Determinants and the Coordinate Plane

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The purpose of this paper is to demonstrate the use of determinants as a tool for the development of aspects of high school geometry and trigonometry. Although this presentation does not presuppose a knowledge of determinants, prior contact with the subject (e.g. study of the properties, Cramer's rule, determinant equations) will unquestionably enhance the study. The writer experimented with such an introduction to determinants for several years as a ninth and tenth year mathematics teacher, and found this approach valuable from several points of view. Primarily, the mathematics is interesting, highly motivating, and lends itself well to student discovery. Secondly, determinants can be presented profitably to the slow learner as well as to the more capable student, since the expansion of a determinant reinforces the fundamental operations in an almost game atmosphere. This reinforcement may involve directed numbers, monomials and polynomials as elements of a determinant. Moreover, problems dealing with determinant factoring, the solution of linear and quadratic determinant equations, have proven fascinating to students at this level. Thirdly, the geometric and trigonometric applications in this algebraic environment provide the student with a real sense of continuity and unity of subject matter. Finally, determinants are important to the development of branches of higher mathematics. Indeed, if we can lay the foundation of this concept while mastering aspects of high school mathematics, it seems we ought to do so.

In order to position our theory sequentially within the framework of our geometry, we first postulate the area relationship of a rectangle as the product of a pair of adjacent sides, viz., $K_{Rect} = bh$. Since a diagonal of a rectangle divides its area into two right triangles of equal area, it follows that $K_{Rt\Delta} = \frac{1}{2}bh$.

Our first major objective will be to derive the (determinant) area formula for a triangle in the coordinate plane. To begin with, we define second- and third-order determinants as square arrays of elements of the form

$$\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

whose values are $x_1y_2 - x_2y_1$ and $x_1y_2z_3 + x_2y_3z_1 + x_3y_1z_2 - x_1y_3z_2 - x_2y_1z_3 - x_3y_2z_1$, respectively.

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the coordinates of the vertices of any triangle ABC in the coordinate plane (see Fig. 1). Through vertices A, B and C, draw lines parallel to the coordinate axes forming rectangle DBEF, so that

$$K_{\Delta ABC} = K_{DBEF} - K_{\Delta I} - K_{\Delta II} - K_{\Delta III}$$

Also,

$$K_{DBEF} = (x_2 - x_1)(y_3 - y_2) = x_2y_3 - x_2y_2 - x_1y_3 + x_1y_2$$

$$K_{\Delta I} = \tfrac{1}{2}(x_2 - x_1)(y_1 - y_2) = \tfrac{1}{2}(x_2y_1 - x_2y_2 - x_1y_1 + x_1y_2)$$

$$K_{\Delta II} = \tfrac{1}{2}(x_2 - x_3)(y_3 - y_2) = \tfrac{1}{2}(x_2y_3 - x_2y_2 - x_3y_3 + x_3y_2)$$

$$K_{\Delta III} = \tfrac{1}{2}(x_3 - x_1)(y_3 - y_1) = \tfrac{1}{2}(x_3y_3 - x_3y_1 - x_1y_3 + x_1y_1)$$

Substituting and collecting terms, we get

$$K_{\Delta ABC} = \tfrac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - x_1y_3 - x_2y_1 - x_3y_2)$$

which, by the definition of third-order determinant, may be expressed as

$$K_{\Delta ABC} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad [A]$$

This relationship produces a positive or negative area as we take the coordinates in a counterclockwise or clockwise order, respectively. To demonstrate the point, take a conveniently placed unit triangle whose coordinates are (0, 0), (0, 1) and (1, 0) as in Fig. 2. Starting at the origin, and taking the points clockwise

$$K_{\Delta} = \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = -\frac{1}{2}$$

If, however, we take the points counterclockwise, then

$$K_{\Delta} = \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = +\frac{1}{2}$$

Since in this elementary treatment of area we will be concerned with positive areas only, we will adopt the convention throughout of taking the points in a counterclockwise direction.[1*]

We now direct our attention to the area of the quadrilateral in the coordinate plane. Given four distinct points $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ and $D(x_4, y_4)$, vertices of convex quadrilateral ABCD. If diagonal AC is drawn, two triangles are formed, the sum of whose areas will be the area of the quadrilateral (see Fig. 3), that is

$$K_{ABCD} = K_{\Delta ABC} + K_{\Delta ACD} .$$

*Footnote references will be found at the conclusion of the article.

