1977 Alberta High School Prize Exam Results

Prize	Amt.	Student	School
Canadian Math. Congress Scholarship	$400	FENSKE, Keith W.	Harry Ainlay High School Edmonton, Alberta
Nickel Foundation Scholarship	$400	BERTRAND, Daniel	Camille Lerouge Collegiate Red Deer, Alberta
Third Highest	$150	PEZZANI, Glenn	Harry Ainlay High School Edmonton, Alberta
Fourth Highest	$150	WONG, David	Lethbridge Collegiate Inst. Lethbridge, Alberta

SPECIAL PROVINCIAL PRIZES

Highest Grade 12 student (below first 4):	$ 75	TOTMAN, Ian W.	Fort Saskatchewan Sr. High Fort Saskatchewan
Highest Grade 10/11 student (below first 4):	$ 75	McINTOSH, Lawrence P.	Bishop Carroll High School Calgary, Alberta

DISTRICT PRIZES

District No.	Amt.	Name	School
1	$50	MORRILL, Cameron	Edwin Parr Composite Athabasca, Alberta
2	$50	FRASER, George	Grand Centre High School Grand Centre, Alberta
3	$50	BAWOL, Rick Allan	Fort Saskatchewan High Fort Saskatchewan, Alberta
4	$50	EISENTRAUT, Matthew A.	Lindsay Thurber Comp. High Red Deer, Alberta
5	$50	MORCK, Adrienne E.	Olds Junior-Senior High Olds, Alberta
6	$50	NEUFELDT, Kevin J.	Kate Andrews High School Coaldale, Alberta
7 (1)	$50	ALI, Syed S.	Harry Ainlay High School Edmonton, Alberta
7 (2)	$50	FORREST, Brian	Eastglen Composite High Edmonton, Alberta
8 (1)	$50	McINTOSH, Richard J.	Bishop Carroll High School Calgary, Alberta
8 (2)	$50	BANMANN, Dan H.	Central Memorial High School Calgary, Alberta

CANADIAN MATHEMATICAL CONGRESS

1977 ALBERTA HIGH SCHOOL

PRIZE EXAMINATION IN MATHEMATICS

PART I ANSWER SHEET

To be filled in by the Candidate.

PRINT: ____
Last Name First Name Initial

____
Candidate's Address Town/City

____
Name of School

____
Grade

ANSWERS:

1	2	3	4	5	6	7	8	9	10

11	12	13	14	15	16	17	18	19	20

To be completed by the Department of Mathematics, University of Alberta:

	Points	Points Correct	Number Wrong
1-20	5	5 × =	1 × =
Totals		C = ______	W = ______

SCORE = C - W =

Do all problems. Each problem is worth five points. *<u>TIME:</u> 60 Minutes*

1. If $a > b > 0$, then

   (A) $\frac{a+1}{a} > \frac{b+1}{b}$ (B) $\frac{a+1}{a} \geq \frac{b+1}{b}$ (C) $\frac{a+1}{a} \leq \frac{b+1}{b}$

   (D) $\frac{a+1}{a} < \frac{b+1}{b}$ (E) none of the preceding are true.

2. Let AB be a diameter of a circle of radius 1 and let C be a point on the circumference such that $\overline{AC} = \overline{BC}$. Then the length $\overline{AC}$ is equal to

   (A) 2 (B) 1/2 (C) $\sqrt{2}$ (D) $1/\sqrt{2}$ (E) none of these.

3. Out of 100 people, 60 report that they receive the daily news by watching television, whereas 70 read the newspaper. Of those that read the newspaper, 70% also watch television. The number not receiving any news by television or newspaper is

   (A) 15 (B) 19 (C) 23 (D) 27 (E) none of these.

4. $(64^{.9})(32)^{-.08}$ equals

   (A) 64 (B) 32 (C) 24 (D) 8 (E) none of these.

5. Let $f(x)$ be a non-constant polynomial with real coefficients. If $f(x) = f(x-1)$ for all x then $f(x)$

   (A) has exactly one root (B) cannot exist

   (C) has exactly two roots (D) has either no roots or an infinite number of roots

   (E) satisfies none of the preceding.

