1978 Alberta High School Prize Examination Results

Prize	Amt.	Student	School
Canadian Mathematics Congress Scholarship	$400	FISHER, Douglas	Strathcona Composite High Edmonton, Alberta
Nickel Foundation Scholarship	$400	DEWAR, Alan	Sir Winston Churchill High Calgary, Alberta
Third Highest	$150	JENSEN, Lawrence	Queen Elizabeth Jr.-Sr. High Calgary, Alberta
Fourth Highest	$150	WONG, Eric	Ross Sheppard Composite High Edmonton, Alberta

Special Provincial Prizes

Highest Grade 12 student (below first 4)	$ 75	LAM, Kay	Harry Ainlay Composite High Edmonton, Alberta
Highest Grade 10/11 student (below first 4)	$ 75	LEUNG, Henry	Bonnie Doon Composite High Edmonton, Alberta

District Prizes

District No.	Amt.	Name	School
1	$50	BANTEL, Darald	Hillside Jr. Sr. High School Valleyview, Alberta
2	$50	WU, Willy	Paul Kane High School St. Albert, Alberta
3	$50	KOCH, Peter	Salisbury Comp. High School Sherwood Park, Alberta
4	$50	NORTON, Duane	Camrose Lutheran College Camrose, Alberta
5	No Award Made.		
6	$50	RODRIGUES, Ivan	Medicine Hat High School Medicine Hat, Alberta
7 (1)	$50	BAUDER, Bob	Harry Ainlay Comp. High School Edmonton, Alberta
7 (2)	$50	TROFIMUK, David	McNally Comp. High School Edmonton, Alberta
8 (1)	$50	HEWITT, Mark	Sir Winston Churchill High School Calgary, Alberta
8 (2)	$50	GORDON, Karen	Sir Winston Churchill High School Calgary, Alberta

356 students from 60 schools in Alberta and the Northwest Territories wrote the 1978 examination. The following students took the first 16 places and are nominated for the Canadian Mathematical Olympiad:

Student	School
BAUDER, Bob	Harry Ainlay Composite High School, Edmonton
CHAMBERLAIN, Martin	Harry Ainlay Composite High School, Edmonton
DENOTTER, Gordon	M.E. Lazerte Composite High School, Edmonton
DEWAR, Alan	Sir Winston Churchill High School, Calgary
FISHER, Douglas	Strathcona Composite High School, Edmonton
GORDON, Daren	Sir Winston Churchill High School, Calgary
HARTWIG, Karen	Jasper Place Composite High School, Edmonton
HAYWARD, Geoffrey	Old Scona Academic High School, Edmonton
HEWITT, Mark	Sir Winston Churchill High School, Calgary
JENSEN, Lawrence	Queen Elizabeth Jr.-Sr. High School, Calgary
LAM, Kay	Harry Ainlay Composite High School, Edmonton
LAMOUREUX, Mike	Archbishop Macdonald, Edmonton
LYNCH, William	Lord Beaverbrook Sr. High School, Calgary
TROFIMUK, David	McNally Composite High School, Edmonton
WILLIS, Ron	St. Francis Xavier Composite, Edmonton
WONG, Eric	Ross Sheppard Composite High School, Edmonton

The following students placed 17-30:

Henry Baragar (Old Scona Academic High School, Edmonton), Catherine Clelland (Eastglen Composite High School, Edmonton), Wallace Chow (Dr. E.P. Scarlett Sr. High School, Calgary), John Haugen (Jasper Place Composite, Edmonton), Mark Herman (Sir Winston Churchill High School, Calgary), Dean Karlen (Jasper Place Composite, Edmonton), Kok Kwan (Eastglen Composite High School, Edmonton), Agnes Lee (Ross Sheppard Composite, Edmonton), Ming Lee(McNally Composite, Edmonton), Henry Leung (Bonnie Doon Composite High School, Edmonton), Mark Salzyn (Harry Ainlay Composite, Edmonton), Glynn Searl (St. Francis High School, Calgary), Kenneth Tsang (Strathcona-Tweedsmuir School, Okotoks), Fred Woslyng (Harry Ainlay Composite, Edmonton)

The following students placed 31-50:

