What's in a Cryptarithm

by *William J. Bruce*

The purpose of this article is to suggest to teachers of mathematics how cryptarithms can be used as a source for design and instruction experiences in arithmetic and algebra.

Cryptarithms have been around for a long time, but for most people they have been little more than a superficial exercise in trial and error problem solving. Invariably these have been additive. By use of an example, the problem may be stated as follows:

Solve on the set {0,1,2,3,4,5,6,7,8,9} the *addition* cryptarithm

$$\begin{array}{r} S\ E\ E \\ \underline{S\ E\ A} \\ E\ B\ B \end{array}$$

in which each different literal number is to be replaced by a different decimal digit and S and E have nonzero replacements.

Examination of this example shows that four different literal numbers are involved. The replacements for these literal numbers must be such that the resulting numbers for S E E and S E A must add up to the number replacement for E B B. Carrying is permitted. By trial and error alone, even an elementary-grade student might find a solution to be one or the other of

$$\begin{array}{r} 255 \\ \underline{256} \\ 511 \end{array} \quad \text{and} \quad \begin{array}{r} 377 \\ \underline{378} \\ 755 \end{array}.$$

In this process, considerable practice in addition with carrying will have been experienced, as well as the thrill of having solved a problem, even though, possibly, only partially. The question should arise as to whether other solutions exist and the student will realize that the method of trial and error is limited.

Junior high school students should be able to pursue the problem of finding the general solution of this cryptarithm. It will be evident that the system

$$\begin{aligned} E + A &= B \\ E + E &= B \\ S + S &= E \end{aligned}$$

must be examined. From the first two relations it follows that E + A = E + E, so that A = E. But this is not possible since the replacements for A and E must be different. Thus, this system does not yield a solution. Note that no carrying has been used.

If carrying is introduced in the first equation, we get

$$\begin{aligned} E + A &= 10 + B \\ E + E + 1 &= B \\ S + S &= E \end{aligned}$$

as another possible system. But from the first two relations it follows that $E + A - 10 = E + E + 1$, so that $A = E + 11$. This implies that $A \geq 12$, since $E > 0$. But none of the replacements can be greater than 9, so the above system cannot have a solution either.

Similarly, if carrying is permitted in the second equation, we have

$$\begin{aligned} E + A &= B \\ E + E &= 10 + B \\ S + S + 1 &= E. \end{aligned}$$

The first two equations now yield $A = E - 10$ and this implies $A < 0$, which is not admissible, so this system has no solution.

There remains only one other possibility, namely, that carrying might occur in both of the first two equations. The system then will be

$$\begin{aligned} E + A &= 10 + B \\ E + E + 1 &= 10 + B \\ S + S + 1 &= E. \end{aligned}$$

From the first two equations, we get $A = E + 1$, which is possible. Since $E + A \geq 10$, and since $2S = E - 1$ from the third equation, the only possibilities are found by using these relations and choosing E first, say, to obtain

E = 5	6	7	8
A = 6	7	8	9
B = 1	3	5	7
S = 2	-	3	-

Therefore, this system yields only two solutions, namely,

$$\begin{array}{r} 255 \\ \underline{256} \\ 511 \end{array} \quad \text{and} \quad \begin{array}{r} 377 \\ \underline{378} \\ 755 \end{array} .$$

Since there are no other systems to solve, these are the only solutions of the cryptarithm. The procedure just described is applicable, with only minor differences, to the problem of finding the general solution of any given cryptarithm.

Students should be encouraged to experience the fun of designing and trying to solve their own addition cryptarithms. They should restrict their choice at first to three-letter words with few unknowns. The ones with three unknowns are not numerous. For example, the silly ones,

$$\begin{array}{r} B\ A\ A \\ \underline{B\ A\ A} \\ E\ B\ B \end{array} \quad \text{and} \quad \begin{array}{r} A\ B\ A \\ \underline{A\ B\ A} \\ B\ E\ B \end{array} ,$$

have three unknowns and two solutions each. Those with four unknowns are quite easily designed. These could be

$$\begin{array}{c} T\ A\ R \\ \underline{F\ A\ T} \\ R\ A\ T \\ \text{(no solution)} \end{array} \quad \begin{array}{c} F\ A\ T \\ \underline{R\ A\ T} \\ T\ A\ R \\ \text{(3 solutions)} \end{array} \quad \begin{array}{c} A\ N\ N \\ \underline{E\ A\ T} \\ T\ E\ A \\ \text{(2 solutions)} \end{array} .$$

As more and more unknowns are included, an even wider variety is possible and in some cases the number of solutions becomes quite great. The following have five unknowns:

A N N	A N N	T O M
T O O	N O T	T O O
F A T	F A T	F A T
(10 solutions)	(no solution)	(11 solutions) .

For those with six unknowns, try

P A T	S A M	T E N	S U N	D A N	E A T
S I P	S I P	P I N	N O T	N O T	B A D
P O P	P O P	B E T	H O T	B A D	D O G .
(no solution)	(many solutions)				

For seven unknowns, attempt

E A T	J A N	T E A	D A N	P A T
H O T	N O T	F O R	E A T	N O T
D O G	B A D	T W O	D O G	B A D
				(320 solutions) .

When cryptarithms are extended to four-letter words, the solutions involve solving systems of four equations but the procedure is the same as before. Consider the addition cryptarithm

J E A N
B A R R
N I C E .

