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MAXIMIZATION OF AREAS OF CIRCLE-INSCRIBED TRIANGLES BY COMPUTERIZED SEQUENCES

William J. Bruce

It has been shown (Bruce) that the limit of the sequence

$$\alpha_n = 60° + (-1/2)^n(\alpha_o - 60°),$$

where α_n represents the vertex angle of any circle-inscribed isosceles triangle that bounds the center of the circle, proves that the triangle of maximum area inscribed in a circle is an equilateral triangle. The formation of each successive isosceles triangle was done by using a side of the previous isosceles triangle as base for the next one (Figure 1). It was shown also that no loss of generality results in starting with an isosceles triangle that bounds the center of the circle.

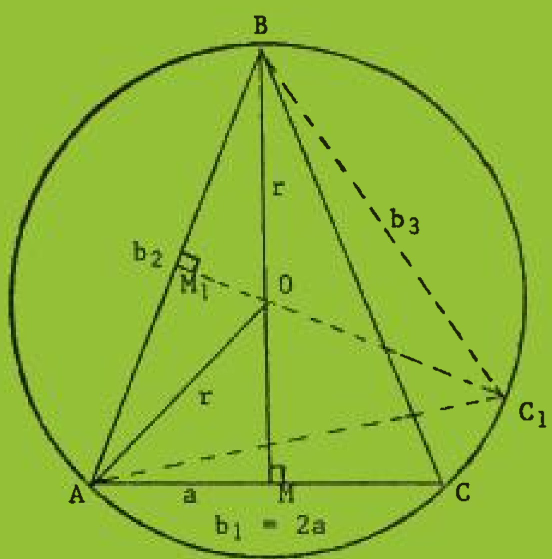

Figure 1.

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Consider now the sequences formed by the lengths of the bases and the altitudes of the successive isosceles triangles. Since the base increases and decreases as the vertex angle increases and decreases, the sequence of bases yields an oscillating sequence, provided, of course, that the triangle bounds the center. Similarly, the sequence of altitudes yields an oscillating sequence. The associated sequence of areas can easily be formed once these two sequences are determined. These algebraic sequences are formed by reference to Figure 1 as follows:

For the bases, we have

$$b_1 = AC = 2a, \quad a < r,$$

$$b_2^2 = (AB)^2 = (MB)^2 + (AM)^2$$

$$= \left(r + \sqrt{r^2 - a^2}\right)^2 + a^2,$$

so

$$b_2 = \sqrt{2r}\,\sqrt{r + \sqrt{r^2 - a^2}}$$

$$= \sqrt{2r}\,\sqrt{r + B_1}, \quad \text{where} \quad B_1 = \sqrt{r^2 - a^2},$$

$$b_3^2 = (M_1C_1)^2 + (BM_1)^2$$

$$= \left(r + \sqrt{r/2}\,\sqrt{r - B_1}\right)^2 + \left(\frac{\sqrt{2r}\,\sqrt{r + B_1}}{2}\right)^2,$$

thus

$$b_3 = \sqrt{2r}\,\sqrt{r + \sqrt{r/2}\,\sqrt{r - B_1}}$$

$$= \sqrt{2r}\,\sqrt{r + B_2}, \quad \text{where} \quad B_2 = \sqrt{r/2}\,\sqrt{r - B_1},$$

$$b_4 = \sqrt{2r}\,\sqrt{r + \sqrt{r/2}\,\sqrt{r - B_2}}$$

$$= \sqrt{2r}\,\sqrt{r + B_3}, \quad \text{where} \quad B_3 = \sqrt{r/2}\,\sqrt{r - B_2},$$

$$\cdots$$

$$b_n = \sqrt{2r}\,\sqrt{r + B_{n-1}}, \quad \text{where} \quad B_{n-1} = \sqrt{r/2}\,\sqrt{r - B_{n-2}}, \quad n \geq 3.$$

The corresponding altitudes are easily found to be

$$h_1 = r + B_1, \quad \text{where} \quad B_1 = \sqrt{r^2 - a^2},$$

$$h_2 = r + B_2, \quad \text{where} \quad B_2 = \sqrt{r/2}\,\sqrt{r - B_1},$$

$$h_3 = r + B_3, \quad \text{where} \quad B_3 = \sqrt{r/2}\,\sqrt{r - B_2},$$

$$\cdots$$

$$h_n = r + B_n, \quad \text{where} \quad B_n = \sqrt{r/2}\,\sqrt{r - B_{n-1}}, \quad n \geq 2.$$

We now have the sequence of areas, namely,

$$K_1 = a(r + B_1), \quad \text{where} \quad B_1 = \sqrt{r^2 - a^2},$$

$$K_2 = \frac{\sqrt{2r}}{2}\sqrt{r + B_1}\,(r + B_2), \quad \text{where} \quad B_2 = \sqrt{r/2}\,\sqrt{r - B_1},$$

$$K_3 = \frac{\sqrt{2r}}{2}\sqrt{r + B_2}\,(r + B_3), \quad \text{where} \quad B_3 = \sqrt{r/2}\,\sqrt{r - B_2},$$

$$\cdots$$

$$K_n = \frac{\sqrt{2r}}{2}\sqrt{r + B_{n-1}}\,(r + B_n), \quad \text{where} \quad B_n = \sqrt{r/2}\,\sqrt{r - B_{n-1}}, \quad n \geq 2.$$

The limit of the area sequence is not easily deduced; however, approximations are determined easily by computer. A computer print-out for the sequence of areas, using $r = 4$ and starting with $a = 3$, is as follows:

$K_1 = 19.937253933194$

$K_2 = 20.583005244258$

$K_3 = 20.732635087590$

$K_4 = 20.771795050671$

$K_5 = 20.781382524594$

$K_6 = 20.783805767667$

$$K_7 = 20.784408347096$$

$$K_8 = 20.784559399988$$

$$K_9 = 20.784597112463$$

$$K_{10} = 20.784606546941$$

$$\cdots$$

$$K_{16} = 20.784609690059 .$$

Note that the sequence tends to settle down to 20.7846 after ten terms. Since it is shown easily that the area of an equilateral triangle inscribed in a circle of radius r is given by $\frac{3}{4}\sqrt{3}\, r^2$, this formula can be used to show that the computer result is indeed an approximation for the area when $r = 4$.

Another pair of sequences, with terms corresponding to those of the above sequences, can be formed by using the trigonometric functions to express the lengths of the bases and the altitudes of the isosceles triangles. With reference to Figure 2, we obtain