Janice Bodnarchuk (Austin O'Brien High School, Edmonton), Cathy Brown (Ross Sheppard Composite, Edmonton), John Chmelicek (Strathcona Composite High School, Edmonton), Stephen Crowe (Ernest Manning High School, Calgary), Dane Douglas (Jasper Place Composite, Edmonton), Rick Eykelbosh (Louis St. Laurent, Edmonton), Duane Foote (F.P. Walshe School, Fort Macleod), Guy Fortier (J.H. Picard High School, Edmonton), Lewis Kay (Ross Sheppard Composite, Edmonton), Yeon Kim (Forest Lawn Sr. High School, Calgary), Gordon Lee (Eastglen Composite High School, Edmonton), David Macpherson (Ross Sheppard Composite High School, Edmonton), Simon McClure (Strathcona Composite High School, Edmonton), Bob McCreight (Sir Winston Churchill High School, Calgary), Andrew McIntosh (Bishop Carroll High School, Calgary), Robert Morewood (Medicine Hat High School, Medicine Hat), Duane Norton (Camrose Lutheran College, Camrose), Karl Pierzchajlo (St. Francis Xavier Composite, Edmonton), Ivan Rodrigues (Medicine Hat High School, Medicine Hat), Michael Whitney (Strathcona Composite High School, Edmonton), Paul Yarema (Archbishop Macdonald, Edmonton).

Part I **Time: 60 Minutes**

ANSWER SHEET

To be filled in by the Candidate.

PRINT:

Last Name First Name Initial

Candidate's Address Town/City

Name of School

Grade

ANSWERS:

1	2	3	4	5	6	7	8	9	10

11	12	13	14	15	16	17	18	19	20

To be completed by the Department of Mathematics, University of Alberta:

	Points	Points Correct	Number Wrong
1-20	5	5 × =	1 × =
	Totals	C = ______	W = ______

SCORE = C - W = ______

Do all problems. Each problem is worth five points.

1. Which of the following inequalities are true for all positive numbers x?

   (A) $x + \frac{1}{x} > 2$ (B) $x + \frac{1}{x} < 2$

   (C) $x + \frac{1}{x} \geq 2$ (D) $x + \frac{1}{x} \leq 2$

   (E) none of the preceding are true for all positive numbers x.

2. A steamer was able to go twenty miles per hour upstream and twenty-five miles per hour downstream. On a return trip the steamer took two hours longer coming upstream than it took coming downstream. The total distance travelled by the steamer was

   (A) 100 miles (B) 200 miles

   (C) 400 miles (D) 800 miles

   (E) 150 miles.

3. If n is a positive integer, then $n^2 + 3n + 1$ is

   (A) always a perfect square (B) never a perfect square

   (C) sometimes a perfect square (D) sometimes an even integer

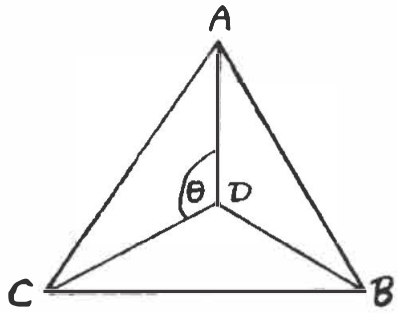
   (E) none of the preceding.

4. The solution set of the inequality $x^2(x^2-1) \leq 0$ is

   (A) an interval (B) two intervals

   (C) a point (D) an interval and a point

   (E) all real numbers.

5. 

In the diagram, ΔABC is an equilateral triangle, ΔBCD is an isosceles triangle, and the angle $C\hat{D}B$ is a right-angle. Then the angle θ is

(A) $45°$ (B) $90°$ (C) $120°$ (D) $135°$

(E) none of the preceding.

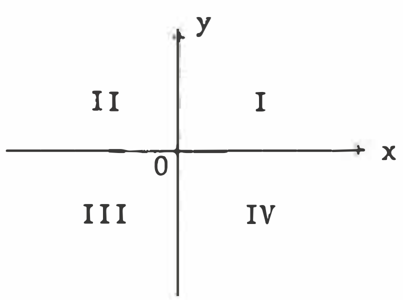
6. Given the binary operation $*$ between two positive integers m, n such that $m * n = mn + 1$ (mn is the usual multiplication of m and n), which of the following does not hold:

(A) commutative law (B) associative law

(C) $m * n$ is a positive integer (D) $m * n \geq 2$

(E) $m * n$ is odd whenever m is even.

