The Fixed Element Match Problem:
An Illustration with Cards

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In probability and statistics units, the following two problems are often considered. Suppose that each of n-persons is given a standard deck of 52 playing cards. At a given time, each person draws at random a card from his/her deck.

1. What is the probability that at least two people draw exactly the same card (same suit, same face value)?

2. If a specific card is designated in advance, what is the probability that at least one person will draw it?

Many students have difficulty in distinguishing between these two problems; let us solve both.

1. Let a group of n-persons be arranged in some order. If each person consecutively draws a card, there can result $\underbrace{52 \cdot 52 \cdot 52 \cdot \cdot \cdot \cdot \cdot 52}_{\text{n factors}}$ or 52^n distinct n-tiples of cards. For each person, each of the cards is equally likely to be drawn. Since the n-persons are drawing their cards independently, each of the 52^n n-tiples is equally likely to occur.

How many of these n-tiples contain *no* matches? Again, visualize the n-persons selecting their cards consecutively. The first person can select any of 52 cards; to avoid a match, the second person can select from 51 cards, the third from 50 cards, and so on. Finally, the n^{th} person can select from any of $(52 - n + 1)$ cards. The number of ways that these n cards can be selected with no matches is thus $52 \cdot 51 \cdot 50 \cdot \cdot \cdot \cdot \cdot (52 - n + 1)$. The probability of no matches is thus $\frac{52 \cdot 51 \cdot 50 \cdot \cdot \cdot \cdot \cdot (52 - n + 1)}{52^n}$; the probability of at least one match is therefore $1 - [\frac{52 \cdot 51 \cdot 50 \cdot \cdot \cdot \cdot \cdot (52 - n + 1)}{52^n}]$.

2. For each draw, the probability of selecting a given card is 1/52. The probability of not selecting a given card is 51/52. Since the n-draws are independent, the probability that none of the n-persons will select the given card is $\underbrace{\frac{51}{52} \cdot \frac{51}{52} \cdot \frac{51}{52} \cdot \cdots \cdot \frac{51}{52}}_{n \text{ factors}}$ or $(\frac{51}{52})^n$. Consequently, the probability that at least one person selects the given card is $1 - (\frac{51}{52})^n$.

The students who think the two problems are substantially the same are very surprised that distinctly different probabilities arise from them. Table I reports these probabilities for varying sizes of n.

TABLE I

n (number of decks)	*Probability of at least one match (Problem 1)*	*Probability of matching a given card (Problem 2)*
1	0	.0192
2	.0192	.0381
3	.0570	.0566
4	.1114	.0747
5	.1793	.0925
6	.2586	.1100
7	.3441	.1271
8	.4324	.1439
9	.5197	.1603
10	.6029	.1765
11	.6792	.1923
12	.7471	.2079
13	.8055	.2231
14	.8541	.2380
15	.8934	.2527
16	.9241	.2671
17	.9475	.2812
18	.9646	.2950
19	.9769	.3085
20	.9853	.3218
21	.9910	.3349
22	.9946	.3477
23	.9969	.3602
24	.9983	.3725
25	.9991	.3846
26	.9995	.3964
27	.9998	.4080
28	.9999	.4194
29	.9999	.4306

n (number of decks)	*Probability of at least one match (Problem 1)*	*Probability of matching a given card (Problem 2)*
30	1.0000 (rounded to 4 decimal places)	.4415
31		.4523
32		.4628
33		.4731
34		.4833
35		.4932
36		.5029

Observe from the table that for Problem 1 only 9 decks are needed before the probability of at least one match exceeds 1/2, while to match a fixed card (Problem 2) 36 decks are needed before the probability of such a match exceeds 1/2.

Logarithms can also be used to achieve the "break-even" value of n in Problem 2. Find the number of decks needed in order that the probability of at least one match with a fixed card exceeds 1/2. Algebraically, solve the following inequality for n: $1 - (\frac{51}{52})^n > \frac{1}{2}$.

Equivalently: $(\frac{51}{52})^n < \frac{1}{2}$

$$n \log_e (\tfrac{51}{52}) < \log_e (\tfrac{1}{2})$$

$$n > \frac{\log_e (\frac{1}{2})}{\log_e (\frac{51}{52})} \quad \text{(Since } \tfrac{51}{52} < 1\text{, the log is negative.)}$$

$$n > \frac{-.6931}{-.0194}$$

$$n > 35.70$$

Consequently, at least 36 decks are needed.

For Problem 1 with 30 decks, the probability of at least one match exceeded .99995 and thus rounded to 1.0000 (4 decimal places). In Problem 2, how many decks are needed so that the probability of matching the fixed card exceeds .99995 and rounds to 1.0000?

$$1 - (\tfrac{51}{52})^n > .99995$$

$$(\tfrac{51}{52})^n < .00005$$

$$n \log_e \left(\frac{51}{52}\right) < \log_e .00005$$

$$n > \frac{\log_e 0.00005}{\log_e \left(\frac{51}{52}\right)}$$

$$n > \frac{-9.9034876}{-.01941809}$$

$$n > 510.01$$

Therefore, 511 decks are needed.

Challenge for the reader: A well-known prabability situation involves comparing birthday anniversaries for each of n-randomly-selected persons. State problems analogous to Problems 1 and 2 in the birthday situations and construct an appropriate table for comparison purposes.