6. If k is a real number such that $0 < k < 1$, then the roots of the quadratic equation $kx^2 - 3x + k = 0$ satisfy

(A) both are positive (B) both are negative (C) both are zero

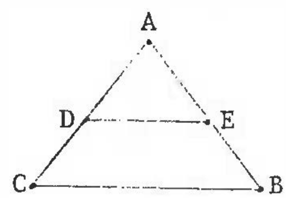
(D) one is positive and one is negative (E) none of the preceding

7. Let ℓ be a line in the real plane passing through the points $(1,1)$ and $(3,5)$. Then ℓ passes through the point $(2,y)$ where

(A) $y = 4$ (B) $y = 2$ (C) $y = 3$ (D) $y = 5$

(E) y is none of the preceding.

8. ΔABC is an equilateral triangle with sides of length 1, and $DE \parallel CB$. If the area of ΔADE is equal to the area of the trapezoid $DEBC$, then the length $\overline{DE}$ equals



(A) $1/2$ (B) $1/3$ (C) $1/\sqrt{2}$ (D) $\dfrac{\sqrt{2}-1}{\sqrt{2}}$ (E) $\dfrac{\sqrt{3}-1}{\sqrt{3}}$

9. The inequality $(x+1)(x-1) \geq x^2$ is valid

(A) for all real x (B) for no real x (C) for all $x \geq 1$

(D) for all $x < 0$ (E) for none of the preceding.

10. Suppose a bowl contains 3 red balls and 3 yellow balls. The probability that two balls drawn out at random without replacement will both be red is

(A) $1/2$ (B) $1/4$ (C) $1/3$ (D) $1/6$ (E) none of these.

11. $\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{1+1}}}$ equals

(A) 1 (B) 1/2 (C) 2/3 (D) 3/5 (E) 5/8

12. $\dfrac{xy - x^2}{xy - y^2} - \dfrac{xy}{x^2-y^2}$ can be simplified to

(A) $\dfrac{x^3}{y^3 - yx^2}$ (B) $\dfrac{x^2}{y^2 - x^2}$ (C) $\dfrac{x^2 + y^2}{x^2 - y^2}$

(D) $\dfrac{x^4 + xy^3}{(x^2-y^2)(xy-y^2)}$ (E) none of these.

13. If the radius of a sphere is increased 100%, the volume is increased by

(A) 100% (B) 200% (C) 300% (D) 400% (E) none of these.

14. $x^4 + 16$ equals

(A) $(x^2+4)(x^2+4)$ (B) $(x^2+4)(x^2-4)$

(C) $(x^2-4x+4)(x^2+4x+4)$ (D) $(x^2-2x\sqrt{2}+4)(x^2+2x\sqrt{2}+4)$

(E) none of these.

15. The price of a book has been reduced by 20%. To restore it to its former value, the last price must be increased by

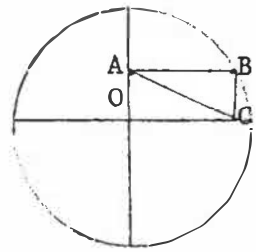
(A) 25% (B) 10% (C) 15% (D) 20% (E) none of these.

16. OABC is a rectangle inscribed in a quadrant of a circle of radius 10. If $\overline{OA} = 5$, then $\overline{AC}$ equals

(A) $5\sqrt{2}$ (B) $\sqrt{75}$

(C) 8 (D) 12

(E) none of these.



17. The lengths of the medians of a right triangle which are drawn from the vertices of the acute angles are $\sqrt{73}$ and $2\sqrt{13}$. The length of the third median is

(A) $\sqrt{73 + 52}$ (B) $\sqrt{73} + 2\sqrt{13}$ (C) 5 (D) 10

(E) none of these.

18. A car travels 240 miles from one town to another at an average speed of 30 miles per hour. On the return trip the average speed is 60 miles per hour. The average speed for the round trip is

(A) 35 mph (B) 40 mph (C) 45 mph (D) 50 mph

(E) 55 mph

19. $\log_3 6 + \log_3(3/2)$ equals

(A) 5/2 (B) 3 (C) 2 (D) 1 (E) 0

20. The slope of the line passing through the points (3,4) and (1,9) is

(A) $-5/2$ (B) 5/2 (C) 5 (D) -2 (E) 6

PART II　　　*TIME: 110 Minutes*

1. Prove that $3^n + 1$ is not divisible by 8 for any positive integer n .

2. Let k be any nonzero real number and let a, b, and c be the roots of the cubic equation $4x^3 - 32x^2 - k^2x + 8k^2 = 0$. Show that the quadratic equation $x^2 + 2kx - abc = 0$ cannot have real roots.