By [A]

$$K_{\Delta ABC} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad \text{and } K_{\Delta ACD} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}$$

Expanding these determinants and substituting,

$$K_{ABCD} = \tfrac{1}{2}[(x_1y_2 + x_3y_1 + x_2y_3 - x_3y_2 - x_2y_1 - x_1y_3) + (x_1y_3 + x_4y_1 + x_3y_4 - x_4y_3 - x_3y_1 - x_1y_4)]$$

$$= \tfrac{1}{2}[x_1(y_2 - y_4) + x_2(y_3 - y_1) + x_3(y_4 - y_2) + x_4(y_1 - y_3)]$$

$$= \tfrac{1}{2}[(x_1 - x_3)(y_2 - y_4) - (x_2 - x_4)(y_1 - y_3)]$$

The binomial in the brackets is of the form ad - bc, which by the definition can be written in the determinant form

$$K_{ABCD} = \frac{1}{2}\begin{vmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_4 & y_2 - y_4 \end{vmatrix}$$

or more compactly

$$K_{ABCD} = \frac{1}{2}\begin{vmatrix} \Delta x_1 & \Delta y_1 \\ \Delta x_2 & \Delta y_2 \end{vmatrix} \quad [B]$$

where the Δ-elements are the differences of the respective x- and y-coordinates of the endpoints of the diagonals of the quadrilateral.

ILLUSTRATIVE EXAMPLE: Show by coordinate methods and determinants the area of a parallelogram is equal to base times height (bh).

Place the parallelogram ABCD in the coordinate plane so that one vertex is at the origin, and one side lies along the x-axis as shown in the figure. The vertices may therefore be expressed as A(0, 0), B(b, 0), C(a + b, h) and D(a, h), where b is a base and h the altitude. (See Fig. 4)

By [B]

$$K_{ABCD} = \frac{1}{2}\begin{vmatrix} (a+b)-0 & h-0 \\ a-b & h-0 \end{vmatrix}$$

$$= \tfrac{1}{2}(bh + ah - ah + bh) = bh$$

Following is a sampling of problems that lend themselves well to solution by this technique:

1) Prove the area of a trapezoid is equal to $\frac{1}{2}h(b_1 + b_2)$.

2) Prove: A line which joins the midpoints of two sides of a triangle cuts off a triangle whose area is one-fourth the area of a given triangle.

3) Prove that a median of a triangle divides it into two triangles of equal area.

4) Prove that a diagonal of a parallelogram divides it into two triangles of equal area.

Since our quadrilateral area formula turns out to be a second-order determinant relationship, we are motivated to re-examine the notion of triangle area. Our purpose is to derive a second-order determinant relationship in places of the third-order form with which we have been involved up to now.

Accordingly, we consider the area of ΔABC with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ as shown in Fig. 5. Let P be any other point with coordinates (x_p, y_p). We first examine the area of quadrilateral ABPC.

By [B]

$$K_{ABPC} = \frac{1}{2}\begin{vmatrix} x_1 - x_p & y_1 - y_p \\ x_2 - x_3 & y_2 - y_3 \end{vmatrix}$$

Move point P to coincide with point B so that a double point exists at B with coordinates (x_2, y_2). The resultant figure is considered from two points of view: (1) as a triangle, viz., ΔABC, and (2), as a degenerated quadrilateral, viz., quad AB(P)C. Notice that side PC of quad AB(P)C falls on BC, which exists here in the dual capacity as a side of the triangle, and as a diagonal of the quadrilateral. Also, diagonal AP falls on side AB. Since the areas of the two figures, ΔABC and quad AB(P)C, are identical, we may write

$$K_{\Delta ABC} = K_{AB(P)C}$$

Relationship [B] applied to the right-hand member results in a new statement for the area of the triangle, that is

$$K_{\Delta ABC} = \frac{1}{2}\begin{vmatrix} x_1 - x_p & y_1 - y_p \\ x_2 - x_3 & y_2 - y_3 \end{vmatrix}$$

Since the vertex P has been moved to B, it follows that $x_p = x_2$ and $y_p = y_2$ and we may rewrite this last relationship in the form

$$K_{\Delta ABC} = \frac{1}{2}\begin{vmatrix} x_1 - x_2 & y_1 - y_2 \\ x_2 - x_3 & y_2 - y_3 \end{vmatrix}$$

or more compactly

$$K_{\Delta ABC} = \frac{1}{2}\begin{vmatrix} \Delta x_1 & \Delta y_1 \\ \Delta x_2 & \Delta y_2 \end{vmatrix} \quad [C]^{2}$$

While the right-handed members of [B] and [C] are apparently identical, it must be made clear that in the former, the Δ-elements represent the differences in the x- and y-coordinates of the end-points of the diagonals, whereas in the latter, the Δ-elements refer to the differences in the x- and y-coordinates of the end-points of the sides of the triangle, taken counterclockwise.

