THE PRODUCT OF SIGNED NUMBERS:
Dissection of an Unmotivated Proof

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Among the many justifications or explanations for why signed numbers behave the way they do under multiplication, the most puzzling is one that relies heavily upon the distributive principle. It looks as if the argument belongs to the domain of legerdemain, for no motivation is provided and we are essentially expected to accept the demonstration on the grounds that the ends justify the means.

The case is quite the opposite in supposedly intuitive demonstrations that make use of patterns rather than the structure of mathematics. Surprisingly enough, it turns out that the legerdemain explanation and a particular intuitive one are linked, and a dissection of their linkage provides illumination for each of them. We turn first to each of the demonstrations.

The Distributive Principle Demonstration

Let us look first at the case of the product of a negative and a positive number. What should (2)(-3) be?

We assume that we know how signed numbers behave under addition, and that we are familiar with basic properties of non-negative reals under multiplication such as:

$ab = ba$
$(ab)c = a(bc)$
$a \times 1 = a$
$a \times 0 = 0$
$a(b + c) = ab + ac$

The demonstration would then be:

$0 = 2(0)$
by the multiplication property of 0 above
$= 2(3 + -3)$
since 0 is $3 + -3$
$= 2 \times 3 + 2(-3)$
if the distributive principle is to hold
$= 6 + 2(-3)$
by a fact of arithmetic
$= 6 + -6$
by the additive-inverse property of addition

The case of the product of two negatives is similarly demonstrated as follows [consider (-2)(-3)]:

$0 = (-2)(0)$
if the multiplication property for zero is to hold
$= (-2)(3 + -3)$
since 0 is $3 + -3$
$= (-2)(3) + (-2)(-3)$
if the distributive property is to hold

$= (3)(-2) + (-2)(-3)$
if the commutative property is to hold for real numbers

$= -6 + (-2)(-3)$
by the previous result

Therefore $(-2)(-3) = 6$ by additive-inverse property.

Completing a Pattern

Compare the unmotivated demonstration above, with the following two intuitive arguments based upon a pattern:

$$\begin{aligned} 2(3) &= 6 \\ 2(2) &= 4 \\ 2(1) &= 2 \\ 2(0) &= 0 \\ 2(-1) &= ? \\ 2(-2) &= ? \\ 2(-3) &= ? \end{aligned}$$

It is obvious that if one is committed to the continuation of a pattern (subtracting 2 in each case) established for the familiar cases (positive integers), then the "?" in each of the bottom three cases could be filled in as follows:

$$\begin{aligned} 2(-1) &= -2 \\ 2(-2) &= -4 \\ 2(-3) &= -6 \end{aligned}$$

Thus $2(-3) = -6$.

Similarly for the case of 2 negatives [for example, $(-2)(-3)$], once we have established the one for the product of a negative and a positive we have:

$$\begin{aligned} (-2)(3) &= -6 \\ (-2)(2) &= -4 \\ (-2)(1) &= -2 \\ (-2)(0) &= 0 \\ (-2)(-1) &= ? \\ (-2)(-2) &= ? \\ (-2)(-3) &= ? \end{aligned}$$

Again, if the pattern of adding 2 in each case to get the answer to the one below is to continue we have:

$$(-2)(-3) = 6$$

A First Approximation in Seeing Linkages

Neither the case of the distributive principle nor the pattern argument provides us with a proof. The reason of course is *not* that we have focused on specific rather than general cases (for we could generalize the arguments without difficulty) but rather that each of the two types of demonstrations shares an important obstacle that could not be overcome by introducing all the variables in the world. The "proofs" are based upon "wishful thinking." That is, there is no God-given reason in the world why the axiomatic structure embedded in the case of non-negatives is *required* to continue as we move to the negatives. It is only *if* we force the distributive law (and others too) to apply in our new set-up that we are led to conventional results. We *cannot* prove that these laws *must* be extended. We merely can investigate the *consequences* of making such extensions.