3. Sketch accurately the set of all points which satisfy $[|x|] + |y| = 2$. Note that for a real number r , $[r]$ is defined to be the largest integer n such that $n \le r$.

4. For any three consecutive natural numbers, prove that the cube of the largest cannot be the sum of the cubes of the other two.

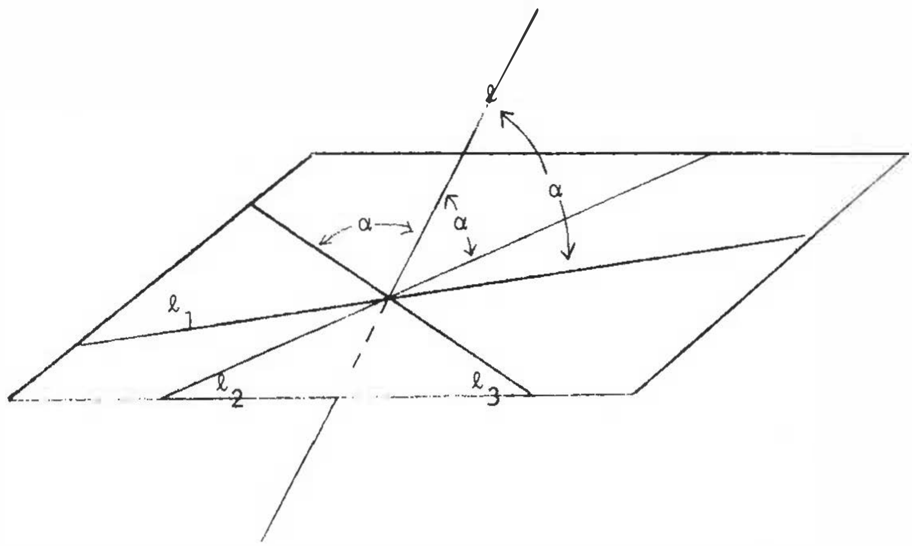
5. Prove that $\sin(\frac{\pi}{24}) = \left(\sqrt{2+\sqrt{2}} - \sqrt{3}\sqrt{2-\sqrt{2}}\right)/4$.

6. Prove that a triangle with sides of lengths 3, 4 and 5 respectively, is a right triangle.

7. If a, b, and c denote the lengths of the sides of a triangle, prove that $\sqrt{a(b+c-a)}$, $\sqrt{b(c+a-b)}$ and $\sqrt{c(a+b-c)}$ are also lengths of the sides of a triangle.

8. Show that the maximum value of $x + y + z$ subject to $x \ge 0$, $y \ge 0$, $z \ge 0$, $x + y \le 2$ and $3x - y + z \le 1$ is 5 .

9. Suppose four lines ℓ, ℓ_1, ℓ_2, and ℓ_3 meet in a point. Suppose ℓ_1, ℓ_2, and ℓ_3 lie in the same plane, and ℓ makes equal angles with ℓ_1, ℓ_2 and ℓ_3. Prove that ℓ is perpendicular to the plane containing ℓ_1, ℓ_2 and ℓ_3 (see diagram).



10. If the real numbers a, b, c, x, y, z satisfy $az - 2by + cx = 0$ and $ac - b^2 > 0$, then prove that $xz - y^2 \le 0$.