ILLUSTRATIVE EXAMPLE: Find the area of the triangle whose vertices are A(1, 0), B(4, 7) and C(6, -2).

$$K_{\Delta ABC} = \frac{1}{2}\begin{vmatrix} 3 & 7 \\ -5 & 2 \end{vmatrix} = \tfrac{1}{2}(6 + 35) = 20\tfrac{1}{2}$$

We now employ the results of the previous section in the development of the linear equation in the determinant form by considering three collinear points as the vertices of a triangle with zero area.

Given two points $A(x_1, y_1)$ and $B(x_2, y_2)$. We wish to consider the equation of the straight line which contains the two points A and B. Take any other point P(x, y) on this line, and consider the figure ABP as a triangle with zero area. It follows from [C] that

$$\begin{vmatrix} \Delta x & \Delta y \\ \Delta x_1 & \Delta y_1 \end{vmatrix} = 0 \quad [D] \qquad \text{where } \Delta x = x - x_1, \text{ and } \Delta x_1 = x_1 - x_2, \text{ etc.}$$

ILLUSTRATIVE EXAMPLE: Write the equation of the line which contains the points A(1, - 3) and B(5, 6).

Consider any other point P(x, y) contained in the line AB.

By [D] $\begin{vmatrix} x - 1 & y - (-3) \\ 1 - 5 & -3 - 6 \end{vmatrix} = 0$ Expressed in standard form $9x - 4y - 21 = 0$.

Another application of this concept is the problem of writing the equation of the line through a point $P(x_3, y_3)$ parallel to the line containing the points $A(x_1, y_1)$ and $B(x_2, y_2)$ as shown in fig. 5.

Consider any other point Q(x, y) contained in the required line.

Since

$QP \parallel AB$, then $K_{\Delta ABQ} = K_{\Delta ABP}$.

By relationship [B], $\begin{vmatrix} \Delta x & \Delta y \\ \Delta x_1 & \Delta y_1 \end{vmatrix} = \begin{vmatrix} \Delta x_1 & \Delta y_1 \\ \Delta x_2 & \Delta y_2 \end{vmatrix}$ [E]3

where $\Delta x = x - x_1$, $\Delta x_1 = x_1 - x_2$, etc.

ILLUSTRATIVE EXAMPLE: Write the equation of the line which passes through the point (3, 2), and is parallel to the line containing the points (-3, 0) and (4, -2).

By [E] $\begin{vmatrix} x - (-3) & y - 0 \\ -3 - 4 & 0 - (-2) \end{vmatrix} = \begin{vmatrix} -3 - 4 & 0 - (-2) \\ 4 - 3 & -2 - 2 \end{vmatrix}$

Expressed in standard form, $2(x + 3) + 7y = 28 - 2$

$$2x + 7y = 20$$

We now direct our attention to the applications of determinants to the subject of trigonometry, more specifically, to the derivation of the trigonometric sum formulas by use of determinants.

Consider the unit circle with the center at the origin, with the angles x, y, and x-y in standard position, and whose rays intersect the circle at A(1, 0), B(cos(x-y), sin(x-y)), C(cos y, sin y) and D(cos x, sin x), as shown in fig. 7. Since angle DOC is equal to x-y, it follows that triangles BOA and DOC are congruent, and thus equal in area. Hence we may write

$$K_{\Delta BOA} = K_{\Delta DOC}$$

By [C]

$$\begin{vmatrix} \cos(x-y) - 0 & \sin(x-y) - 0 \\ 0 - 1 & 0 - 0 \end{vmatrix} = \begin{vmatrix} \cos x - 0 & \sin x - 0 \\ 0 - \cos y & 0 - \sin y \end{vmatrix}$$

so that

$$\begin{vmatrix} \cos(x-y) & \sin(x-y) \\ -1 & 0 \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\cos y & -\sin y \end{vmatrix}$$

which, when expressed in standard form, produces

$$\sin(x-y) = \sin x \cos y - \cos x \sin y.$$

To derive the cosine difference formula we consider triangles DOB and COA in Fig. 7. As in the previous case, these triangles are congruent, and thus equal in area. Hence we may write

$$K_{\Delta DOB} = K_{\Delta COA}$$

By [C]

$$\begin{vmatrix} \cos x - 0 & \sin x - 0 \\ 0 - \cos(x-y) & 0 - \sin(x-y) \end{vmatrix} = \begin{vmatrix} \cos y - 0 & \sin y - 0 \\ 0 - 1 & 0 - 0 \end{vmatrix}$$

so that

$$\begin{vmatrix} \cos x & \sin x \\ -\cos(x-y) & -\sin(x-y) \end{vmatrix} = \begin{vmatrix} \cos y & \sin y \\ -1 & 0 \end{vmatrix}$$