7. Label the four quadrants of the (x,y)-plane as follows:



Then the solution set of the simultaneous inequalities $x^2 - y < 0$, $x^2 + y^2 < 1$ lies entirely in quadrants

(A) I and II (B) II and III (C) III and IV

(D) IV and I (E) none of the preceding are correct.

8. If $f(n) = n^2$, where n is an integer, then

$$\frac{f\big(f(n+1)\big) - f\big(f(n-1)\big)}{f(n+1) - f(n-1)} \quad \text{equals}$$

(A) n^2 (B) $2n^2 + 2$ (C) $n^2 + 1$ (D) $n^4 + 1$

(E) none of the preceding.

9. Which of the following inequalities hold for all pairs of real numbers x, y?

(A) $\sqrt{x^2 + y^2} \le x + y$ (B) $\sqrt{x^2 + y^2} \le x^2 + y^2$

(C) $\sqrt{x^2 + y^2} \le xy$ (D) $\sqrt{x^2 + y^2} \le |x| + |y|$

(E) none of the preceding are true for all real numbers x, y.

10. Two similarly proportioned boxes have their surface areas in the ratio 4:1. Their volumes are in the ratio

(A) 9:1 (B) 8:1 (C) 3:1 (D) 2:1

(E) none of the preceding.

11. The roots of the quadratic polynomial $2x^2 + kx + 1$ are r and s. Which of the following are impossible?

(A) $r = s$ (B) $r \cdot s = 1$ (C) $r + s = 1$ (D) $r + s = 0$

(E) all of the preceding are possible.

12. A hat contains three slips of paper, of which one bears the name John, one bears the name Diana and the other bears both names. If John and

Diana each draw a slip, the probability that they each draw a slip with their own name on is

(A) $\frac{1}{9}$ (B) $\frac{1}{6}$ (C) $\frac{1}{4}$ (D) $\frac{1}{3}$ (E) none of the preceding.

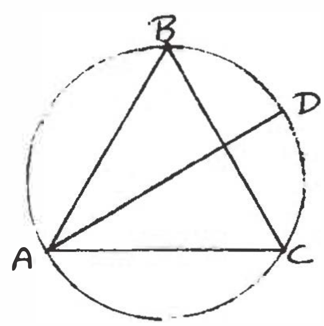
13. The value of k such that

$$x^6 - kx^4 + kx^2 - kx + 4k + 6 \text{ is divisible by } x - 2 \text{ is}$$

(A) 1 (B) 5 (C) 7 (D) 11

(E) there is no such value of k.

14.



ΔABC is an equilateral triangle inscribed in a circle of diameter 1. If AD is a diameter of the circle, then the length $\overline{BD}$ is

(A) $\frac{1}{2}$ (B) 1 (C) 2 (D) $\frac{1}{2}\sqrt{3}$ (E) $1/\sqrt{2}$.

15. The equation of the line through the point $(1,1)$ that is perpendicular to the line $y = -2x - 3$ is

(A) $y = \frac{2}{3}x + \frac{1}{3}$ (B) $y = \frac{1}{3}x + \frac{2}{3}$

(C) $y = 2x - 1$ (D) $y = \frac{1}{2}x + \frac{1}{2}$

(E) none of the preceding.

16. If $\log_a b = c$, then $\log_a(b^c) =$

(A) bc (B) 6^c (C) c^c (D) c^2 (E) $2c$.

17. A circle and a square can <u>never</u> intersect in

(A) one point (B) two points (C) three points

(D) four points (E) all of the preceding are possible.

18. $a_1, a_2, a_3, \cdots$ is a sequence of real numbers such that the sum of the first n of them is $n^2 + n$. Then a_n is equal to

(A) n (B) $2n - 1$ (C) $2n + 1$ (D) 1

(E) none of the preceding.

19. The domain of the function

$$f(x) = \sqrt{1 - \sqrt{1 - x^2}} \quad \text{is}$$

(A) a single point (B) an infinite interval

(C) a finite interval (D) an infinite interval with a point deleted

(E) none of the preceding.

20. A polynomial which passes through the points $(-1,7)$, $(1,0)$, $(2,0)$ is

(A) $x + 8$ (B) $x^2 - 3x + 2$ (C) $x^2 + 9$ (D) $x^3 - x^2 + x - 1$

(E) none of the preceding.