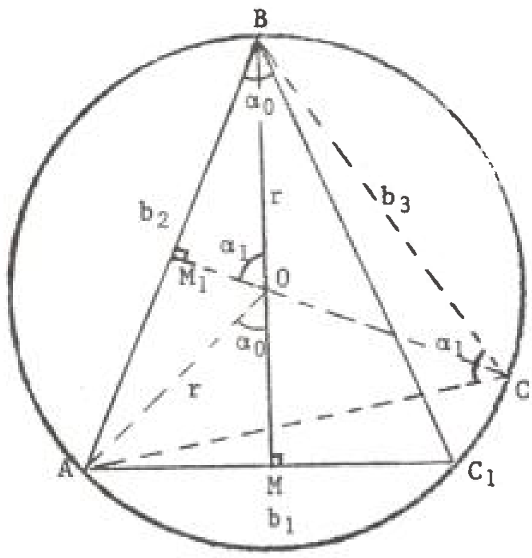


Figure 2

$$b_1 = 2r \sin \alpha_o ,$$

$$b_2 = 2r \cos \frac{\alpha_o}{2} ,$$

$$b_3 = 2r \cos \frac{\alpha_1}{2}$$

$$= 2r \sqrt{\frac{1 + \cos \alpha_1}{2}} = \sqrt{2}\, r \sqrt{1 + \sin \frac{\alpha_o}{2}} , \quad \text{since} \quad \cos \alpha_1 = \sin \frac{\alpha_o}{2} ,$$

$$b_4 = 2r \cos \frac{\alpha_2}{2}$$

$$= 2r \sqrt{\frac{1 + \cos \alpha_2}{2}} = \sqrt{2}\, r \sqrt{1 + \sin \frac{\alpha_1}{2}} = \sqrt{2}\, r \sqrt{1 + \sqrt{\frac{1 - \cos \alpha_1}{2}}}$$

$$= \sqrt{2}\, r \sqrt{1 + \sqrt{\frac{1 - \sin \frac{\alpha_o}{2}}{2}}} = \sqrt{2}\, r \sqrt{1 + B_1}, \quad \text{where} \quad B_1 = \sqrt{\frac{1 - \sin \frac{\alpha_o}{2}}{2}} ,$$

$$b_5 = 2r \cos \frac{\alpha_3}{2} = \sqrt{2}\, r \sqrt{1 + B_2}, \quad \text{where} \quad B_2 = \sqrt{\frac{1 - B_1}{2}} ,$$

$$\cdots$$

$$b_n = \sqrt{2}\, r \sqrt{1 + B_{n-3}}, \quad \text{where} \quad B_{n-3} = \sqrt{\frac{1 - B_{n-4}}{2}} , \quad n \geq 5.$$

The corresponding altitudes are given by

$$h_1 = r + r \cos \alpha_o = r(1 + \cos \alpha_o),$$

$$h_2 = r(1 + \cos \alpha_1) = r\left(1 + \sin \frac{\alpha_o}{2}\right) ,$$

$$h_3 = r\left(1 + \sin \frac{\alpha_1}{2}\right) = r\left(1 + \sqrt{\frac{1 - \sin \frac{\alpha_o}{2}}{2}}\right)$$

$$= r(1 + B_1), \quad \text{where} \quad B_1 = \sqrt{\frac{1 - \sin \frac{\alpha_o}{2}}{2}} ,$$

$$h_4 = r\left(1 + \sin\frac{\alpha_2}{2}\right) = r(1 + B_2), \quad \text{where} \quad B_2 = \sqrt{\frac{1 - B_1}{2}} ,$$

$$\cdots$$

$$h_n = r(1 + B_{n-2}), \quad \text{where} \quad B_{n-2} = \sqrt{\frac{1 - B_{n-3}}{2}} , \quad n \geq 4.$$

We now have the sequence of areas, namely,

$$K_1 = r^2 \sin\alpha_o(1 + \cos\alpha_o),$$

$$K_2 = r^2 \cos\frac{\alpha_o}{2}\left(1 + \sin\frac{\alpha_o}{2}\right) ,$$

$$K_3 = \frac{\sqrt{2}}{2} r^2 \sqrt{1 + \sin\frac{\alpha_o}{2}}(1 + B_1), \quad \text{where} \quad B_1 = \sqrt{\frac{1 - \sin\frac{\alpha_o}{2}}{2}} ,$$

$$K_4 = \frac{\sqrt{2}}{2} r^2 \sqrt{1 + B_1}(1 + B_2), \quad \text{where} \quad B_2 = \sqrt{\frac{1 - B_1}{2}} ,$$

$$\cdots$$

$$K_n = \frac{\sqrt{2}}{2} r^2 \sqrt{1 + B_{n-3}}(1 + B_{n-2}), \quad \text{where} \quad B_{n-2} = \sqrt{\frac{1 - B_{n-3}}{2}} , \quad n \geq 4.$$

The computer print-out for this sequence of areas, using $r = 4$ and starting with $\alpha_o = \arcsin 0.75$, is as follows:

$K_1 = 19.937253933194$

$K_2 = 20.583005244258$

$K_3 = 20.732635087590$

$K_4 = 20.771795050671$

$K_5 = 20.781382524594$

$K_6 = 20.783805767667$

$K_7 = 20.784408347096$

$K_8 = 20.784559399988$

$$K_9 = 20.784597112463$$

$$K_{10} = 20.784606546941$$

$$\cdots$$

$$K_{16} = 20.784609690059$$

Note again that this sequence tends to 20.7846. Its terms have the same values as those of the other area sequence because we chose $\alpha_o = \arcsin 0.75$, which is equivalent to choosing $a = 3$ in the algebraic sequence. This indicates that the two different area sequences are probably correct as presented and also serves as a check on the computer programming and calculations.

That the area sequences are monotonically increasing can be shown easily. From Figure 2, which is representative of all such successive pairs of isosceles triangles, we have $\alpha_1 + \frac{\alpha_o}{2} = 90°$ and that

$$\text{Area } \Delta ABC = r^2 \sin \alpha_o (1 + \cos \alpha_o)$$

$$\text{Area } \Delta ABC_1 = r^2 \sin \alpha_1 (1 + \cos \alpha_1).$$

Now

$$\frac{\text{Area } \Delta ABC_1}{\text{Area } \Delta ABC} = \frac{\sin \alpha_1 (1 + \cos \alpha_1)}{\sin \alpha_o (1 + \cos \alpha_o)}$$

$$= \frac{\cos \frac{\alpha_o}{2} \left(1 + \sin \frac{\alpha_o}{2}\right)}{4 \sin \frac{\alpha_o}{2} \cos \frac{\alpha_o}{2} \left(1 - \sin^2 \frac{\alpha_o}{2}\right)}$$

$$= \frac{1}{4 \sin \frac{\alpha_o}{2} \left(1 - \sin \frac{\alpha_o}{2}\right)}$$

> 1 unless $\sin \frac{\alpha_o}{2} = 1/2$, that is, $\alpha_o = 60°$.

This can be seen readily by considering $4x(1 - x)$, where $x = \sin \frac{\alpha_o}{2}$. By the elementary method of completing the square, we find that 1 is the maximum value of $4x(1 - x)$ and this occurs when $x = 1/2$. Thus 1 is the minimum value of the reciprocal function and the above result follows, which proves that the area sequences do increase monotonically.

REFERENCE

Bruce, William J., "Maximization of Areas of Circle-Inscribed Triangles by an Oscillating Algebraic Sequence," *Delta-K*, Vol. XV, No. 3, February 1976, Edmonton, AB: Mathematics Council of The Alberta Teachers' Association.

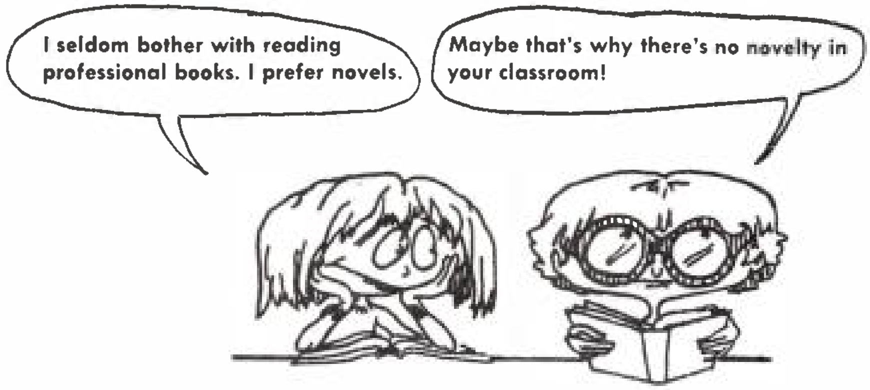
NCTM Increases Services

A new section, "Sharing Teaching Ideas," and the addition of a September issue are new features of the *Mathematics Teacher*. "Sharing Teaching Ideas" is designed to inform readers of classroom-tested ideas on topics related to the secondary curriculum. A regular September *Mathematics Teacher* means nine issues per membership and subscription year.