Expressing this in standard form

$$(-\sin(x-y))\cos x + (\cos(x-y))\sin x = \sin y.$$

Since $\sin(x-y) = \sin x \cos y - \cos x \sin y$, we get

$$-(\sin x \cos y - \cos x \sin y)\cos x + (\cos(x-y))\sin x = \sin y$$

and $(\cos(x-y))\sin x = \sin y + \sin x \cos x \cos y - \cos^2 x \sin y$

$$= \sin x \cos x \cos y + \sin y \sin^2 x$$

so that $\cos(x-y) = \cos x \cos y + \sin x \sin y.$

The sum formulas for $\sin(x + y)$ and $\cos(x + y)$ are derived in appropriately analogous ways (see fig. 8).

REFERENCES

1. For a complete discussion of positive and negative areas and volumes see Felix Klein, Elementary Mathematics From an Advanced Standpoint-Geometry, Dover Publications, Inc. 1939, pp. 3-9.

2. This result is a geometric interpretation of the elementary properties of determinants, which allow for the subtraction of rows (or columns) and the reduction of an nth order determinant to an (n - 1)th order determinant, viz.,

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} x_1 - x_2 & y_1 - y_2 & 0 \\ x_2 - x_3 & y_2 - y_3 & 0 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} x_1 - x_2 & y_1 - y_2 \\ x_2 - x_3 & y_2 - y_3 \end{vmatrix}$$

See also <u>Higher Algebra</u>, Hall and Knight, Macmillan and Company, Ltd. 1946, pp. 415-16.

3. If, based on the area equality, we write [E] in an equivalent form

$$\begin{vmatrix} \Delta x & \Delta y \\ \Delta x_1 & \Delta y_1 \end{vmatrix} = \begin{vmatrix} x_3 - x_1 & y_3 - y_1 \\ \Delta x_1 & \Delta y_1 \end{vmatrix}$$

so that

$$\begin{vmatrix} \Delta x & \Delta y \\ \Delta x_1 & \Delta y_1 \end{vmatrix} - \begin{vmatrix} x_3 - x_1 & y_3 - y_1 \\ \Delta x_1 & \Delta y_1 \end{vmatrix} = 0,$$

and if we apply the property of determinants

$$\begin{vmatrix} a & b \\ e & f \end{vmatrix} - \begin{vmatrix} c & d \\ e & f \end{vmatrix} = \begin{vmatrix} a - c & b - d \\ e & f \end{vmatrix}$$

then [E] may be restated in the simpler form

$$\begin{vmatrix} x - x_3 & y - y_3 \\ \Delta x_1 & \Delta y_1 \end{vmatrix} = 0$$

It is suggested the reader do the illustrative problem using this relationship.

