring devices would also get smaller and smaller; consequently, our universe, from our conception of it, would appear to be unbounded but, in fact, it is bounded. But physical space is locally Euclidean which means that in the neighborhood of a point, space is Euclidean. In contradistinction, universal space may be Euclidean, elliptic or hyperbolic and notwithstanding Poincaré, Einstein's general theory of relativity predicts hyperbolic space.

Non-Euclidean geometry transmogrified deductive reasoning. It was the basis of viewing mathematics as a creation of postulates by mathematicians. The advance of mathematics is gradual as Bell points out in "The Development of Mathematics" so even by 1945 some still adhered to the Platonic doctrine of mathematical truths. But the greatest iconoclasm of non-Euclidean geometry, nevertheless, was its destruction of the myth of absolute truth in mathematics. This, then, was an important precursor to the notion that a mathematical system could have an intrinsic independence of any notion of physical reality.

The self-consistency concept of a mathematical system was consonant with the prevailing scientific philosophy of the late nineteenth and early twentieth centuries. The first step on the ladder of maturity is the view of mathematics as a science independent of physical reality, a science whose entities are themselves abstract systems capable of self-consistency without being true of any particular reality but necessarily of some other structure. So Frege's conception of a foundation for mathematics - because mathematics is independent of physical reality, its truths must also be independent of this reality - was ultimately an outgrowth of the existing conception of the universe as suggested by non-Euclidean geometry.

It was also during the late nineteenth and early twentieth centuries, after the foundations of geometry had been examined extensively, that satisfactory postulate sets emerged for Euclidean geometry. It is not the purpose of this paper to ferret out all the inconsistencies in Euclid's Elements but we shall mention a few that gave rise to the further advance of geometry. Postulate two which was referred to earlier, that a straight line may be produced indefinitely, does not mean necessarily that a straight line is infinite. Riemann in 1854 pointed out that distinction. In Proposition I 21 Euclid made an assumption that Moritz Pasch (1843-1930) recognized must be made explicit. So he supplied what is sometimes called Pasch's Axiom: If a line l intersects AB, one side of a triangle ABC, it intersects either BC or AC in a point between B and C or in a point between A and C.



Dedekind supplied a continuity postulate sometimes called Dedekind's Axiom: If a line segment connects a point inside a circle to a point outside a circle (in the same plane) then the line segment will intersect the circle. This postulate was necessary to the logical analysis of Proposition II where it is assumed that circles with centres at the ends of a line segment and having the line segment as a common radius intersect. To tidy up some of these and other criticisms of Euclid such men as Veblen, Hilbert, Pieri and Hüntington suggested other postulates and other undefined terms. So it was the quest for a logically acceptable postulate set for Euclidean geometry coupled with the apocalypse of consistent hyperbolic, parabolic and elliptic geometries that advanced the development of axiomatics.

We now descend to a more profound level of Pasch's work where we observe antecedents of the postulational method in geometry. He effectively obliterated both the Newtonian conception of space as the absolute ultimate and the Leibnizian idea of space as a labyrinth of relations. Pasch, like Peirce before him, thought of geometry as an hypothetico-deductive system in which the elemental terms like points, lines, and planes remained undefined. It is important for us to note that Pasch's lucid representation of geometry was the first after Euclid's in the postulational tradition, though he went beyond Euclid, as we have explained, in his ferreting out of covert assumptions. That our present conception of geometry is close to that of Pasch is testimony to support his profound influence on the subject.

In the early 1890s, Peano began the monumental task of restructuring all of mathematics, including geometry, to a precise symbolism. Consonant with this approach were postulate sets that were necessary and sufficient conditions for proofs. The nineteenth century witnessed an unprecedented development of mathematical shorthand, especially in the work of Boole. It was this movement, spearheaded by Peano and culminated by David Hilbert (1862-1943), that probably more than anything else laid the foundation to mathematical logic. Also consonant with the Peanoian approach was the view of geometry as an abstract, purely formal system without any intrinsic content save that implied by the postulates. In the penultimate year of the nineteenth century the ultimate advance was made in Hilbert's classical *Grundlagen der Geometrie*. Once and for all the postulational method was established not only for geometry but for most of mathematics to come, if we are permitted the liberty of a hindsight. Here was the genesis of the realization that axiomatics were not peculiar to geometry.

The adoption of the axiomatic method as a general foundational device was slow in coming. The method *ipso facto* was not fully accepted until the nineteenth century. Then the beginnings of the power of the method not only as a means of generalizing mathematical concepts but also as a research tool were realized. The stage was set for the prominent role that logic was to assume in the Edwardian period. While it is not the purpose of this paper to consider the role of axiomatics in logic, it is, nevertheless, interesting to note the parallel between the destruction of the uniqueness of Euclidean geometry by the invention of the non-Euclidean geometries and the destruction of the uniqueness of mathematical logic by the axiomatic analysis of logic.

Having traveled the non-Euclidean highway so far we should now like to glance over the route and note some other geometric landmarks that contributed to the development of axiomatics. Mathematicians were profoundly influenced in the development of modern mathematical thought by projective geometry and the attendant elements such as the law of duality. J.D. Gergonne (1771-1859) noticed that if point and line were interchanged in some plane geometry theorems it would be possible to create independently provable but dual propositions. He suggested that the original theorem is a sufficient condition for the dual. Further, Gergonne reasoned analogously in three-dimensional space that point and plane were duals. J.V. Poncelet (1788-1867) published his *Traité des propriétés projectives des figures* in 1822 which was a classic of the synthetic method and a definitive precursor to the conception of geometry as an hypothetico-deductive system. In observing that certain characteristics of a plane configuration, for instance, collinearity (Pascal's theorem), remain invariant under projection, Poncelet defined the projective properties of figures. But it was Julius Plücker (1801-1868) who generalized the classic duality for configurations of points and lines in plane geometry. While Gergonne may have believed his duality an absolute attribute of "space" born of intuition, the "space" of elementary projective geometry for Plücker's geometry was a trivial consequence of a narrow way of choosing systems of coordinates. In fact, it was Plücker's abandonment of visual intuition for an algebraic and analytic treatment that finished something that the non-Euclidean geometries only began. It was this kind of mathematical construction of "spaces" and "geometries" that finally demolished Kant's conception of the nature of mathematics. In any case, there is no question that the residue of the work of Plücker was another testimony that geometry was fast becoming an abstract formal discipline.

The development of axiomatics was profoundly influenced by a number of geometric themes: abstract geometry in the laws of duality, Plücker's coordinates and hyperspace, non-Euclidean geometry, and the criticism of the fundamental principles of geometry in reference to physical facts. It was all of these and more that precipitated the so-called hypothetico-deductive system of contemporary mathematical theory. These themes have retained their vitality and interest although particular concepts may have lost their attractiveness for those trained in newer habits of thought for which those very concepts were partly responsible. But in seeking the things that have endured in mathematics, we are led to processes and ways of thinking rather than to their products in any one epoch. However, the creation of a set of axioms that is fecund in profound results, small in number, platitudinous in difficulty, self-evident to reasonable people and demonstrably independent will enhance the aesthetic appearance of the creation but the axiom system can never be proved consistent by methods formalizable within the system itself. Consequently, the closer we approach the foundations of mathematics the more illusory is our grasp on the axiomatic method, until, in the final analysis, when we transcend the threshold of mathematics, the heuristics of axiomatics disappear and the focus ought to change to new methods of exploration.

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