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MAXIMIZATION OF AREAS OF CIRCLE-INSCRIBED TRIANGLES BY COMPUTERIZED SEQUENCES

William J. Bruce

It has been shown (Bruce) that the limit of the sequence

$$\alpha_n = 60° + (-1/2)^n(\alpha_o - 60°),$$

where α_n represents the vertex angle of any circle-inscribed isosceles triangle that bounds the center of the circle, proves that the triangle of maximum area inscribed in a circle is an equilateral triangle. The formation of each successive isosceles triangle was done by using a side of the previous isosceles triangle as base for the next one (Figure 1). It was shown also that no loss of generality results in starting with an isosceles triangle that bounds the center of the circle.

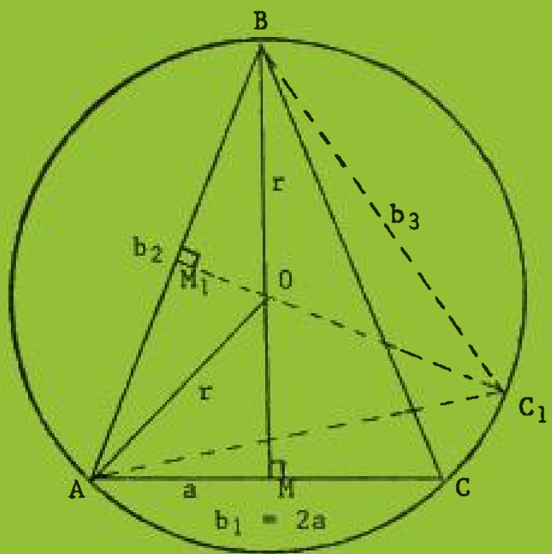

Figure 1.

IN THIS ISSUE

Consider now the sequences formed by the lengths of the bases and the altitudes of the successive isosceles triangles. Since the base increases and decreases as the vertex angle increases and decreases, the sequence of bases yields an oscillating sequence, provided, of course, that the triangle bounds the center. Similarly, the sequence of altitudes yields an oscillating sequence. The associated sequence of areas can easily be formed once these two sequences are determined. These algebraic sequences are formed by reference to Figure 1 as follows:

For the bases, we have

$$b_1 = AC = 2a, \quad a < r,$$

$$b_2^2 = (AB)^2 = (MB)^2 + (AM)^2$$

$$= \left(r + \sqrt{r^2 - a^2}\right)^2 + a^2,$$

so

$$b_2 = \sqrt{2r}\,\sqrt{r + \sqrt{r^2 - a^2}}$$

$$= \sqrt{2r}\,\sqrt{r + B_1}, \quad \text{where} \quad B_1 = \sqrt{r^2 - a^2},$$

$$b_3^2 = (M_1C_1)^2 + (BM_1)^2$$

$$= \left(r + \sqrt{r/2}\,\sqrt{r - B_1}\right)^2 + \left(\frac{\sqrt{2r}\,\sqrt{r + B_1}}{2}\right)^2,$$

thus

$$b_3 = \sqrt{2r}\,\sqrt{r + \sqrt{r/2}\,\sqrt{r - B_1}}$$

$$= \sqrt{2r}\,\sqrt{r + B_2}, \quad \text{where} \quad B_2 = \sqrt{r/2}\,\sqrt{r - B_1},$$

$$b_4 = \sqrt{2r}\,\sqrt{r + \sqrt{r/2}\,\sqrt{r - B_2}}$$

$$= \sqrt{2r}\,\sqrt{r + B_3}, \quad \text{where} \quad B_3 = \sqrt{r/2}\,\sqrt{r - B_2},$$

$$\cdots$$

$$b_n = \sqrt{2r}\,\sqrt{r + B_{n-1}}, \quad \text{where} \quad B_{n-1} = \sqrt{r/2}\,\sqrt{r - B_{n-2}}, \quad n \geq 3.$$

The corresponding altitudes are easily found to be

$$h_1 = r + B_1, \quad \text{where} \quad B_1 = \sqrt{r^2 - a^2},$$

$$h_2 = r + B_2, \quad \text{where} \quad B_2 = \sqrt{r/2}\,\sqrt{r - B_1},$$

$$h_3 = r + B_3, \quad \text{where} \quad B_3 = \sqrt{r/2}\,\sqrt{r - B_2},$$

$$\cdots$$

$$h_n = r + B_n, \quad \text{where} \quad B_n = \sqrt{r/2}\,\sqrt{r - B_{n-1}}, \quad n \geq 2.$$

We now have the sequence of areas, namely,

$$K_1 = a(r + B_1), \quad \text{where} \quad B_1 = \sqrt{r^2 - a^2},$$

$$K_2 = \frac{\sqrt{2r}}{2}\sqrt{r + B_1}\,(r + B_2), \quad \text{where} \quad B_2 = \sqrt{r/2}\,\sqrt{r - B_1},$$

$$K_3 = \frac{\sqrt{2r}}{2}\sqrt{r + B_2}\,(r + B_3), \quad \text{where} \quad B_3 = \sqrt{r/2}\,\sqrt{r - B_2},$$

$$\cdots$$

$$K_n = \frac{\sqrt{2r}}{2}\sqrt{r + B_{n-1}}\,(r + B_n), \quad \text{where} \quad B_n = \sqrt{r/2}\,\sqrt{r - B_{n-1}}, \quad n \geq 2.$$