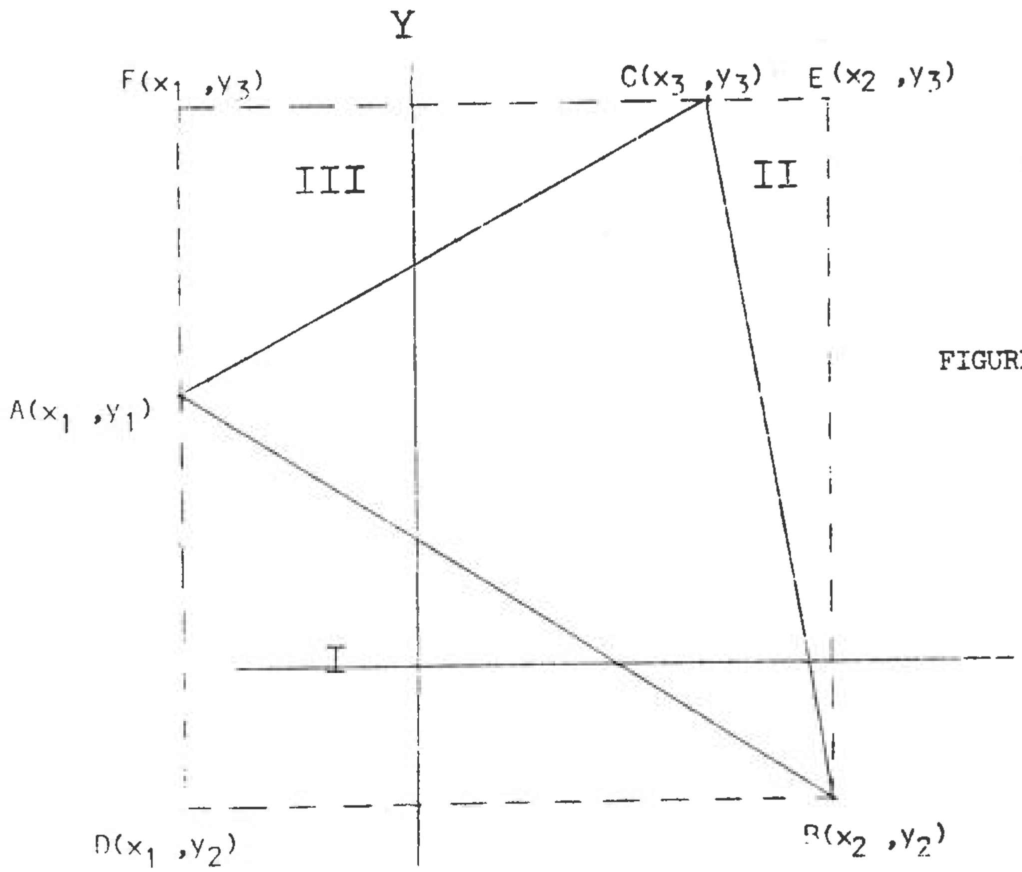


FIGURE 1

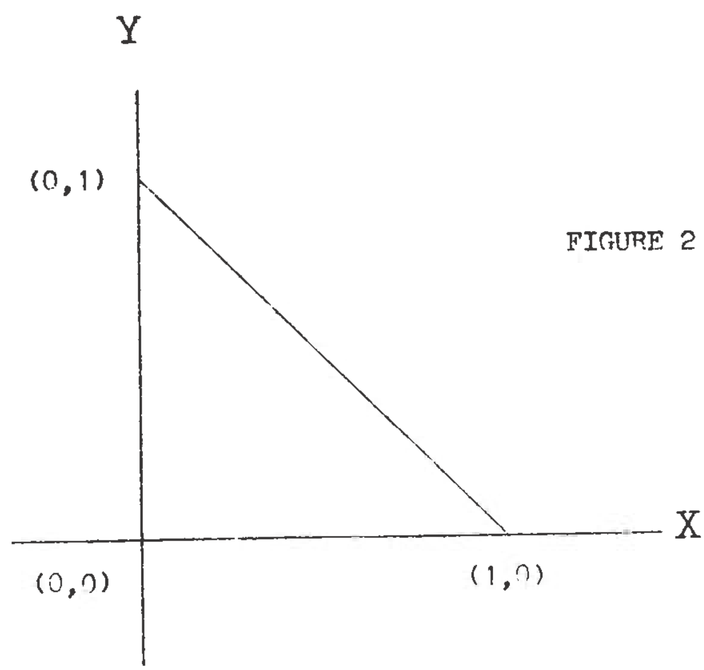