Part II

Time: 110 Minutes

<u>Marks</u>

20 1. Determine all angles θ with $0 \le \theta \le 2\pi$ such that $\sin^6\theta + \cos^3\theta = 1$.

2. (a) A corner reflector consists of two straight lines, perpendicular to each other, which are assumed capable of reflecting
5 a ray of light which is in the same plane as the lines. If a ray of light reflects successively off each of the lines, prove that the exit ray is parallel in the opposite sense to the entering ray.

(b) [3-dimensional version]. This time the corner reflector consists of three plane mirrors which are mutually perpendicular.
15 If a ray of light reflects successively off each of the three mirrors, in any order, prove that the exit ray is parallel in the opposite sense to the entering ray.

20 3. S is a finite set of positive integers, not necessarily different from each other, such that for any three members a,b,c of S, $a+b$ is divisible by c. Classify all possible such sets S.

20 4. Let i be a square root of -1 and for real numbers x, y, write the complex number $\dfrac{x+iy+1}{x+iy-1}$ in the form $a+ib$, where a, b are real numbers. Find the set of points (x,y) for which $a \le 0$.

20 5. Prove that for any integers m, n, $mn(m^4-n^4)$ is divisible by 30.

20 6. For which values of k do the polynomial equations

$$x^6 + x^5 + x^4 + x^3 + x^2 + x - 3k = 0$$
$$x^6 - x^5 + x^4 - x^3 + x^2 - x - k = 0$$

have a common root?

20 7. 36 points are placed inside a square whose sides have length 3. Show that there are 3 points which determine a triangle of area no greater than $\frac{1}{2}$.

20 8. You are given a set of 21 dominoes $\boxed{a \mid b}$ where a and b are integers from 1 to 6, each pair occurring once (note $\boxed{a \mid b}$ is to be considered the same as $\boxed{b \mid a}$). Any number of dominoes can be joined to form a chain if they have matching numbers at each join. For example, a 3-chain is given by $\boxed{1 \mid 1}$ $\boxed{1 \mid 6}$ $\boxed{6 \mid 4}$. Show that it is <u>not</u> possible to form a 21-chain.

Solutions to Part I

C	C	B	A	D	B	A	B	D	B
1	2	3	4	5	6	7	8	9	10

B	E	C	A	D	D	E	E	C	E
11	12	13	14	15	16	17	18	19	20

Solutions to Part II

1. Since $|\sin\theta| \le 1$ and $|\cos\theta| \le 1$, we have $\sin^6\theta + \cos^3\theta \le \sin^2\theta + \cos^2\theta = 1$, with equality only if $\sin\theta = \pm 1$ and $\cos\theta = 0$, or $\sin\theta = 0$ and $\cos\theta = 1$. For $0 \le \theta \le 2\pi$, this gives $\theta = 0, \pi/2, 3\pi/2, 2\pi$.

2. Using vectors, let $\underline{i}$, $\underline{j}$ be unit vectors perpendicular to the two mirrors, and let the incident ray have the direction $a\underline{i} + b\underline{j}$. Reflection at the first mirror will cause the ray to have direction $-a\underline{i} + b\underline{j}$, then reflection at the second mirror will change this to $-a\underline{i} - b\underline{j}$, i.e. the exit ray is parallel in the opposite sense to the entering ray. This argument is easily extended to the 3-dimensional situation.

3. If $a,b,c \in S$ with $a \le b < c$ or $a < b \le c$, then $a + b < 2c$. Now $a + b$ is divisible by c, and so we must have $a + b = c$. It follows that S cannot have four distinct integers, for if $a,b,c,d \in S$ with $a < b < c < d$, we would have $a + b = c$, $a + b = d$, i.e. $c = d$. So S has at most three <u>distinct</u> integers.

   <u>Case (i).</u> S has three distinct integers a, b, c, with $a < b < c$. If d is any other member of S, it must be equal to one of a, b, c. But $d = a$ implies $d + a = b = c$; $d = b$ implies $a + d = b$, i.e. $a = 0$; $d = c$ implies $b + c = d$, i.e. $b = 0$. All of these are impossible and so there can be no other member of S.
   Now $a < b < c$ implies $a + b = c$, and $b < a + c = 2a + b < 3b$. Since $a + c$ is divisible by b, we must have $a + c = 2b$. $a + b = c$ and $a + c = 2b$ imply that $c = a + b = 2b - a$, i.e. $b = 2a$, $c = a + b = 3a$. So the only sets S with three distinct integers are of the form $S = \{a,2a,3a\}$.

