Estimating Roots of Quadratic Equations by Templet and Grid

William J. Bruce

Consider any quadratic function defined by the equation $y = x^2 + ax + b$, and assume that it is known that the graph of this equation is a parabola. In order that our proposed device be operable, it is essential that the given quadratic equation be uniquely determined by knowing the axis of symmetry of the parabola and one point on the Y-axis. This is shown easily.

Let $(0,y_1)$ be the coordinates of a point on the Y-axis and let $x = h$ be the equation of the axis of symmetry of the parabola. Because of this symmetry, $(2h,y_1)$ are the coordinates of another point on the parabola? Substitute the coordinates $(0,y_1)$ into the equation $y = x^2 + ax + b$ and obtain $b = y_1$ so that

$$y = x^2 + ax + y_1.$$

The coordinates $(2h,y_1)$ are substituted next to give

$$y_1 = 4h^2 + 2ah + y_1$$

or

$$4h^2 + 2ah = 0$$

from which we find, since $h \neq 0$, that $a = -2h$. Finally the quadratic equation becomes uniquely

$$y = x^2 - 2hx + y_1.$$

For any quadratic equation $y = x^2 + ax + b$, it is obvious that the point with coordinates $(0,b)$ lies on the graph. Complete the square and obtain

$$y = (x+a/2)^2 + b - a^2/4$$

from which we have that

$$x = -a/2$$

is the equation of the axis of symmetry of the parabola. For example, if $y = x^2 - 5x + 6$, we find that $(0,6)$ are the coordinates of the point on the Y-axis and $x = 5/2$ is the equation of the axis of symmetry.

Since neither horizontal nor vertical translation change the shape of a parabola, all parabolas represented by the equation $y = x^2 + ax + b$ have the same shape. Because $y = x^2$ is the simplest form of this equation, it suffices to cut a templet by using a truncated portion of the graph of this equation. All other parabolas referred to can be drawn by shifting such a templet, without rotation, on a Cartesian grid and tracing along the parabolic boundary. Figure 1 illustrates how to obtain a pattern from which this templet can be cut. Note that equal scale units are used but are not essential. The smaller graph shown is that of $y = x^2$ drawn with considerable accuracy. Included is a larger graph of $y = x^2$ made possible by simply shifting the origin downward on the grid to allow for the plotting of more points. If a higher degree of accuracy is needed, the parabolic plots can be done by computer.