FIGURE 2

FIGURE 3

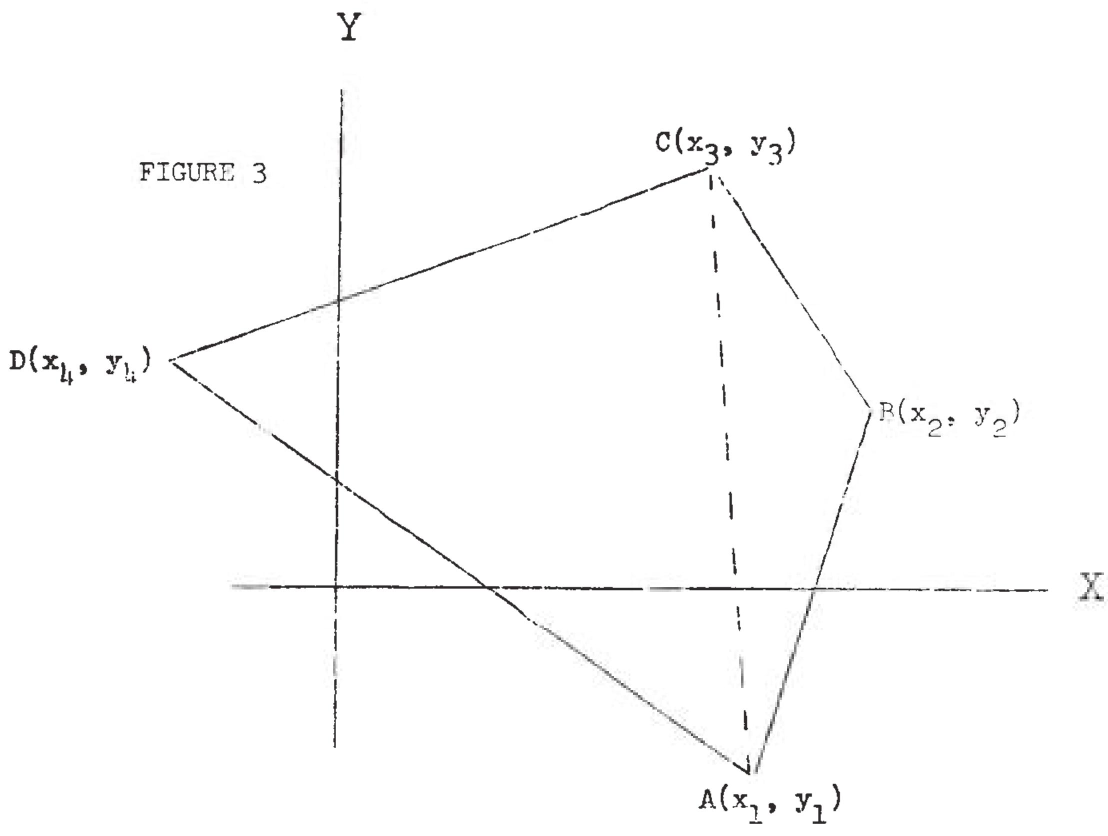
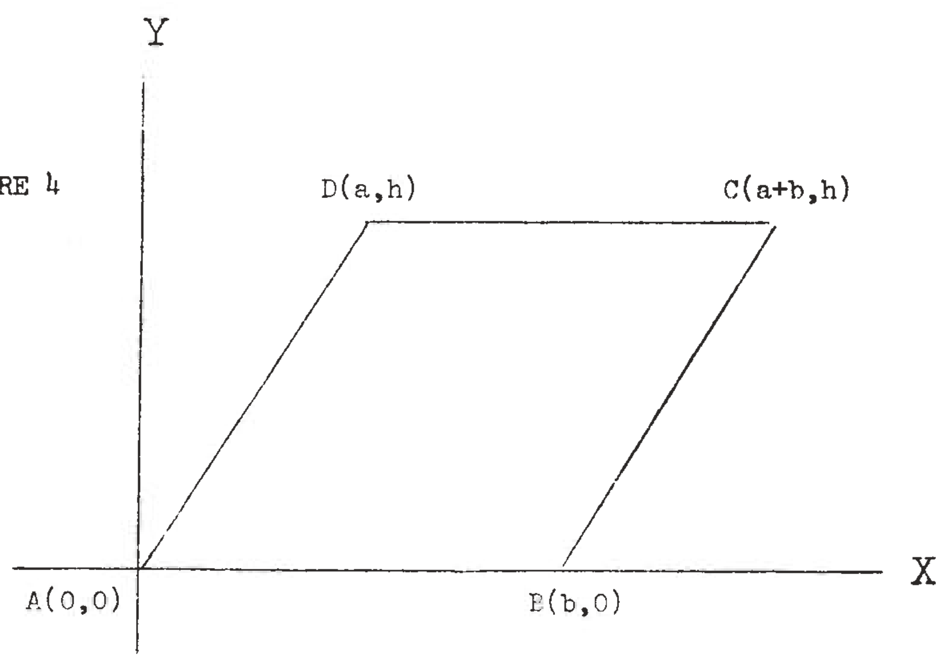