Case (ii). S has two distinct integers a, b with $a < b$. If c is any third member of S, we have $a = c < b$ or $a < c = b$, and so $a + c = b$. So c must equal a, and the only sets S with two distinct integers are of the form $S = \{a,a,\cdots,a,2a\}$.

Case (iii). Finally, S may be of the form $\{a,a,\cdots,a\}$.

4. Note that $\frac{x+iy+1}{x+iy-1}$ is undefined for $x = 1$, $y = 0$.

$$\frac{x+iy+1}{x+iy-1} = \left(\frac{x+iy+1}{x+iy-1}\right)\left(\frac{x-iy-1}{x-iy-1}\right)$$

$$= \frac{x^2+y^2-1-2iy}{(x-1)^2+y^2} = a + ib, \quad \text{where}$$

$$a = \frac{x^2+y^2-1}{(x-1)^2+y^2}, \quad b = \frac{-2y}{(x-1)^2+y^2}.$$

$a \le 0 \Leftrightarrow x^2+y^2 \le 1$ and $(x,y) \neq (1,0)$, i.e. the set of points (x,y) for which $a \le 0$ is the circle, center $(0,0)$, radius 1, with the point $(1,0)$ removed.

5. $mn(m^4-n^4) = mn(m^2-n^2)(m^2+n^2) = mn(m-n)(m+n)(m^2+n^2)$. To show divisibility by 30, we need to show divisibility by each of the prime factors $2, 3, 5$.

Divisibility by 2: Either one of m, n is divisible by 2 or their sum is divisible by 2.

Divisibility by 3: Either one of $m, n, m+n$ is divisible by 3, or two of them leave the same remainder on dividing by 3. In the latter case, the difference of these two integers will be divisible by 3, which implies that one of the integers $m, n, m-n$ is divisible by 3.

Divisibility by 5: If neither m nor n is divisible by 5, then each of them has a remainder 1, 2, 3 or 4 when divided by 5, and their squares will have a remainder 1 or 4. These remainders are either the same, in which case $m^2 - n^2$ is divisible by 5, or they add up to 5, in which case $m^2 + n^2$ is divisible by 5.

6. $x^6 + x^5 + x^4 + x^3 + x^2 + x = 3k$ — (1)

$x^6 - x^5 + x^4 - x^3 + x^2 - x = k$ — (2).

Subtracting (2) from (1) gives $2x^5 + 2x^3 + 2x = 2k$, i.e.

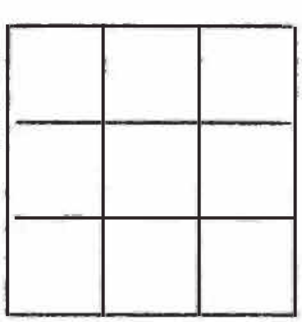
$$k = x(x^4+x^2+1) \text{ — } (3).$$

Multiplying (2) by (3) and subtracting (1) gives $2x^6 - 4x^5 + 2x^4 - 4x^3 + 2x^2 - 4x = 0$, which simplifies first to $2x^2(x^4+x^2+1) - 4x(x^4+x^2+1) = 0$ and then to

$$x(x-2)(x^4+x^2+1) = 0, \text{ i.e.}$$

$$x = 0 \text{ or } 2, \quad \text{or} \quad x^4 + x^2 + 1 = 0.$$

The first and third of these options give $\underline{k = 0}$ (using (3)), the second gives $\underline{k = 42}$.

7. 

Subdivide the large square into 9 squares of side 1. Since there are 36 points, at least one of the small squares will contain 4 (or more) points. These 4 points determine at least 2 non-overlapping triangles lying inside the small square. Since the small square has area 1, one of the triangles must have area no greater than 1/2.

8. Except at the extreme ends of a dominoe chain, each time an integer is introduced into the chain, it must be "paired" by the next dominoe placed in the chain. Therefore each of the integers 1 through 6 must be used an even number of times, except possibly those occurring at the ends of the chain. However, each integer occurs seven times in the complete set of 21 dominoes and so at least four integers cannot be used up in any chain, which means that at least two dominoes will not be used. Thus a 19-chain is the longest possible (and this can be achieved). ■