SOLUTIONS

1977 Alberta High School

PRIZE EXAMINATION IN MATHEMATICS

PART I - KEY

1	2	3	4	5	6	7	8	9	10
D	C	B	B	B	A	C	C	B	E

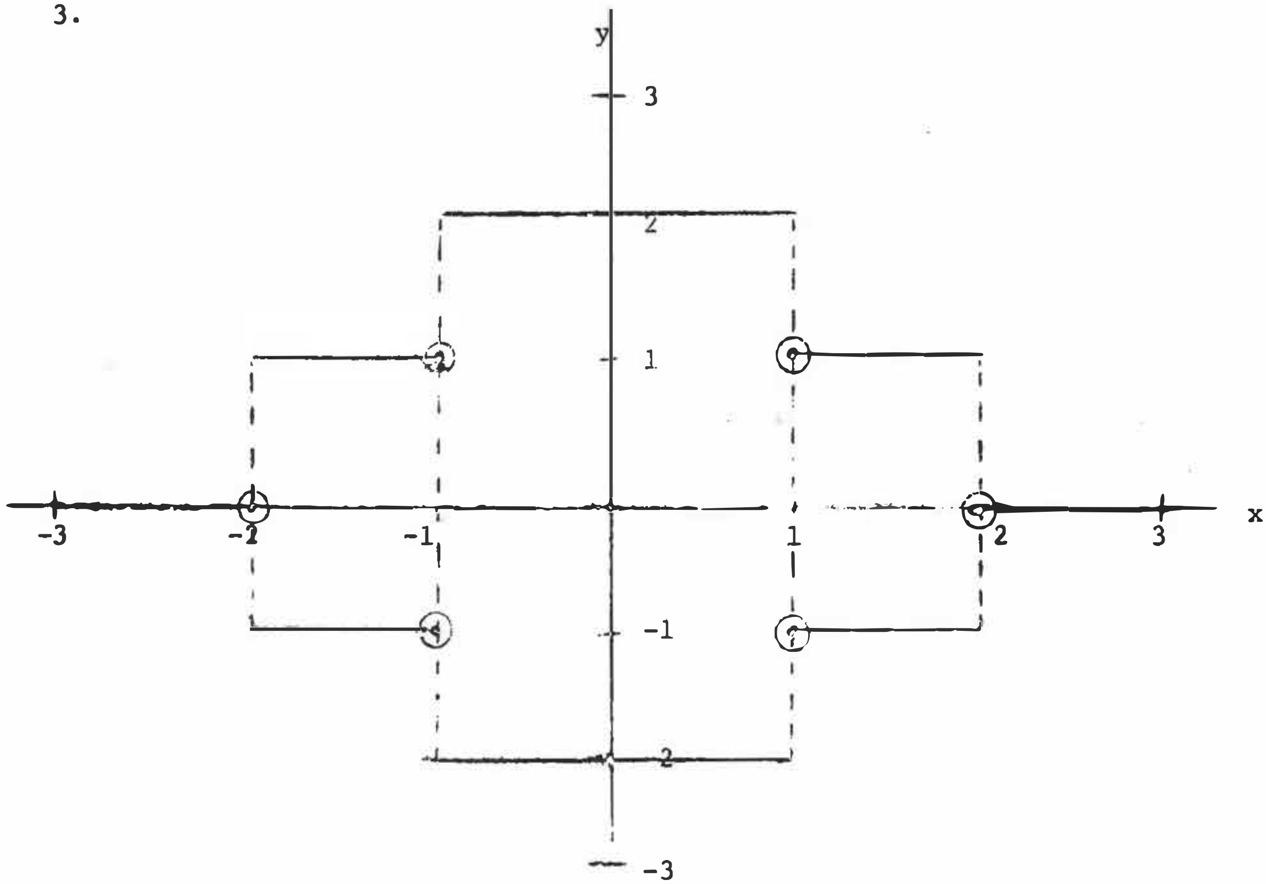
D	A	E	D	A	E	C	B	C	A
11	12	13	14	15	16	17	18	19	20

PART II - ANSWERS

1. Clearly $3^n + 1$ is not divisible by 8 for $n = 1,2$. Let m be the smallest positive integer for which $3^m + 1$ is divisible by 8 . Thus there is an integer k such that $3^m + 1 = 8k$. Adding 8 to both sides gives $3^m + 9 = 8k + 8$, or $9(3^{m-2}+1) = 8(k+1)$. This shows that $3^{m-2} + 1$ is also divisible by 8 , so that either m was not the smallest such integer or m is 1 or 2 . This contradiction proves the original assertion.

2. The cubic $4x^3 - 32x^2 - k^2 + 8k^2 = 0$ factors into $(4x^2-k^2)(x-8) = 0$. The roots of this equation are $x = 8$, $x = \frac{k}{2}$ and $x = \frac{-k}{2}$. The product of the roots is $abc = -2k^2$. This means that the quadratic equation become $x^2 + 2kx + 2k^2 = 0$. The roots of this equation are $-k \pm k\sqrt{-1}$ which are not real for any nonzero real value of k .