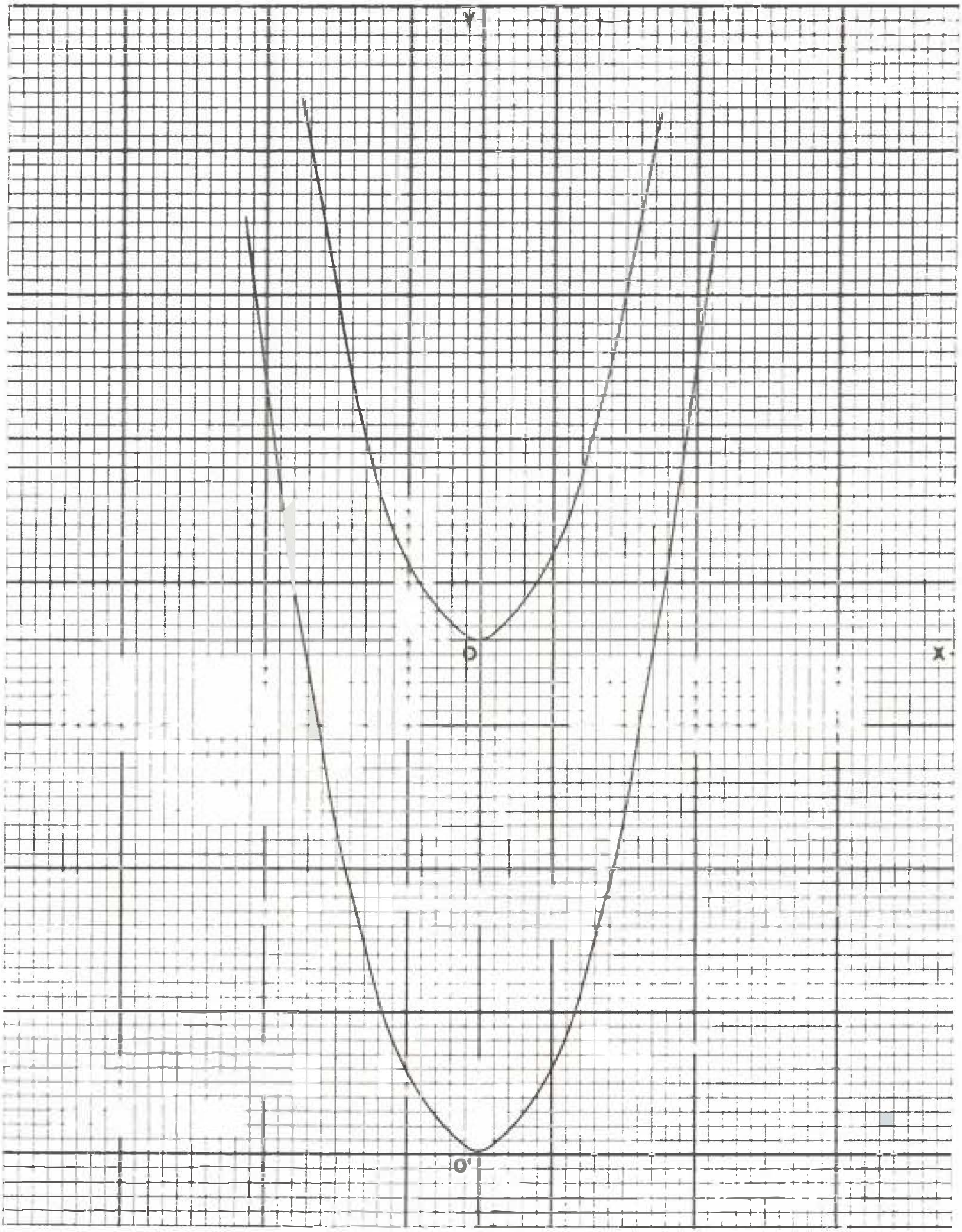

Figure 1

A templet can be prepared in the following manner. Place a piece of fairly rigid cardboard behind the graph chosen, and with a long sharp pin with a beaded head carefully punch along the parabola at frequent intervals. One extra punch will be needed to locate the axis of symmetry if this is desired. In using the templet it will be found that it is not absolutely necessary to have the axis of symmetry as long as the vertex is accurately punched and the parabolic region is truncated on a line perpendicular to the axis of symmetry. Carefully cutting along the pin-pricked path, one can produce a reasonably accurate templet.

In order to use a templet to solve quadratic equations of the form $x^2 + ax + b = 0$, we shall need a square grid ruled on a durable sheet of cardboard, plastic or wood. This grid must be prepared so as to match the parabolic templet. Coordinate axes might be pre-marked or could be omitted for greater flexibility and chosen where needed. Some device for marking the axes in the latter case would be necessary. The size of the templet and the grid can be chosen as needed. Standard page size for the grid likely would be adequate for most purposes. Figure 2 shows a grid that might be used.

The following examples illustrate how to use the templet with a grid to estimate the roots of a quadratic equation. At first we shall restrict these examples to equations that have real roots. Figure 2 is used in each case.

(1) Solve $x^2 - 4x + 4 = 0$.

Let $y = x^2 - 4x + 4$. This is of the form $y = x^2 + ax + b$ so that $b = 4$ is the y-intercept while $x = -a/2 = 2$ is the equation of the axis of symmetry. Place the templet on the grid so that its edge cuts the Y-axis at 4 and so that its axis of symmetry lies on the line represented by $x = 2$. In this case the