FIGURE 4



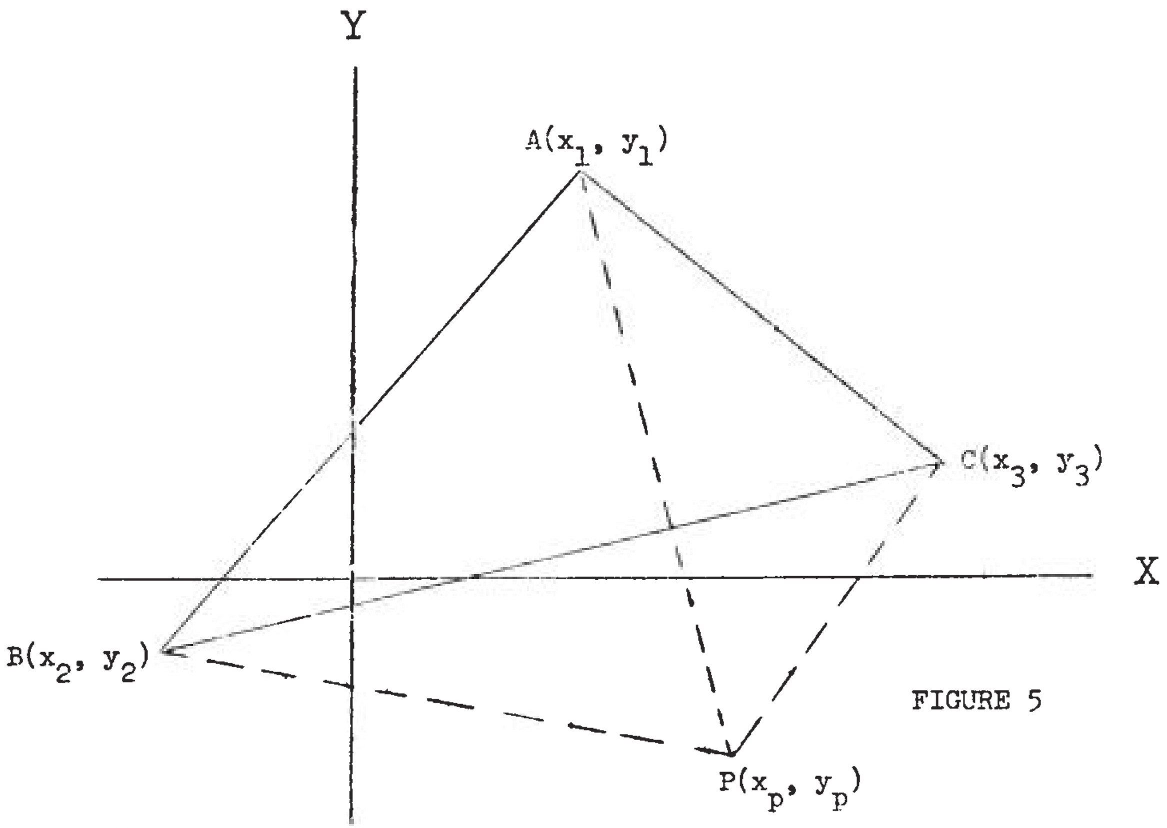


FIGURE 5

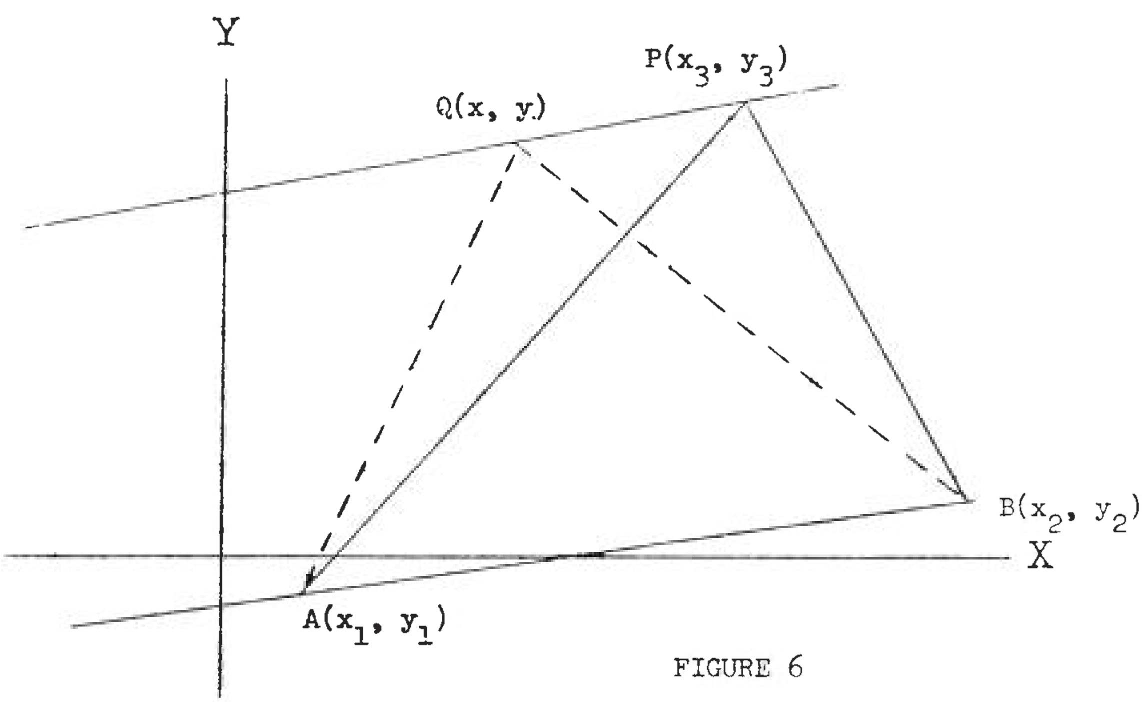
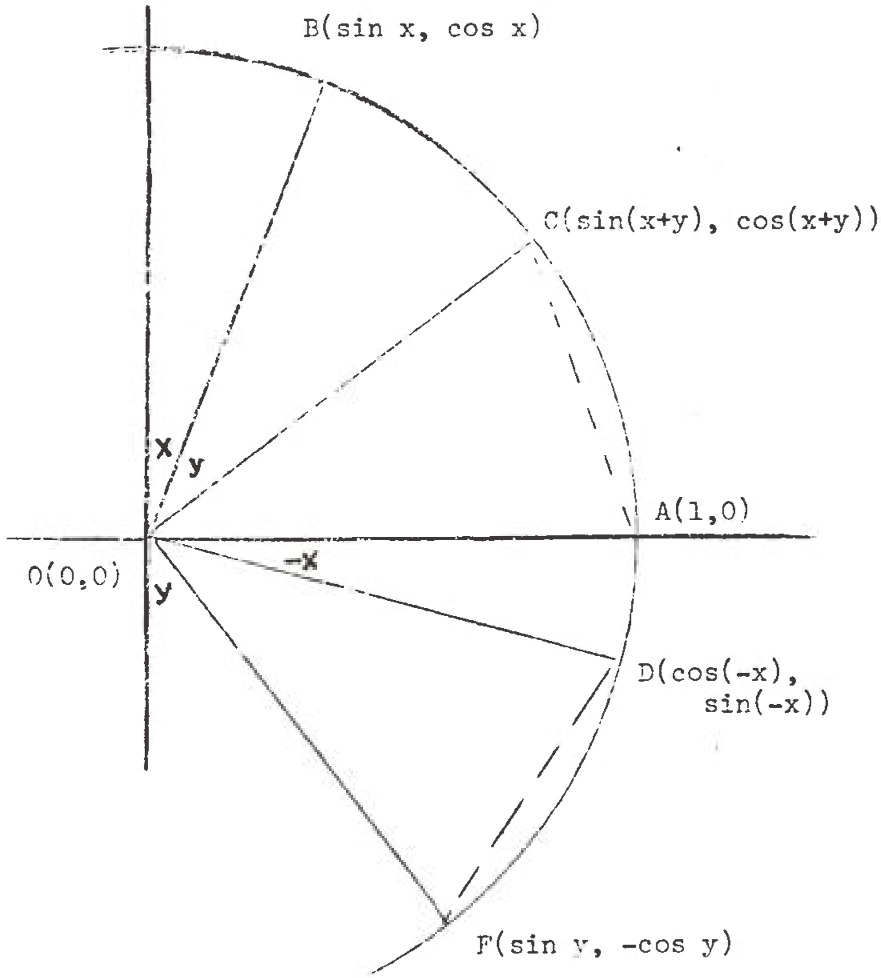