3.



4. Let n be a natural number such that $n^3 + (n+1)^3 = (n+2)^3$. Thus

$n^3 + n^3 + 3n^2 + 3n + 1 = n^3 + 6n^2 + 12n + 8$ and n satisfies

$n^3 - 3n^2 - 9n - 7 = 0$. This shows that $\frac{7}{n} = n^2 - 3n - 9$ is an

integer so that n , being a natural number must be 1 or 7 .

However $1^3 + 2^3 = 9 \neq 3^3 = 27$

and $7^3 + 8^3 = 855 \neq 9^3 = 729$, proving the assertion.

5. $$\sin\left(\frac{\pi}{24}\right) = \sin\left(\frac{\pi}{6} - \frac{\pi}{8}\right) = \sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{8}\right) - \cos\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{8}\right)$$

and so

$$\sin\left(\frac{\pi}{24}\right) = \frac{1}{2}\cos\left(\frac{\pi}{8}\right) - \frac{\sqrt{3}}{2}\sin\left(\frac{\pi}{8}\right) .$$

But

$$\cos^2(\tfrac{\pi}{8}) = \cos^2(\tfrac{1}{2}\,\tfrac{\pi}{4}) = \frac{1 + \cos(\frac{\pi}{4})}{2} = \frac{1 + \frac{\sqrt{2}}{2}}{2} = \frac{2 + \sqrt{2}}{4}$$

and so

$$\sin^2(\tfrac{\pi}{8}) = 1 - \cos^2(\tfrac{\pi}{8}) = 1 - \frac{2 + \sqrt{2}}{4} = \frac{2 - \sqrt{2}}{4} \quad .$$

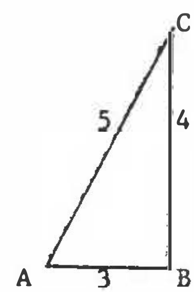
Since $\cos(\frac{\pi}{8})$ and $\sin(\frac{\pi}{8})$ are positive,

$$\cos(\tfrac{\pi}{8}) = \frac{\sqrt{2 + \sqrt{2}}}{2} \quad \text{and} \quad \sin(\tfrac{\pi}{8}) = \frac{\sqrt{2 - \sqrt{2}}}{2} \quad .$$

Therefore,

$$\sin(\tfrac{\pi}{24}) = \frac{1}{2}\,\frac{\sqrt{2 + \sqrt{2}}}{2} - \frac{\sqrt{3}}{2}\,\frac{\sqrt{2 - \sqrt{2}}}{2} = \frac{\sqrt{2 + \sqrt{2}} - \sqrt{3}\,\sqrt{2 - \sqrt{2}}}{4}$$

6. Let ΔABC have sides AB, BC, and CA of lengths 3, 4, and 5 respectively. By the law of cosines



$$\cos(\sphericalangle ABC) = \frac{1}{2\,\overline{AB}\,\overline{BC}}\left|(\overline{AC})^2 - (\overline{BC})^2 - (\overline{AB})^2\right|$$

$$= \frac{1}{2\cdot 3\cdot 4}\,(5^2 - 4^2 - 3^2) = 0$$

Since $0 < \sphericalangle ABC < 180^\circ$, ABC must be 90°.

7. Three positive real numbers x, y and z are lengths of the sides of a triangle if and only if $x + y > z$, $x + z > y$ and $y + z > x$. Suppose that x^2, y^2 and z^2 are lengths of the sides of a triangle.

$$(x+y)^2 = x^2 + y^2 + xy > x^2 + y^2 > z^2 \ .$$

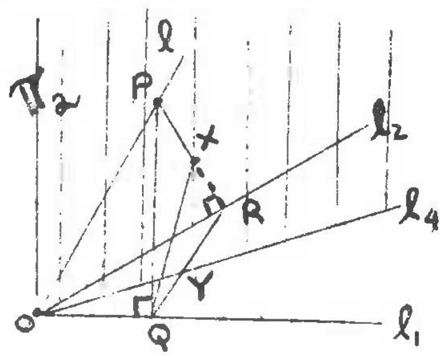
Since x, y and z are positive, $x + y > z$. Similarly $x + z > z$