The *Arithmetic Teacher* has added a new feature, "From the File," which provides ideas in file card format for elementary classrooms - teaching ideas that have been used with good results by other teachers. Also watch for the new *Arithmetic Teacher* size, 21.5cm x 28cm, beginning with the October 1977 issue.

The number of issues of the *Journal for Research in Mathematics Education* was increased to five last year. This year the *Mathematics Student* will be published eight times during the school year (October - May), double the number of issues previously published.

Reprinted from NCTM Bulletin for Leaders, *June 1977*.



NCTM's Newest Nine

Reprinted from Newsletter, *National Council of Teachers of Mathematics, May 1977.*

We've borrowed these cartoon characters from one of our delightful new books, *How to Study Mathematics*, to call your attention to NCTM's Newest Nine - nine new books packed with classroom helps for you and your students. Perhaps one or several - or all - will be just what you need on your personal or classroom bookshelf or what your school needs in its library. Peruse the reviews, use the convenient order blank following to let us know what to send you, and enjoy!

Four-Dimensional Geometry - an Introduction, by Adrien L. Hess, contains everything you've always wanted to know about four-dimensional geometry but didn't quite know how to ask! This 32-page booklet presents a history and a definition of four-dimensional geometry, selected drawings and models, and instructions on how to study the configurations. $1.60

The *Mathematics Projects Handbook* is also by Adrien L. Hess. This book is useful for helping teachers help students come up with intriguing ideas for projects. Many ideas are incorporated in the text of the 48-page handbook, and an extensive bibliography points the way to materials that will stimulate further ideas. $1.70

Guidelines for the Tutor of Mathematics, by Henry S. Kepner, Jr., and David R. Johnson, is full of ideas and encouragement for the student who has been asked to tutor a fellow student in mathematics. Although the prospective tutors will undoubtedly be successful in mathematics, they may lack teaching skills, and that is the gap that this useful 32-page booklet proposes to fill. It can be read in a single sitting and then used as a reference when specific questions or problems arise. $1.30

How to Study Mathematics: A Handbook for Students, by James Margenau and Michael Sentlowitz, is a readable self-help for the struggling but earnest mathematics student. Captivating cartoons enliven the text. A valuable feature is the list of diagnoses ("All those quizzes and tests make me nervous") and prescriptions. A helpful addition to the school library or mathematics classroom, the lively 32-page booklet sells for $1.30.

Calculus: Readings from the "Mathematics Teacher" is edited by Louise S. Grinstein and Brenda Michaels. Grinstein and Michaels have scoured back issues of the *Mathematics Teacher* from its first issue in 1908 through 1974 for articles on calculus. The editors include an annotated bibliography of further readings at the close of each section of the book, as well as helpful author and subject indexes. With 225 pages, *Calculus* sells for $4.90.

An In-Service Handbook for Mathematics Education is edited by Alan Osborne. It is a study of in-service education: the reasons for it; a reporting of the "what-is and the what-ought-to-be of in-service education" according to teachers and supervisors; an analysis of policies, processes, and procedures for in-service education in terms of the rules and responsibilities of individuals and institutions participating in the in-service effort; and a consideration of the future of in-service education for mathematics teachers. This 256-page handbook sells for $4.80.

Bulletin Board Ideas for Elementary and Middle School Mathematics is by Seaton E. Smith, Jr. Any teacher who has ever struggled to come up with an idea for a bulletin-board display will appreciate this booklet, which is rich in ideas and in full-color photographs of real teacher-made bulletin boards. This valuable 56-page book sells for $3.00.

How to Draw a Straight Line, by A.B. Kempe, was first published in London in 1877 as an expanded version of a "Lecture on Linkages" by the author. Now NCTM has reprinted this venerable classic, an elegant 64-page hardback, as the sixth in its Classics in Mathematics Education series. $4.90

NCTM's 1977 Yearbook, *Organizing for Mathematics Instruction,* edited by Joe Crosswhite and Robert Reys, discusses such alternative teaching approaches as individualization, survival groups, simulations, open schools, and others. Seventeen authors writing on 12 topics tell how to organize for such approaches and give specific illustrative examples. A timely, non-thematic essay on hand-held calculators is included. Hardback, $8.50 (as usual, NCTM members are entitled to a $2 discount on one copy of the yearbook).

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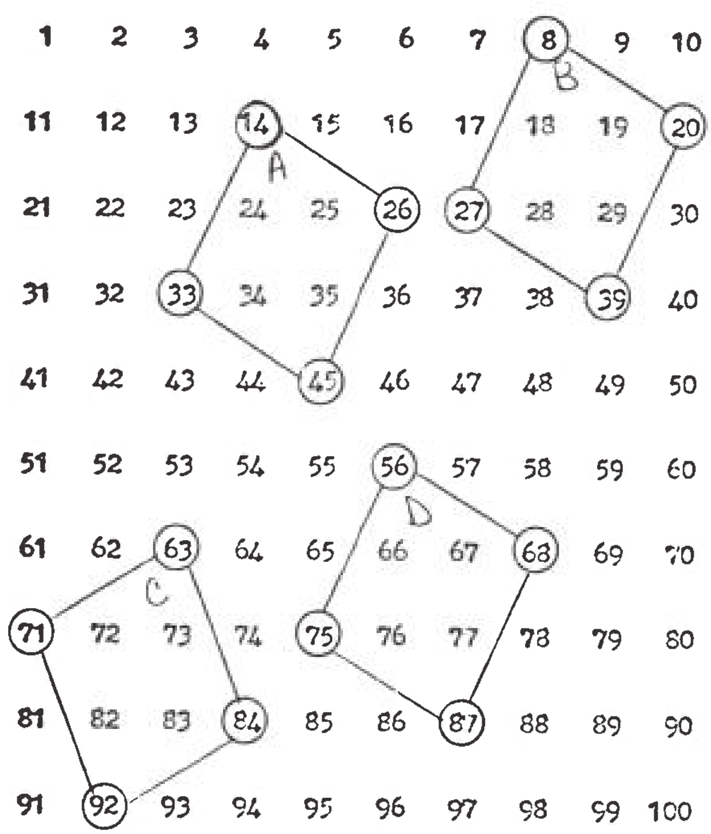
Rhombi on the Hundred Square: Sums, Patterns, and Proof

David R. Duncan and *Bonnie H. Litwiller*
Professors of Mathematics
University of Northern Iowa
Cedar Falls, Iowa

The Hundred Square is a rich source of number patterns. Seeking these patterns gives the student experience in formalizing and verifying conjectures. A valuable by-product is the maintenance of computational skills.

Activity I: Consider rhombi drawn on the Hundred Square as shown in Figure I.

Figure I



1. Add the four numbers which represent the vertices of each rhombus; call the sum V.

2. Add the four interior numbers of each rhombus; call this sum I.

The results of the above calculations appear in Table I.

