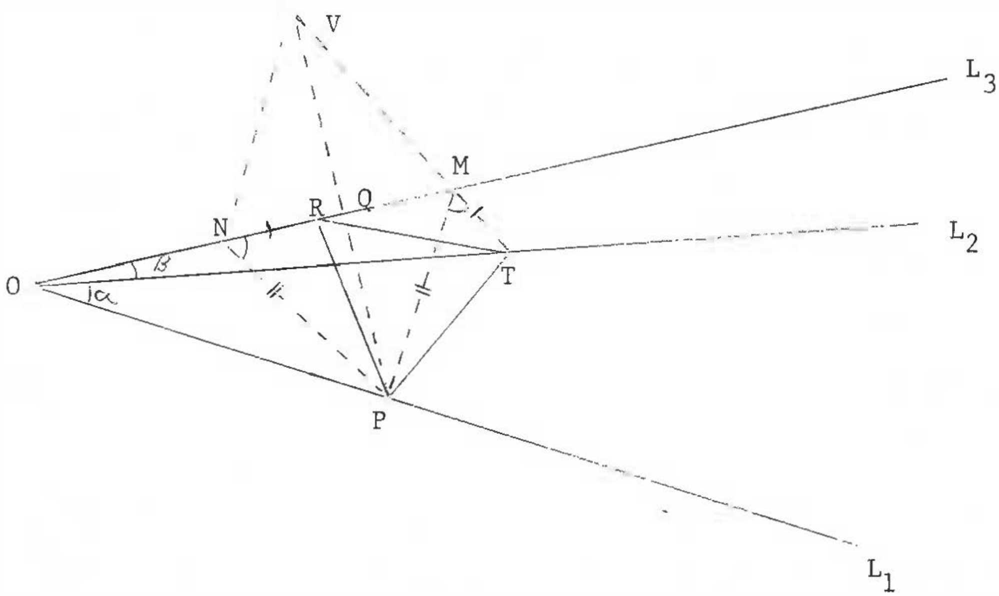
A STRAIGHT EDGE AND COMPASS CONSTRUCTION, by William J. Bruce and Edgar Phibbs

Editors' Note: Professors Bruce and Phibbs of the University of Alberta, Edmonton, found a construction in plane geometry rather challenging and they pass their discoveries along in an article.

Problem: L_1, L_2 and L_3 are three rays emanating from O. P is any point on L_1. Locate T and R on L_2 and L_3, respectively, so that $\triangle$ PRT is equilateral.

Solution I. Consider the general case with rays L_1, L_2 and L_3 such that $\alpha = \beta$ as shown.



Construction:

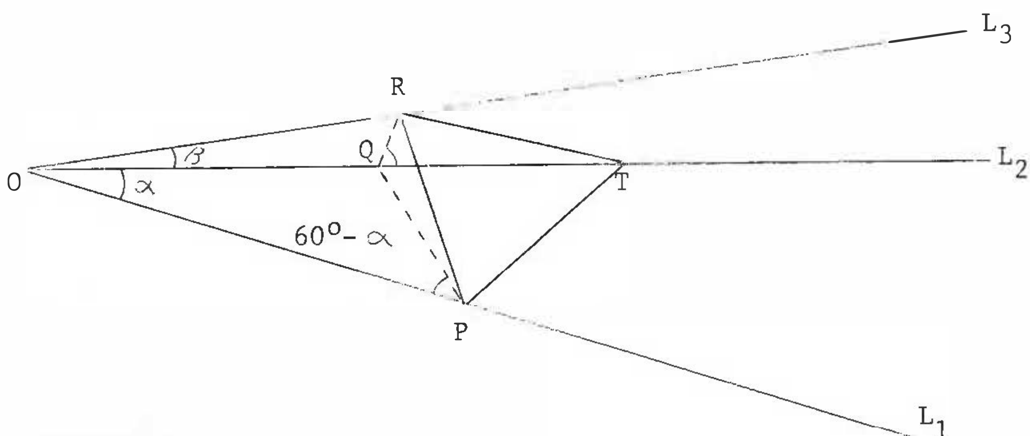
1. Construct $PQ \perp L_3$.
2. Construct $\angle MPQ = \angle NPQ = 30^o$.
3. Construct the equilateral $\triangle$ NVM.
4. Produce VM to cut L_2 at T.
5. Construct NR = MT.
6. $\triangle$ PRT is equilateral.

Proof:

1. $\triangle$'s PNR and PMT are congruent by construction (s.a.s.)
2. Hence PR = PT and $\angle$ NPR = $\angle$ MPT.
3. $\angle$ RPT = 60^o since $\angle$ NPM = 60^o by construction and equal angles NPR and MPT are subtracted therefrom and added thereto respectively.
4. Hence $\blacktriangle$ PTR is equilateral.

Note: If VN is produced to cut L_2 and steps corresponding to the above are followed, another such equilateral triangle results but with T on the opposite side of PR.

Solution II. Consider the general case with rays L_1, L_2 and L_3 such that $\alpha = \beta$ as shown.



Construction:

1. Construct $\angle$ OPQ $= 60^o - \alpha$.
2. Construct $\angle$ RQT $= 60^o$.
3. Draw a circle through P, Q and R cutting OL_2 at T.
4. $\triangle$ PRT is equilateral.

Proof:

1. $\angle$ PQT $= 60^o$.
2. $\angle$ TRP $= \angle$ PQT $= 60^o$. (Subtended by the same chord TP)
3. $\angle$ TPR $= \angle$ TQR $= 60^o$. (Subtended by the same chord TR)
4. Hence $\triangle$ PTR is equilateral.

Note: If $\angle L_1PQ$ is constructed to be $60^o + \alpha$, another such equilateral triangle results but with T on the opposite side of PR.

Special Cases and Remarks

When $\alpha + \beta \leq 60^o$ two triangles are possible without considering the backward extension of the rays. If $\alpha + \beta = 60^o$, one vertex lies at 0. If in addition to this, $\alpha = \beta$, the construction is trivial.

When $60^o < \alpha + \beta \leq 120^o$ two triangles are possible by permitting the backward extension of the rays. In this case, one triangle lies on the forward extension, the other on the backward extension. If $\alpha + \beta = 120^o$, one vertex lies at 0. Again, if in addition to this, $\alpha = \beta$, the construction is trivial.

For $120^o < \alpha + \beta < 360^o$ two triangles are generally possible by permitting the backward extension of the rays. If $\alpha = \beta = 90^o$, the construction is trivial with one triangle on the backward extension. If $\alpha = \beta = 120^o$, the construction is also trivial with three triangles possible. One of these triangles is on the forward extensions and the other two on the backward extensions.