and $y + z > x$. The conclusion is that if x, y, and z are positive, and if x^2, y^2, and z^2 are the lengths of the sides of a triangle, then x, y and z are the length of the sides of a triangle. The problem will be complete if we now show that $a(b+c-a) = (\sqrt{a(b+c-a)})^2$, $b(c+a-b) = (\sqrt{b(c+a-b)})^2$, and $c(a+b-c) = (\sqrt{c(a+b-c)})^2$ are lengths of the sides of a triangle whenever a, b and c are. However $a > 0$ $b + c > a$ implies $a(b+c-a) > 0$. Similarly $b(c+a-b) > 0$ and $c(a+b-c) > 0$. Finally $a(b+c-a) + b(c+a-b) - c(a+b-c) = c^2 - (a-b)^2$. Since $c > a - b$ then $c^2 - (a-b)^2 > 0$, and
$a(b+c-a) + b(c+a-b) > c(a+b-c)$. Similarly
$a(b+c-a) + c(a+b-c) > b(c+a-b)$ and
$b(c+a-b) + c(a+b-c) > a(b+c-a)$, proving the problem.

8. Suppose x, y and z satisfy the constraints $x \geq 0$, $y \geq 0$, $z \geq 0$, $x + y \leq 2$ and $3x - y + z \leq 1$. Adding the last two inequalities gives $4x + z \leq 3$ so that $z \leq 3 - 4x$. Since $x \geq 0$, this implies that $z \leq 3$. Since $x + y \leq 2$, $x + y + z \leq 5$. However the values $x = 0$, $y = 2$, and $z = 3$ satisfy the given conditions and sum to 5. Therefore the maximum value of $x + y + z$ subject to the given conditions is 5.

9. Let π_1 be the plane containing ℓ_1, ℓ_2 and ℓ_3, and let 0 be the point of intersection of ℓ_1, ℓ_2, ℓ_3 and ℓ. Let ℓ_4 be a line in π_1 bisecting the angle between ℓ_1 and ℓ_2, and let π_2 be the plane through ℓ_4 perpendicular to π_1. We claim that ℓ lies on π_1. Let $P \neq 0$ be a point on ℓ.

Drop perpendiculars from P to ℓ_1 and ℓ_2 respectively, and suppose these perpendiculars meet ℓ_1 and ℓ_2 at Q and R respectively. π_2 must intersect one of the segments PR and PQ so suppose π_2 intersects PR at the point X .



Since $\sphericalangle POQ = \sphericalangle POR$, $\sphericalangle OQP = \sphericalangle ORP = 90^\circ$ and $\overline{OP} = \overline{OP}$, ΔOPQ is congruent to ΔOPR . Thus $\overline{OQ} = \overline{OR}$, and $\overline{PQ} = \overline{PR}$. Let RQ intersect ℓ_4 at Y . Since ΔOYQ is congruent to ΔORY, $\overline{YQ} = \overline{YR}$. Since $\sphericalangle XYQ = \sphericalangle XYR = 90^\circ$, $\overline{XY} = \overline{XY}$ and $\overline{YQ} = \overline{YR}$, ΔXYR is congruent to ΔXYQ so $\overline{XR} = \overline{XQ}$. However $\overline{PR} = \overline{PX} + \overline{XR} = \overline{QP}$, so $\overline{QP} = \overline{PX} + \overline{ZQ}$. This can only occur if $X = P$, proving that ℓ is on π_2 . If π_3 is the plane meeting the bisector of the angle between the lines ℓ_2 and ℓ_3 , and perpendicular to π_1 , then a similar argument shows that ℓ is on π_3 . Since two distinct planes meet in a line, ℓ is thus the intersection of π_2 and π_3 which is perpendicular to π_1 .

10. Given: $az - 2by + cx = 0$ (1)

$ac > b^2$ (2)

Suppose $xz > y^2$ (3)

From (2) and (3) $acxz > b^2y^2$ (4)

From (1) $(az+cx)^2 = 4b^2y^2$ which by (4) $(az+cx)^2 < 4acxz$.

Thus $(az+cx)^2 - 4acxz = (az-cx)^2 < 0$. This is impossible since $az - cx$ is a real number. Thus (3) does not hold and $xz - y^2 \leq 0$.