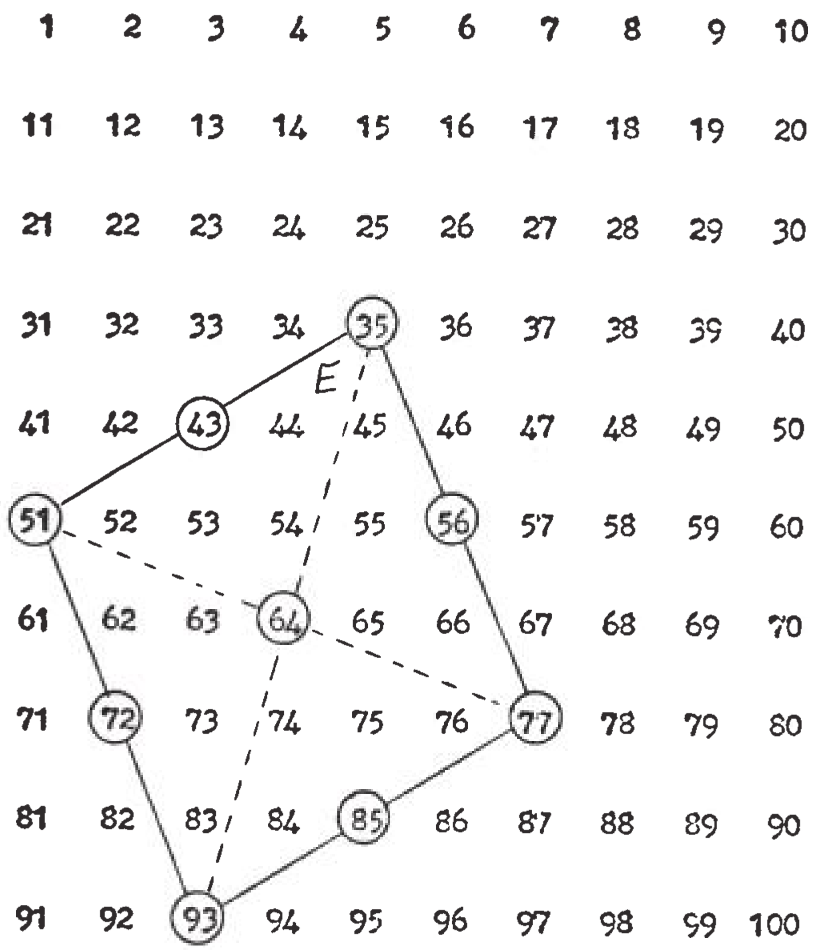
Table I

Rhombus	V	I
A	118	118
B	94	94
C	310	310
D	286	286

Observe that for each rhombus, the sums V and I are the same; the ratio of these corresponding sums is 1/1. To see if this pattern is true for all rhombi of this size, check other rhombi positioned differently on the Hundred Square.

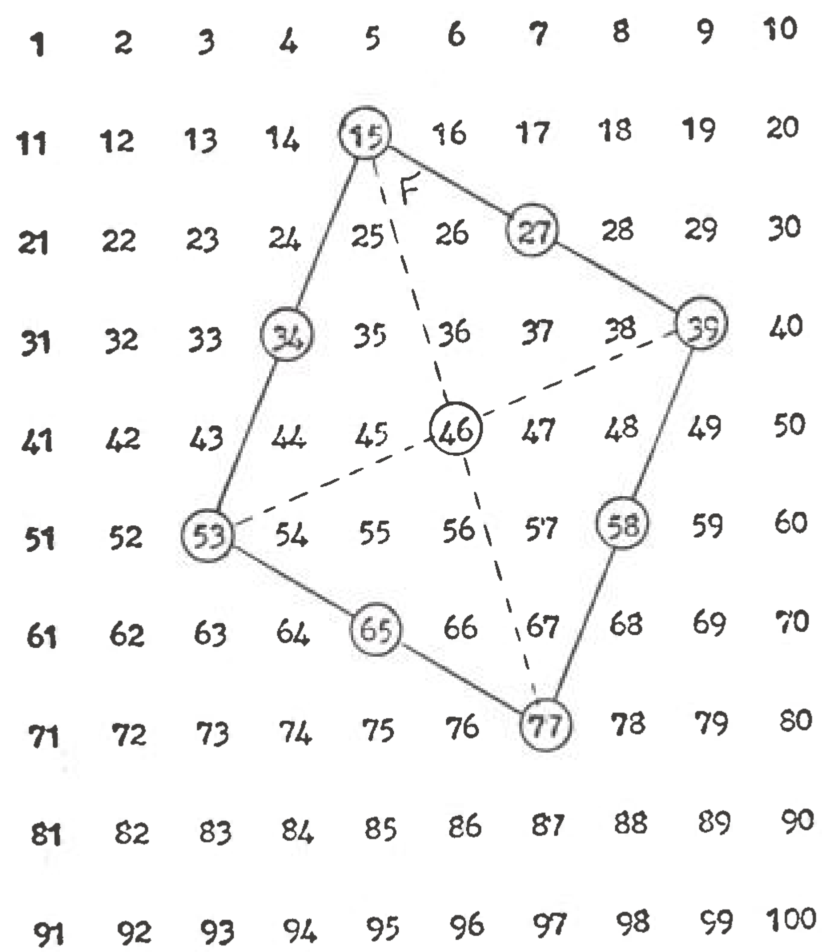
Activity II: Consider the rhombi drawn on Figures II and III. Note that these rhombi have three circled numbers per side.

Figure II



1 2 3 4 5 6 7 8 9 10
11 12 13 14 15 16 17 18 19 20
21 22 23 24 25 26 27 28 29 30
31 32 33 34 35 36 37 38 39 40
E
41 42 43 44 45 46 47 48 49 50
51 52 53 54 55 56 57 58 59 60
61 62 63 64 65 66 67 68 69 70
71 72 73 74 75 76 77 78 79 80
81 82 83 84 85 86 87 88 89 90
91 92 93 94 95 96 97 98 99 100

Figure III



1. Add the four numbers which represent the vertices of each rhombus; call this sum V.

2. Add the eight numbers which lie on the perimeter; call this sum P.

3. Add the seventeen interior numbers; call this sum I.

4. Let C be the "center number" of each rhombus.

5. Find I - C.

The results of these calculations appear in Table II.

Table II

Rhombus	V	P	I	C	I - C
E	256	512	1088	64	1024
F	184	368	782	46	736

Observations:

1. $V = 4C$ or $\frac{V}{C} = \frac{4}{1}$; the sum of the vertices is four times the center number.

2. $\frac{P}{I} = \frac{8}{17}$. Both 512/1088 and 368/782 can be renamed 8/17. Note that there are eight numbers on the perimeter of the rhombus and 17 numbers in the interior of the rhombus.

3. $\frac{I - C}{C} = \frac{16}{1}$. Both 1024/64 and 736/46 can be renamed 16/1. Note that there are 16 numbers in the interior of the rhombus excluding the center number.

Be sure to check these patterns for other rhombi of this size positioned elsewhere on the Hundred Square.

We shall verify pattern 3; that is, $\frac{I - C}{C} = 16$.