In the above argument, we have applied a heuristic that is used generally in extending mathematical concepts - the *preservation principle*. The principle asserts that if we wish to extend a mathematical concept beyond its original domain, then that candidate ought to be chosen which leaves as many principles of the old system intact as possible. The preservation principle is, however, an *aesthetic* and not a logical one. The mathematical world would not collapse if we were to modify drastically old principles when we extend to new systems. As a matter of fact, we frequently *must* relinquish some old principles when we extend our domain, for we may be led to contradictions otherwise. (See for example what havoc is played if we try to relate $\sqrt{-1}$ to zero as part of an ordered-field structure as we move from the reals to the complex numbers.)

It is important to see that such an aesthetic argument is made in the case of the pattern demonstration as well. There is no God-given reason why the terms *must* decrease by 2 in the new domain as they do in the old.

$$\begin{aligned}2(2) &= 4\\ 2(1) &= 2\\ 2(0) &= 0\\ 2(-1) &= ?\end{aligned}$$

We are of course familiar with a function that behaves quite differently - the absolute value function. We *could* force the pattern to revise itself below zero:

$$\begin{aligned}2(2) &= 4\\ 2(1) &= 2\\ 2(0) &= 0\\ 2(-1) &= 2\\ 2(-2) &= 4\end{aligned}$$

It would then be interesting to investigate what principle in the system might have to be modified, based upon this new extension.

Putting a Fine Point to It

Let us now take a closer look at the pattern argument to see just what it is we are attempting to preserve as we continue the pattern. Let us leave the answers on the right hand side in unsimplified form:

$$\begin{aligned}2(3) &= 2(3)\\ 2(2) &= 2(2)\\ 2(1) &= 2(1)\\ 2(0) &= 2(0)\\ 2(-1) &= ?\\ 2(-2) &= ?\\ 2(-3) &= ?\end{aligned}$$

Notice that as we move upward from 2(0), we add a multiple of two each time. If that is the pattern we want to preserve, then in the case of 2(-1), we want to be able to add 2 in order to get to the next level, 2(0); for 2(-2), we want to be able to add a multiple of 2 two times to get the 2(0) level; for 2(-3), we want to be able to add a multiple of 2 three times to get to the 2(0) level.

Thus, merely to preserve the *pattern* that we already have for multiplication of non-negative signed numbers, we would want:

$$2(-3) + 2(3) = 2(0) \tag{1}$$

But justification of the above equation (coming strictly from the pattern) *is tantamount to extending the distributive principle!* That is, now that the *pattern* has motivated us to strive for (1), how might we achieve it by looking strictly at the axiomatic structure of the number system? It is obvious that we could achieve the equality if we were allowed to distribute the left side of that equation, that is,

if $2(-3) + 2(3) = 2(-3 + 3)$.

It might be possible to view the desired result slightly differently. Since we want $2(-3) + 2(3)$ to be 0, we really are requesting that $2(-3)$ act like the additive inverse of $2(3)$; that is, we want $2(-3)$ to act like $-[(2)(3)]$. But that perspective sends us back immediately to the analysis we have just completed, for to say that $2(-3) = -[(2)(3)]$ is equivalent to asserting that $2(-3) + 2(3) = 0$.

Extension of the distributive principle thus provides the desired missing link and we have accomplished two things at once:

1. We can use the intuitive pattern argument to *motivate* the more axiomatically-based argument.

2. We see that the intuitive pattern argument does - in a disguised way - assume exactly what we felt we could bypass by moving away from an axiomatic approach.[1]

It should be a source of consolation rather than distress that - as Morris Kline has been trying to tell us for a long time - rigorous formulations of a problem and intuitive ones not only do not belong to different moral planes but may in fact have more in common logically than we generally concede.

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[1]See Stephen I. Brown, "Multiplication, Addition and Duality," in *The Mathematics Teacher*, October 1966, pp.543-51, for an analysis of why it is that $a(-n) = -[(a)(n)]$ belongs to the class of equations that *require* the distributive principle in their proofs.