The limit of the area sequence is not easily deduced; however, approximations are determined easily by computer. A computer print-out for the sequence of areas, using $r = 4$ and starting with $a = 3$, is as follows:

$K_1 = 19.937253933194$

$K_2 = 20.583005244258$

$K_3 = 20.732635087590$

$K_4 = 20.771795050671$

$K_5 = 20.781382524594$

$K_6 = 20.783805767667$

$$K_7 = 20.784408347096$$
$$K_8 = 20.784559399988$$
$$K_9 = 20.784597112463$$
$$K_{10} = 20.784606546941$$
$$\cdots$$
$$K_{16} = 20.784609690059 \ .$$

Note that the sequence tends to settle down to 20.7846 after ten terms. Since it is shown easily that the area of an equilateral triangle inscribed in a circle of radius r is given by $\frac{3}{4}\sqrt{3}\,r^2$, this formula can be used to show that the computer result is indeed an approximation for the area when $r = 4$.

Another pair of sequences, with terms corresponding to those of the above sequences, can be formed by using the trigonometric functions to express the lengths of the bases and the altitudes of the isosceles triangles. With reference to Figure 2, we obtain

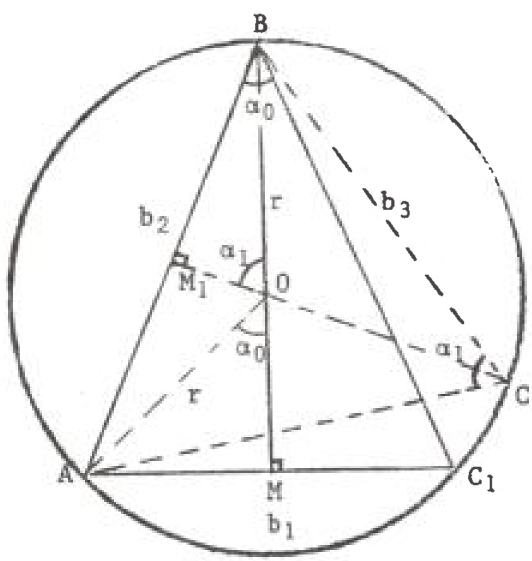

Figure 2

$$b_1 = 2r \sin \alpha_o ,$$

$$b_2 = 2r \cos \frac{\alpha_o}{2} ,$$

$$b_3 = 2r \cos \frac{\alpha_1}{2}$$

$$= 2r \sqrt{\frac{1 + \cos \alpha_1}{2}} = \sqrt{2}\, r \sqrt{1 + \sin \frac{\alpha_o}{2}} , \quad \text{since} \quad \cos \alpha_1 = \sin \frac{\alpha_o}{2} ,$$

$$b_4 = 2r \cos \frac{\alpha_2}{2}$$

$$= 2r \sqrt{\frac{1 + \cos \alpha_2}{2}} = \sqrt{2}\, r \sqrt{1 + \sin \frac{\alpha_1}{2}} = \sqrt{2}\, r \sqrt{1 + \sqrt{\frac{1 - \cos \alpha_1}{2}}}$$

$$= \sqrt{2}\, r \sqrt{1 + \sqrt{\frac{1 - \sin \frac{\alpha_o}{2}}{2}}} = \sqrt{2}\, r \sqrt{1 + B_1}, \quad \text{where} \quad B_1 = \sqrt{\frac{1 - \sin \frac{\alpha_o}{2}}{2}} ,$$

$$b_5 = 2r \cos \frac{\alpha_3}{2} = \sqrt{2}\, r \sqrt{1 + B_2}, \quad \text{where} \quad B_2 = \sqrt{\frac{1 - B_1}{2}} ,$$

$$\cdots$$

$$b_n = \sqrt{2}\, r \sqrt{1 + B_{n-3}}, \quad \text{where} \quad B_{n-3} = \sqrt{\frac{1 - B_{n-4}}{2}} , \quad n \geq 5.$$

The corresponding altitudes are given by

$$h_1 = r + r \cos \alpha_o = r(1 + \cos \alpha_o),$$

$$h_2 = r(1 + \cos \alpha_1) = r\left(1 + \sin \frac{\alpha_o}{2}\right) ,$$

$$h_3 = r\left(1 + \sin \frac{\alpha_1}{2}\right) = r\left(1 + \sqrt{\frac{1 - \sin \frac{\alpha_o}{2}}{2}}\right)$$

$$= r(1 + B_1), \quad \text{where} \quad B_1 = \sqrt{\frac{1 - \sin \frac{\alpha_o}{2}}{2}} ,$$

$$h_4 = r\left(1 + \sin\frac{\alpha_2}{2}\right) = r(1 + B_2), \quad \text{where} \quad B_2 = \sqrt{\frac{1 - B_1}{2}},$$

$$\cdots$$

$$h_n = r(1 + B_{n-2}), \quad \text{where} \quad B_{n-2} = \sqrt{\frac{1 - B_{n-3}}{2}}, \quad n \geq 4.$$