Figure IV: A General Hundred Square

1 11 . . .	b						
11							
.							
.							
.							
10a+1	10a+b	10a+(b+1)	10a+(b+2)	10a+(b+3)	10a+(b+4)	10a+(b+5)	10a+(b+6)
	10(a+1)+b	10(a+1)+(b+1)	10(a+1)+(b+2)	10(a+1)+(b+3)	10(a+1)+(b+4)	10(a+1)+(b+5)	10(a+1)+(b+6)
	10(a+2)+b	10(a+2)+(b+1)	10(a+2)+(b+2)	10(a+2)+(b+3)	10(a+2)+(b+4)	10(a+2)+(b+5)	10(a+2)+(b+6)
	10(a+3)+b	10(a+3)+(b+1)	10(a+3)+(b+2)	10(a+3)+(b+3)	10(a+3)+(b+4)	10(a+3)+(b+5)	10(a+3)+(b+6)
	10(a+4)+b	10(a+4)+(b+1)	10(a+4)+(b+2)	10(a+4)+(b+3)	10(a+4)+(b+4)	10(a+4)+(b+5)	10(a+4)+(b+6)
	10(a+5)+b	10(a+5)+(b+1)	10(a+5)+(b+2)	10(a+5)+(b+3)	10(a+5)+(b+4)	10(a+5)+(b+5)	10(a+5)+(b+6)
	10(a+6)+b	10(a+6)+(b+1)	10(a+6)+(b+2)	10(a+6)+(b+3)	10(a+6)+(b+4)	10(a+6)+(b+5)	10(a+6)+(b+6)

The sum I - C is depicted as follows:

$$
\begin{array}{l}
10(a + 1) + (b + 2) \\
10(a + 1) + (b + 3) \\
10(a + 2) + (b + 2) \\
10(a + 2) + (b + 3) \\
10(a + 2) + (b + 4) \\
10(a + 2) + (b + 5) \\
10(a + 3) + (b + 1) \\
10(a + 3) + (b + 2) \\
10(a + 3) + (b + 4) \\
10(a + 3) + (b + 5) \\
10(a + 4) + (b + 1) \\
10(a + 4) + (b + 2) \\
10(a + 4) + (b + 3) \\
10(a + 4) + (b + 4) \\
10(a + 5) + (b + 3) \\
10(a + 5) + (b + 4) \\
\hline
160a + 480 + (16b + 48)
\end{array}
$$

$$
\begin{aligned}
\frac{I - C}{C} &= \frac{160a + 480 + 16b + 48}{10(a + 3) + (b + 3)} \\
&= \frac{16(10a + b + 33)}{10a + b + 33} \\
&= 16
\end{aligned}
$$

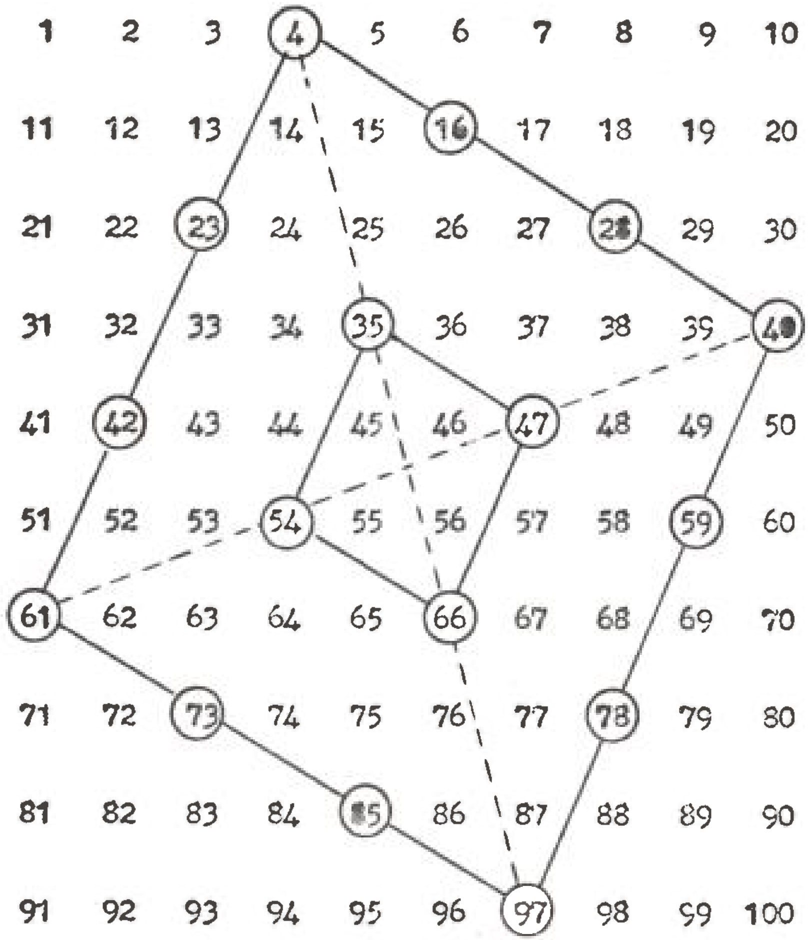
Questions for the reader and his/her students:

1. Do the patterns found in Activities I and II hold on the Addition Table, Multiplication Table, and the Subtraction Table?

2. Can you prove the other patterns?

3. Can you find similar patterns on the following nested rhombus configuration? There are four circled numbers per side. There is no center number; there is a center rhombus.

Figure V


1 2 3 4 5 6 7 8 9 10
11 12 13 14 15 16 17 18 19 20
21 22 23 24 25 26 27 28 29 30
31 32 33 34 35 36 37 38 39 40
41 42 43 44 45 46 47 48 49 50
51 52 53 54 55 56 57 58 59 60
61 62 63 64 65 66 67 68 69 70
71 72 73 74 75 76 77 78 79 80
81 82 83 84 85 86 87 88 89 90
91 92 93 94 95 96 97 98 99 100

Teaching Aids

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Gr. VI to IX	$5.75;	$12.75

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The four series of supplementary books all follow the same format: puzzles, mazes, line drawings, decodes, optical illusions, and others. They are designed to help, especially, the slow, gifted, bilingual, and average students. The pages are in "TEAR OUT" form and can be easily reproduced or laminated. The grade level in all books is not easily identified allowing use at the elementary or high school level, depending on the level of the student.

Each series covers the area of math as the name indicates. Measurement in the Geometry series is metric.

Authors in the four series are: Alice Clack and Carol Leitch.

Report from Last Annual NCTM Meeting

Bob Holt
Vice President
Mathematics Council, ATA

The following are a few statements that were made at sessions of the 55th Annual NCTM Conference in Cincinnati. I am not implying either agreement or disagreement with the statements, but merely that some of these statements and their implications might provide us with food-for-thought.

- Areas that have adopted minimum standards have found that they have also become maximum standards, especially if there is testing involved.
- New Jersey has state minimum standards and New Jersey has a mess.
- A Grade VI study showed no significant difference between males and females in likes-dislikes of Math.
- Grades VI and VII are the times when students decide whether they like or dislike Math.
- Many different metric systems are developing in the U.S.A.
- There is a movement in the U.S. to use "L" for litre instead of "ℓ."
- Teacher organizations are not as concerned with improving instruction as they used to be.
- Politicians have taken over education.
- Political decisions give only lip service to the student in making decisions.
- In Louisiana, 48 percent of college freshmen require remedial math courses.
- Southern Louisiana Bell Telephone requires job applicants to write an exam - 49 percent are rejected because they fail the mathematical part of the exam.
- Chief superintendents in Louisiana are elected.
- On the average, a person measures 10 times a day.
- For a new word to become part of you, 28 experiences are needed with it.
- One school system stopped social promotions - the first year they had 1200 failures, the second 1100, and the third 700.
- Individual work is working at own slow rate with own slow amount of learning; but do these equal zero?
- Put a calculator in the student's hands and let him join the world.