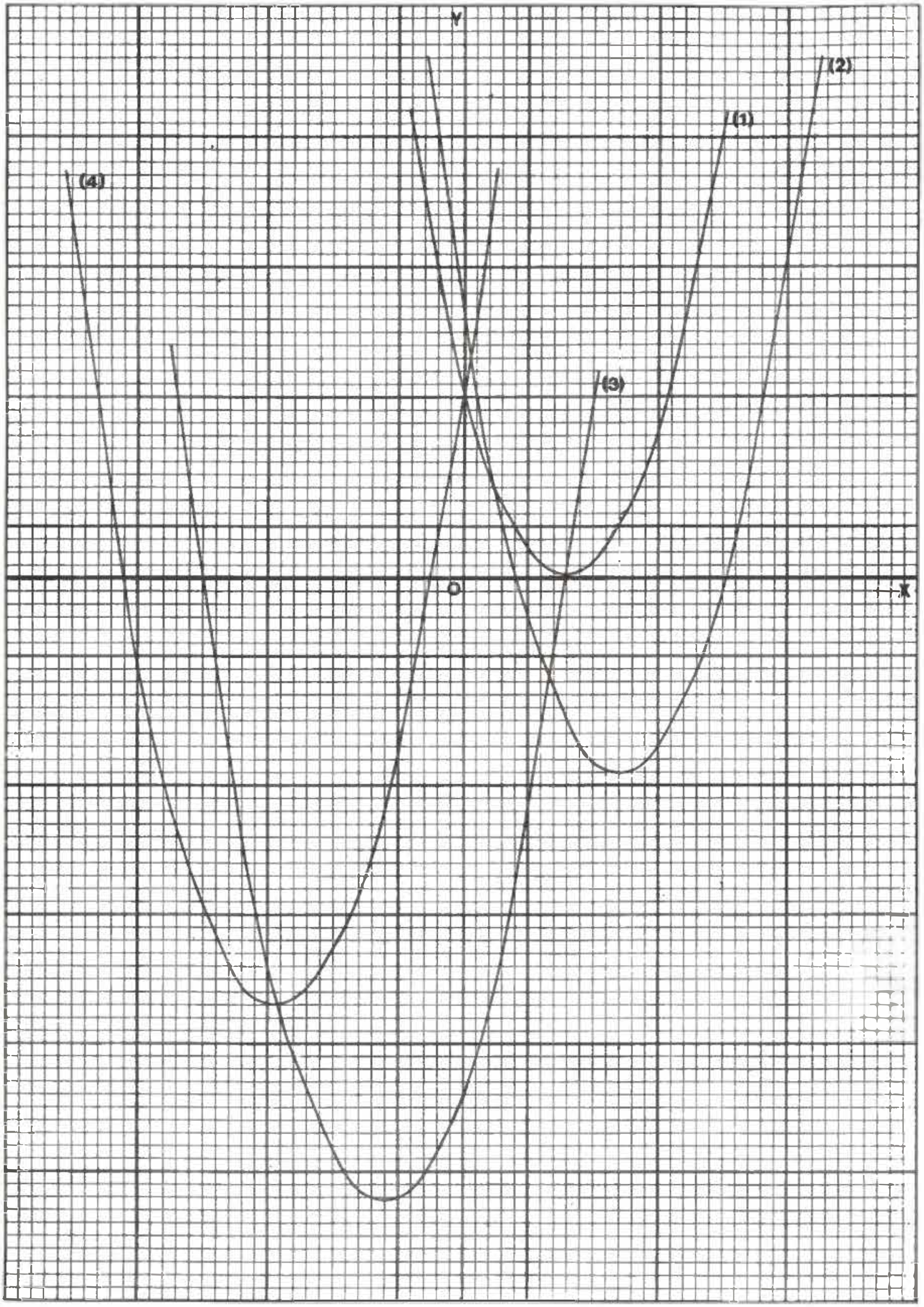

Figure 2

vertex of the parabola touches the X-axis at 2. If r_1 and r_2 are the roots of the quadratic equation, we have that $r_1 = r_2 = 2$.

(2) Solve $x^2 - 6x + 5 = 0$.

Let $y = x^2 - 6x + 5$. Here $b = 5$ is the y-intercept and $x = 3$ is the equation of the axis of symmetry. Place the templet on the grid carefully and note that its edge crosses the X-axis at 1 and 5. Hence $r_1 = 1$ and $r_2 = 5$ are the roots.

(3) Solve $x^2 + 3x - 10 = 0$.

Let $y = x^2 + 3x - 10$. Here $b = -10$ is the y-intercept and $x = -3/2$ is the equation of the axis of symmetry. Place the templet on the grid so that its edge cuts the Y-axis at -10 and so that its axis of symmetry lies on the graph of $x = -3/2$. The parabola cuts the X-axis at -5 and 2. Hence $r_1 = -5$ and $r_2 = 2$ are the roots.

(4) Solve $x^2 + 7x + 4 = 0$.

Let $y = x^2 + 7x + 4$. Here $b = 4$ is the y-intercept and $x = -7/2$ is the equation of the axis of symmetry. In this case the templet doesn't cut the X-axis at integral values so we have to estimate the roots more carefully than in the previous examples. These appear to be approximately $r_1 = -6.4$ and $r_2 = -0.6$.

For equations with leading coefficients that are not unity, it is sufficient to divide by this coefficient at the outset to get an equivalent equation and then solve the latter as before. If the templet doesn't cut the X-axis, this will indicate that the roots are not real, in which case they cannot be found by the method described above. For such nonreal roots, the method of completing the square can be used to form the well-known quadratic formula.

The presence of a negative discriminant in the latter also indicates that the roots are nonreal.

There are two ways by which one might prepare a broader templet. One of these is to shorten the vertical scale units. For example, the ratio of vertical to horizontal scale units might be taken as 1 to 2. Of course, both the templet and the grid must conform to the same units in all cases. Another way to accomplish this is to retain equal scale units on both scales, but to use a fractional multiplier for the quadratic function. For example, one might produce the templet from the graph of $y = \frac{1}{2}x^2$ and always write the equation in the form $\frac{1}{2}(x^2+ax+b) = 0$ so that the graph would be that of $y = \frac{1}{2}(x^2+ax+b)$ when the templet is shifted about on the grid. The roots of the quadratic equation will not change if variations such as these are used. ■