The systems that must be considered are

CASE I. *No carrying*

$N + R = E$
$A + R = C$
$E + A = I$
$J + B = N$ (no solution)

CASE II. *Carrying in one equation*

(a) $N + R = 10 + E$
$A + R + 1 = C$
$E + A = I$
$J + B = N$
(4 solutions)

(b) $N + R = E$
$A + R = 10 + C$
$E + A + 1 = I$
$J + B = N$
(no solution)

(c) $N + R = E$
$A + R = C$
$E + A = 10 + I$
$J + B + 1 = N$
(10 solutions)

CASE III. *Carrying in two equations*

(a) $N + R = 10 + E$
$A + R + 1 = 10 + C$
$E + A + 1 = I$
$J + B = N$
(26 solutions)

(b) $N + R = 10 + E$
$A + R + 1 = C$
$E + A = 10 + I$
$J + B + 1 = N$
(no solution)

(c) $N + R = E$
$A + R = 10 + C$
$E + A + 1 = 10 + I$
$J + B + 1 = N$
(2 solutions)

CASE IV. *Carrying in three equations*

$N + R = 10 + E$
$A + R + 1 = 10 + C$
$E + A + 1 = 10 + I$
$J + B + 1 = N$
(10 solutions)

A total of 52 solutions were found for this addition cryptarithm, but an independent check has not been done.

The four-letter word addition cryptarithms, namely,

$$\begin{array}{c} \mathrm{D\ E\ A\ R} \\ \underline{\mathrm{E\ D\ N\ A}} \\ \mathrm{M\ E\ A\ N} \end{array} \quad \text{and} \quad \begin{array}{c} \mathrm{C\ A\ S\ H} \\ \underline{\mathrm{B\ U\ Y\ S}} \\ \mathrm{C\ A\ R\ S} \end{array},$$

have no solutions, but

$$\begin{array}{c} \mathrm{D\ E\ A\ R} \\ \underline{\mathrm{E\ D\ N\ A}} \\ \mathrm{N\ I\ C\ E} \end{array} \quad \text{and} \quad \begin{array}{c} \mathrm{L\ I\ V\ E} \\ \underline{\mathrm{V\ I\ L\ E}} \\ \mathrm{E\ V\ I\ L} \end{array}$$

do have solutions. The last one is special in that it has only one solution.

Five-letter ones are very common. Try

$$\begin{array}{c} \mathrm{W\ R\ O\ N\ G} \\ \underline{\mathrm{W\ R\ O\ N\ G}} \\ \mathrm{R\ I\ G\ H\ T} \\ \text{(20 solutions)} \end{array} \quad \begin{array}{c} \mathrm{R\ I\ G\ H\ T} \\ \underline{\mathrm{R\ I\ G\ H\ T}} \\ \mathrm{W\ R\ O\ N\ G} \\ \end{array} \quad \begin{array}{c} \mathrm{P\ O\ L\ A\ R} \\ \underline{\mathrm{B\ E\ A\ R\ S}} \\ \mathrm{L\ O\ O\ S\ E} \\ \end{array} \quad \begin{array}{c} \mathrm{T\ A\ R\ T\ Y} \\ \underline{\mathrm{P\ A\ R\ T\ Y}} \\ \mathrm{F\ O\ R\ A\ Y} \\ \end{array} \quad \begin{array}{c} \mathrm{A\ P\ P\ L\ E} \\ \underline{\mathrm{M\ A\ P\ L\ E}} \\ \mathrm{B\ E\ E\ C\ H} \\ \end{array} \quad \begin{array}{c} \mathrm{T\ I\ T\ H\ E} \\ \underline{\mathrm{T\ I\ T\ H\ E}} \\ \mathrm{F\ I\ F\ T\ H} \\ \text{(one solution)} \end{array}$$

in which the number of solutions is given only for the first and the last ones.

Subtraction cryptarithms can be designed and solved in much the same way as addition ones. Consider the *subtraction* cryptarithm

$$\begin{array}{c} \mathrm{E\ B\ B} \\ \underline{\mathrm{S\ E\ A}} \\ \mathrm{S\ E\ E} \end{array}$$

in which each different literal number is to be replaced by a different decimal digit and S and E have nonzero replacements. The systems that must be considered are as follows:

CASE I. *No borrowing*

$B - A = E$
$B - E = E$
$E - S = S$

No solution is possible for this case because the first two equations imply $A = E$.

CASE II. *Borrowing in one equation*

(a) $B + 10 - A = E$
$(B - 1) - E = E$
$E - S = S$
(no solution)

(b) $B - A = E$
$B + 10 - E = E$
$(E - 1) - S = S$
(no solution)

CASE III. *Borrowing in two equations*

$B + 10 - A = E$
$(B - 1) + 10 - E = E$
$(E - 1) - S = S$

This system yields two solutions, namely,

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511        755
256  and   378
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255        377
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Note that the borrowing of 1 in the tens and hundreds places is indicated by B - 1 and E - 1, respectively, followed by the addition of 10 to each of the left members of the previous equations. Once the system is formed, the solution sets are determined as in the addition cases.

Rearrangement of each system for the *subtraction* cryptarithm will yield the corresponding system for the equivalent *addition* cryptarithm

```
S E E
S E A
-----
E B B .
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This will be true always, so all *subtraction* cryptarithms can be solved first as *addition* cryptarithms to obtain the solution sets.

Teachers are urged to attempt a school project on the design and solution of both addition and subtraction cryptarithms. Unusual ones and original ones that are found are usually accepted for publication in both mathematics and science journals.

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