- There is a calculator on the market that will handle the basic operations for fractions and mixed numbers ($24.95).
- You don't require good penmanship from a student before he can take typing, so why should you require good calculation ability before you let a student use a calculator.
- Calculators are here to stay.
- Are we wasting a lot of time and money helping students who are beyond help?
- We have no control over the calculator outside of school.
- How many of your students have one or more calculators in their homes?
- It is not whether to use a calculator, but where.
- Research shows students of 1972 can add as well as students of 1937.
- The operation of a calculator should be a basic - look at the people who are operating the cash registers.
- "Why can't Johnny add?" - in the '70s the public is saying the problem is the "New Math"; in the '80s the public will blame the calculator.
- What are the basics in Math for everyday life, for other subject areas, for learning more Math, for job entry, for job advancement ...

New Minicalculator Publication

The Iowa Council of Teachers of Mathematics (ICTM) has published its Monograph - *1977, High School Activities for the Calculator.* Copies are available for $2.00 (ICTM member) or $2.50 (nonmember) from Ann Robinson, 2712 Cedar Heights Drive, Cedar Falls, IA 50613.

Reprinted from NCTM Bulletin for Leaders, *June 1977.*

MCTM Minicalculator Publication

The Michigan Council of Teachers of Mathematics (MCTM) has published the monograph *Uses of the Calculator in School Mathematics, K-12,* which emphasizes the practical uses of the calculator in grades K-12.

Copies are available for $1.50 from the MCTM, Box 16124, Lansing, MI 48902. Make cheques payable to MCTM.

Reprinted from NCTM Bulletin for Leaders, *July 1977.*

PUZZLES & PROBLEMS

1. If 3 cats eat 3 mice in 3 minutes, how long will it take 50 cats to eat 50 mice?

2. I have many coins in coins in my pocket, but I can't make change for a nickel, a dime, a quarter, a fifty-cent piece, or a dollar. What is the largest sum of money I can have?

3. How may 100 be expressed using only 4 like numerals? For example, $100 = 99 + \frac{9}{9}$ (using four 9s). There are more than 20 ways to do this. Can you find 10 ways?

4. Find numerals for as many numbers as you can from 1 to 100 using four 4s. For example,

$$12 = \frac{4}{0.4} + \frac{4}{\sqrt{4}} \qquad 35 = \frac{44}{4} + 4!$$

5. What is the largest number that can be formed using three 2s? Using three 4s?

6. Alice, Betty, and Carol set off simultaneously from the same point on a circular track 3 km in circumference. Alice cycles at 14 km/h, Betty runs at 8 km/h, and Carol walks at 5 km/h, all in the same direction. How long will it be before they all come together again at some point on the track?

7. The boys in Grade 5 rate 2 spears as worth 3 fishhooks and a knife, and will give 25¢ for 3 spears, 2 knives, and a fishhook together. How much will they pay for each article separately?

8. A hungry hunter came upon two shepherds, one of whom had 3 small loaves of bread, and the other 5, all of the same size. The loaves were equally divided among the three, and the hunter paid 80¢ for his share. How should the shepherds divide the money?

9. The following is a problem in addition. Each letter is a placeholder for a digit, and each time a letter appears, it represents the same digit. Can you determine what each letter represents?

$$\begin{array}{r} \text{S E N D} \\ \text{M O R E} \\ \hline \text{M O N E Y} \end{array}$$

10. A man drove his car 1 km to the top of a mountain at 30 km/h. How fast must he drive 1 km down the other side in order to average 60 km/h for the 2 km trip?

11. A clock gains 12 min/h. It is correct at 17 00 h. What is the correct time when it reads midnight?

12. In a large department store a father and son mounted a moving escalator to toytown. Both walked up, but the son in his excitement took two steps to each one taken by his father. Father noted that his son took 28 steps while he took 21 steps to reach the top. How many steps were in sight at one time on the escalator?

13. Three men, let us call them A, B, and C, went out to dinner in Montreal. One was English, one Scottish, and one Welsh. The waiter was asked to guess their nationalities. He said that A was English, B was not English, and C was not Welsh. Only one of his answers was correct. What were the nationalities of A, B, and C?

14. If a bag contained twelve balls, four red, four white, and four blue, what is the smallest number you would have to withdraw to ensure that you had:
    i. two balls of the same color?
    ii. one ball of each color?
    iii. three red balls?

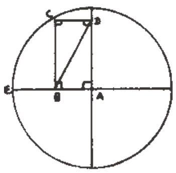
15. Frankie and Johnny went for a long cycle ride. Johnny maintained a speed of 15 km/h but Frankie could only manage a steady 10 km/h. After riding for 5 h, Johnny turned back. How far from their starting point would the two meet?

16. Three glasses sit on a counter -- two inverted, and one upright.
    a. Invert two glasses simultaneously so that all the glasses are upright.
    b. Can you get all the glasses upright in exactly two moves, inverting two glasses each time?
    c. Can you do it in three moves? Four moves?
    d. Can you find a general strategy for any number of moves?

I GOTTCHA!

contributed by Jay Caturay, Prince Rupert Secondary School

I wonder how many of our readers can give the answers to the following problems within 10 seconds for each question?

1. What is 32% of 25?
2. 3/8 + 2/3 - 1/24 =
3. Thirty divided by a half.
4. 

Given.

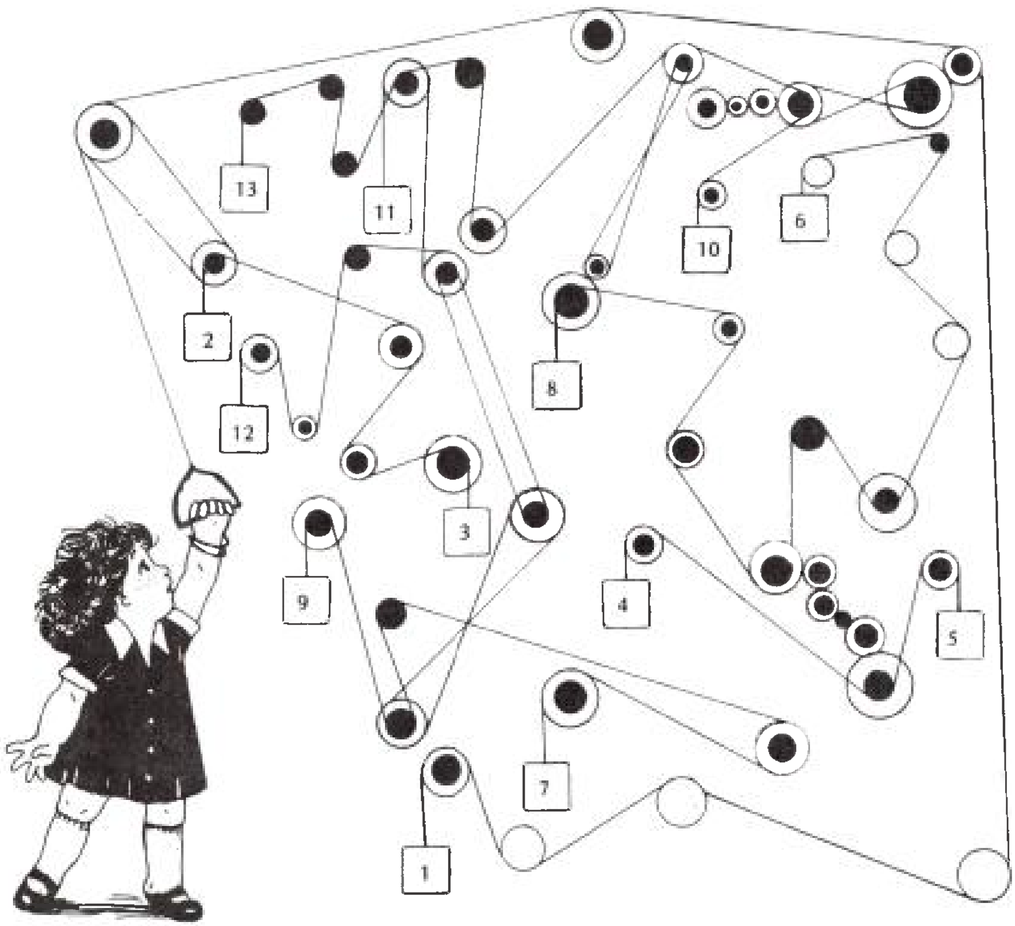
radius $\overline{AE}$ = 10 cm

$\overline{BE}$ = 2 cm

Find: $\overline{BD}$

Answers:

1. 8 32/100 x 25 = 32 x 25/100 = 32 x 1/4 = 8
2. 1 a/b + c/d = ad + bc/bd
   3/8 + 2/3 = 9 + 16/24 = 25/24
3. 60, not 15
   $\frac{30}{1/2} = \frac{30 \times 2/1}{1/2 \times 2/1} = 60$
4. $\overline{BD}$ = 10 cm Draw AC $\overline{AE} = \overline{AC}$, $\overline{AC} = \overline{BD}$
   Hence $\overline{AE} = \overline{BD}$ = 10 cm



PULLEY PROBLEM

When Josie pulls down on the lever, the whole system of loads, belts, and pulleys will begin to move. Which loads will go up, and which loads will come down?

Remember: wheels belted to each other revolve in the same direction, unless the belt crosses itself. In that case, the wheels revolve in opposite directions. Wheels in contact with each other on their outer rims will rotate in opposite directions.

NATIONAL COUNCIL OF SUPERVISORS OF MATHEMATICS POSITION PAPER ON BASIC MATHEMATICAL SKILLS

INTRODUCTION

The currently popular slogan "Back to the Basics" has become a rallying cry of many who perceive a need for certain changes in education. The result is a trend that has gained considerable momentum and has initiated demands for programs and evaluations which emphasize narrowly defined skills.

Mathematics educators find themselves under considerable pressure from boards of education, legislatures, and citizens' groups who are demanding instructional programs which will guarantee acquisition of computational skills. Leaders in mathematics education have expressed a need for clarifying what are the basic skills needed by students who hope to participate successfully in adult society.

The narrow definition of basic skills which equates mathematical competence with computational ability has evolved as a result of several forces:

1. Declining scores on standardized achievement tests and college entrance examinations;
2. Reactions to the results of the National Assessment of Educational Progress;
3. Rising costs of education and increasing demands for accountability;
4. Shifting emphasis in mathematics education from curriculum content to instructional methods and alternatives;
5. Increased awareness of the need to provide remedial and compensatory programs;
6. The widespread publicity given to each of the above by the media.

This widespread publicity, in particular, has generated a call for action from governmental agencies, educational organizations, and community groups. In responding to these calls, the National Institute of Education adopted the area of basic skills as a major priority. This resulted in a Conference on Basic Mathematical Skills and Learning, held in Euclid, Ohio, in October, 1975.

The National Council of Supervisors of Mathematics (NCSM), during the 1976 Annual Meeting in Atlanta, Georgia, met in a special session to discuss the Euclid Conference Report. More than 100 members participating in that session expressed the need for a unified position on basic mathematical skills which would enable them to provide more effective leadership within their respective school systems, to give adequate rationale and direction in their tasks of implementing basic mathematics programs, and to appropriately expand the definition of basic skills. Hence, by an overwhelming majority, they mandated the NCSM to establish a task force to formulate a position on basic mathematical skills. This statement is the result of that effort.

RATIONALE FOR THE EXPANDED DEFINITION

There are many reasons why basic skills must include more than computation. The present technological society requires daily use of such skills as estimating, problem solving, interpreting data, organizing data, measuring, predicting, and applying mathematics to everyday situations. The changing needs of society, the explosion of the amount of quantitative data, and the availability of computers and calculators demand a redefining of the priorities for basic mathematics skills. In recognition of the inadequacy of computation alone, NCSM is going on record as providing both a general list of basic mathematical skills and a clarification of the need for such an expanded definition of basic skills.

Any list of basic skills must include computation. However, the role of computational skills in mathematics must be seen in the light of the contributions they make to one's ability to use mathematics in everyday living. In isolation, computational skills contribute little to one's ability to participate in mainstream society. Combined effectively with the other skill areas, they provide the learner with the basic mathematical ability needed by adults.

DEFINING BASIC SKILLS

The NCSM views basic mathematical skills as falling under ten vital areas. The ten skill areas are interrelated and many overlap with each other and with other disciplines. All are basic to pupils' development of the ability to reason effectively in varied situations.

This expanded list is presented with the conviction that mathematics education must not emphasize computational skills to the neglect of other critical areas of mathematics. The ten components of basic mathematical skills are listed below, but the order of their listing should not be interpreted as indicating either a priority of importance or a sequence for teaching and learning.

Furthermore, as society changes our ideas about which skills are basic also change. For example, today our students should learn to measure in both the customary and metric systems, but in the future the significance of the customary system will be mostly historical. There will also be increasing emphasis on when and how to use hand-held calculators and other electronic devices in mathematics.

TEN BASIC SKILL AREAS

Problem Solving

Learning to solve problems is the principal reason for studying mathematics. Problem solving is the process of applying previously acquired knowledge to new and unfamiliar situations. Solving word problems in texts is one form of problem solving, but students also should be faced with non-textbook problems. Problem-solving strategies involve posing questions, analyzing situations, translating results, illustrating results, drawing diagrams, and using trial and error. In solving problems, students need to be able to apply the rules of logic necessary to arrive at valid conclusions. They must be able to determine which facts are relevant. They should be unfearful of arriving at tentative conclusions and they must be willing to subject these conclusions to scrutiny.

Applying Mathematics to Everyday Situations

The use of mathematics is interrelated with all computation activities. Students should be encouraged to take everyday situations, translate them into mathematical expressions, solve the mathematics, and interpret the results in light of the initial situation.

Alertness to the Reasonableness of Results

Due to arithmetic errors or other mistakes, results of mathematical work are sometimes wrong. Students should learn to inspect all results and to check for reasonableness in terms of the original problem. With the increase in the use of calculating devices in society, this skill is essential.

Estimation and Approximation

Students should be able to carry out rapid approximate calculations by first rounding off numbers. They should acquire some simple techniques for estimating quantity, length, distance, weight, etc. It is also necessary to decide when a particular result is precise enough for the purpose at hand.

Appropriate Computational Skills

Students should gain facility with addition, subtraction, multiplication, and division with whole numbers and decimals. Today it must be recognized that long, complicated computations will usually be done with a calculator. Knowledge of single-digit number facts is essential and mental arithmetic is a valuable skill. Moreover, there are everyday situations which demand recognition of, and simple computation with, common fractions.

Because consumers continually deal with many situations that involve percentage, the ability to recognize and use percents should be developed and maintained.