FIGURE 6

FIGURE 8



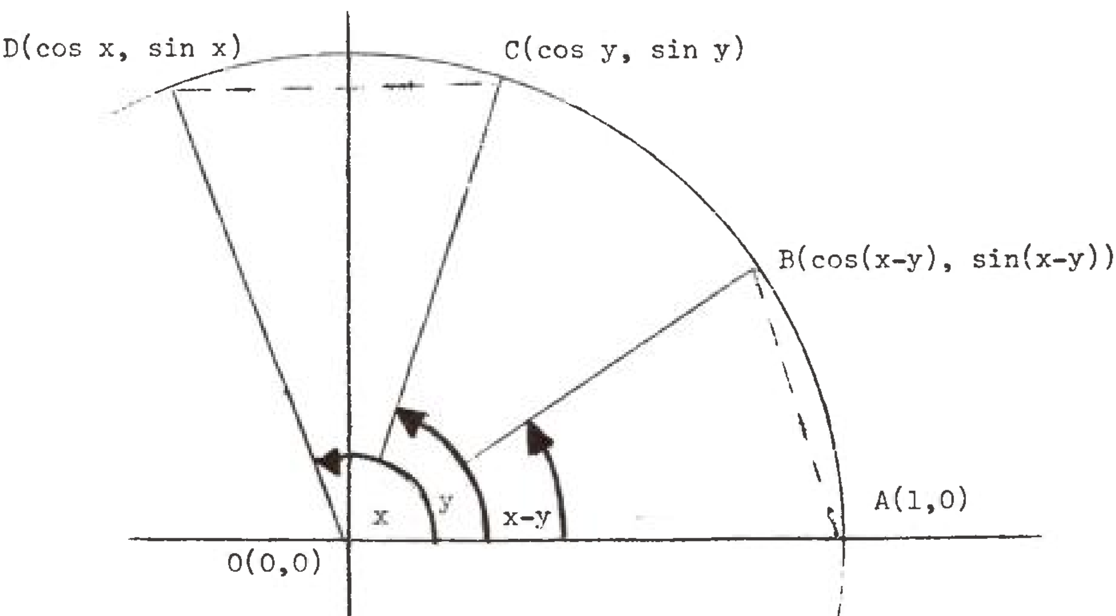


FIGURE 7