We now have the sequence of areas, namely,

$$K_1 = r^2 \sin\alpha_o (1 + \cos\alpha_o),$$

$$K_2 = r^2 \cos\frac{\alpha_o}{2}\left(1 + \sin\frac{\alpha_o}{2}\right),$$

$$K_3 = \frac{\sqrt{2}}{2}\, r^2 \sqrt{1 + \sin\frac{\alpha_o}{2}}\,(1 + B_1), \quad \text{where} \quad B_1 = \sqrt{\frac{1 - \sin\frac{\alpha_o}{2}}{2}},$$

$$K_4 = \frac{\sqrt{2}}{2}\, r^2 \sqrt{1 + B_1}\,(1 + B_2), \quad \text{where} \quad B_2 = \sqrt{\frac{1 - B_1}{2}},$$

$$\cdots$$

$$K_n = \frac{\sqrt{2}}{2}\, r^2 \sqrt{1 + B_{n-3}}\,(1 + B_{n-2}), \quad \text{where} \quad B_{n-2} = \sqrt{\frac{1 - B_{n-3}}{2}}, \quad n \geq 4.$$

The computer print-out for this sequence of areas, using $r = 4$ and starting with $\alpha_o = \arcsin 0.75$, is as follows:

$K_1 = 19.937253933194$

$K_2 = 20.583005244258$

$K_3 = 20.732635087590$

$K_4 = 20.771795050671$

$K_5 = 20.781382524594$

$K_6 = 20.783805767667$

$K_7 = 20.784408347096$

$K_8 = 20.784559399988$

$$K_9 = 20.784597112463$$

$$K_{10} = 20.784606546941$$

$$\cdots$$

$$K_{16} = 20.784609690059$$

Note again that this sequence tends to 20.7846. Its terms have the same values as those of the other area sequence because we chose $\alpha_o = \arcsin 0.75$, which is equivalent to choosing $a = 3$ in the algebraic sequence. This indicates that the two different area sequences are probably correct as presented and also serves as a check on the computer programming and calculations.

That the area sequences are monotonically increasing can be shown easily. From Figure 2, which is representative of all such successive pairs of isosceles triangles, we have $\alpha_1 + \frac{\alpha_o}{2} = 90°$ and that

$$\text{Area } \Delta ABC = r^2 \sin \alpha_o (1 + \cos \alpha_o)$$

$$\text{Area } \Delta ABC_1 = r^2 \sin \alpha_1 (1 + \cos \alpha_1).$$

Now

$$\frac{\text{Area } \Delta ABC_1}{\text{Area } \Delta ABC} = \frac{\sin \alpha_1 (1 + \cos \alpha_1)}{\sin \alpha_o (1 + \cos \alpha_o)}$$

$$= \frac{\cos \frac{\alpha_o}{2} \left(1 + \sin \frac{\alpha_o}{2}\right)}{4 \sin \frac{\alpha_o}{2} \cos \frac{\alpha_o}{2} \left(1 - \sin^2 \frac{\alpha_o}{2}\right)}$$

$$= \frac{1}{4 \sin \frac{\alpha_o}{2} \left(1 - \sin \frac{\alpha_o}{2}\right)}$$

$$> 1 \quad \text{unless} \quad \sin \frac{\alpha_o}{2} = 1/2, \quad \text{that is,} \quad \alpha_o = 60°.$$

This can be seen readily by considering $4x(1 - x)$, where $x = \sin \frac{\alpha_o}{2}$. By the elementary method of completing the square, we find that 1 is the maximum value of $4x(1 - x)$ and this occurs when $x = 1/2$. Thus 1 is the minimum value of the reciprocal function and the above result follows, which proves that the area sequences do increase monotonically.

REFERENCE

Bruce, William J., "Maximization of Areas of Circle-Inscribed Triangles by an Oscillating Algebraic Sequence," *Delta-K*, Vol. XV, No. 3, February 1976, Edmonton, AB: Mathematics Council of The Alberta Teachers' Association.

NCTM Increases Services

A new section, "Sharing Teaching Ideas," and the addition of a September issue are new features of the *Mathematics Teacher*. "Sharing Teaching Ideas" is designed to inform readers of classroom-tested ideas on topics related to the secondary curriculum. A regular September *Mathematics Teacher* means nine issues per membership and subscription year.

The *Arithmetic Teacher* has added a new feature, "From the File," which provides ideas in file card format for elementary classrooms - teaching ideas that have been used with good results by other teachers. Also watch for the new *Arithmetic Teacher* size, 21.5cm x 28cm, beginning with the October 1977 issue.

The number of issues of the *Journal for Research in Mathematics Education* was increased to five last year. This year the *Mathematics Student* will be published eight times during the school year (October - May), double the number of issues previously published.

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