Geometry

Students should learn the geometric concepts they will need to function effectively in the 3-dimensional world. They should have knowledge of concepts such as point, line, plane, parallel, and perpendicular. They should know basic properties of simple geometric figures, particularly those properties which relate to measurement and problem-solving skills. They also must be able to recognize similarities and differences among objects.

Measurement

As a minimum skill, students should be able to measure distance, weight, time, capacity, and temperature. Measurement of angles and calculations of simple areas and volumes are also essential. Students should be able to perform measurement in both metric and customary systems using the appropriate tools.

Reading, Interpreting, and Constructing Tables, Charts, and Graphs

Students should know how to read and draw conclusions from simple tables, maps, charts, and graphs. They should be able to condense numerical information into more manageable or meaningful terms by setting up simple tables, charts, and graphs.

Using Mathematics to Predict

Students should learn how elementary notions of probability are used to determine the likelihood of future events. They should learn to identify situations where immediate past experience does not affect the likelihood of future events. They should become familiar with how mathematics is used to help make predictions such as election forecasts.

Computer Literacy

It is important for all citizens to understand what computers can and cannot do. Students should be aware of the many uses of computers in society, such as their use in teaching/learning, financial transactions, and information storage and retrieval. The "mystique" surrounding computers is disturbing and can put persons with no understanding of computers at a disadvantage. The increasing use of computers by government, industry, and business demands an awareness of computer uses and limitations.

BASIC SKILLS AND THE STUDENT'S FUTURE

Anyone adopting a definition of basic skills should consider the "door-opening/door-closing" implications of the list. The following diagram illustrates expected outcomes associated with various amounts of skill development.

Scope of Skill Development	Expected Outcomes
EXPANDED SKILLS Mathematical skills beyond those described here plus a desire to learn more.	**POTENTIAL LEADERS** Employment and educational opportunities will continue to increase as mathematical skills continue to grow.
BASIC SKILLS The skills described here.	**EMPLOYMENT VERY LIKELY** Employment opportunities are predictable. Doors to further education opportunities are open.
MINIMAL SKILLS Limited skills, primarily computation. Little exposure to the other skill areas described here.	**LIMITED OPPORTUNITIES** Unemployment likely. Potential generally limited to low-level jobs.

MINIMUM ESSENTIALS FOR HIGH-SCHOOL GRADUATION

Today some school boards and state legislatures are starting to mandate mastery of minimum essential skills in reading and mathematics as a requirement for high-school graduation. In the process, they should consider the potential pitfalls of doing this without an appropriate definition of "basic skills." If the mathematics requirements are set inordinately high, then a significant number of students may not be able to graduate. On the other hand, if the mathematics requirements are set too low and mathematical skills are too narrowly defined, the result could be a sterile mathematics program concentrating exclusively on learning of low-level mathematical skills. This position paper neither recommends nor condemns minimal competencies for high-school graduation. However, the ten components of basic skills stated here can serve as guidelines for state and local school systems that are considering the establishment of minimum essential graduation requirements.

DEVELOPING THE BASIC SKILLS

One individual difference among students is style or way of learning. In offering opportunities to learn the basic skills, options must be provided to meet these varying learning styles. The present "back-to-basics" movement may lead to an emphasis on drill and practice as a way to learn.

Certainly drill and practice is a viable option, but it is only one of many possible ways to bring about learning and to create interest and motivation in students. Learning centers, contracts, tutorial sessions, individual and small-group projects, games, simulations and community-based activities are some of the other options that can provide the opportunity to learn basic skills. Furthermore, to help students fully understand basic mathematical concepts, teachers should utilize the full range of activities and materials available, including objects the students can actually handle.

The learning of basic mathematical skills is a continuing process which extends through all of the years a student is in school. In particular, a tendency to emphasize computation while neglecting the other nine skill areas at the elementary level must be avoided.

EVALUATING AND REPORTING STUDENT PROGRESS

Any systematic attempt to develop basic skills must necessarily be concerned with evaluating and reporting pupil progress.

In evaluation, test results are used to judge the effectiveness of the instructional process and to make needed adjustments in the curriculum and instruction for the individual student. In general, both educators and the public have accepted and emphasized an overuse of and overconfidence in the results of standardized tests. Standardized tests yield comparisons between students and can provide a rank ordering of individuals, schools, or districts. However, standardized tests have several limitations including the following:

a. Items are not necessarily generated to measure a specific objective or instructional aim.
b. The tests measure only a sample of the content that makes up a program; certain outcomes are not measured at all.

Because they do not supply sufficient information about how much mathematics a student knows, standardized tests are not the best instruments available for reporting individual pupil growth. Other alternatives such as criterion tests or competency tests must be considered. In criterion tests, items are generated which measure the specific objectives of the program and which establish the student's level of mastery of these objectives. Competency tests are designed to determine if the individual has mastered the skills necessary for a certain purpose such as entry into the job market. There is also need for open-ended assessments such as observations, interviews, and manipulative tasks to assess skills which paper and pencil tests do not measure adequately.

Reports of pupil progress will surely be made. But, while standardized tests will probably continue to dominate the testing scene for several years, there is an urgent need to begin reporting pupil progress in other terms, such as criterion tests and competency measures. This will also demand an immediate and extensive program of inservice education to instruct the general public on the meaning and interpretation of such data and to enable teachers to use testing as a vital part of the instructional process.

Large scale testing, whether involving all students or a random sample, can result in interpretations which have great influence on curriculum revisions and development. Test results can indicate, for example, that a particular mathematical topic is being taught at the wrong time in the student's development and that it might better be introduced later or earlier in the curriculum. Or, the results might indicate that students are confused about some topic as a result of inappropriate teaching procedures. In any case, test results should be carefully examined by educators with special skills in the area of curriculum development.

CONCLUSION

The present paper represents a preliminary attempt by the National Council of Supervisors of Mathematics to clarify and communicate its position on basic mathematical skills. The NCSM position establishes a framework within which decisions on program planning and implementation can be made. It also sets forth the underlying rationale for identifying and developing basic skills and for evaluating pupils' acquisition of these competencies. The NCSM position underscores the fundamental belief of the National Council of Supervisors of Mathematics that any effective program of basic mathematical skills must be directed not "back" but *forward* to the essential needs of adults in the present and future.

Meetings of the National Council of Teachers of Mathematics

MISSOULA MEETING
16-18 March 1978
Missoula, Montana

56th ANNUAL MEETING
12-15 April 1978
San Diego, California

(Bob Holt, 1977-78 President, Mathematics Council, ATA, is on the Planning Committee for this conference.)

CHEYENNE MEETING
28-30 September 1978
Cheyenne, Wyoming

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Mathematics Council, ATA
17th Annual Conference

October 14 & 15 - Red Deer Lodge

Feature Speakers & Sessions-

ROBERT WIRTZ, Lecturer and writer from Carmel, California, will present the Friday evening session and a session for elementary teachers on Saturday.

DR. ERIC MACPHERSON, Dean of the Faculty of Education, University of Manitoba, will present the luncheon address and a session for secondary teachers.

Idea Exchanges - nine mini-sessions at all levels.

Pre-Registration- (Prior to September 30)

Detach and mail to: D. Hinde, Box 741 Lacombe, Alberta, TOC 1SO

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MATHEMATICS COUNCIL, ATA —— 17th ANNUAL CONFERENCE

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(including luncheon):

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Non-Member (incl. membership)	$23.50	___	$25.50	___
Student Members	$ 7.00	___	$ 9.00	___
Non-Members (incl. membership)	$11.00	___	$13.00	___
Additional luncheon tickets	$ 7.00 each			______

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