TRIGONOMETRIC FUNCTIONS OF SUMS

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The following analytic method of launching a study of trigonometric functions of sums involves the unit circle, the distance formula and a simple rotation; it also furnishes quite briefly a completely general result in a rather elegant manner.

Consider any two angles α and β in standard position with P_1 and P_2, respectively, points on their terminal sides on a unit circle. The coordinates of P_1 are thus $(\cos\alpha, \sin\alpha)$ and those of P_2 are $(\cos\beta, \sin\beta)$ - see Figure 1. By the distance formula, $d^2 = |P_1P_2|^2$ is given by

$$
\begin{aligned}
d^2 &= (\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 \\
&= \cos^2\alpha + \cos^2\beta - 2\cos\alpha\cos\beta + \sin^2\alpha + \\
&\qquad \sin^2\beta - 2\sin\alpha\sin\beta \\
&= 2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)
\end{aligned}
$$

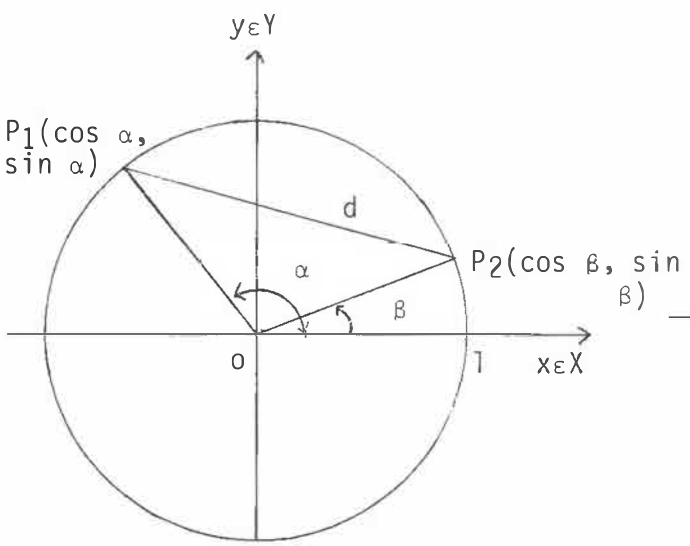

Fig. 1

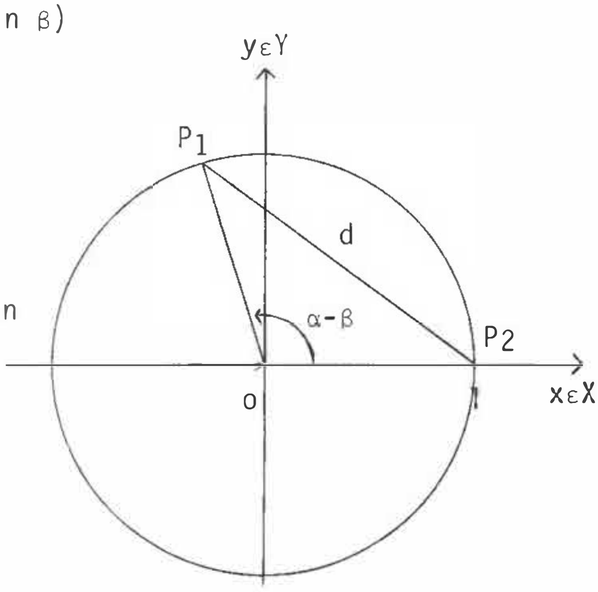

Fig. 2

Rotate P_1P_2 about O, through the angle $-\beta$, so that $P_1(\cos\alpha, \sin\alpha)$ becomes $P_1[\cos(\alpha-\beta), \sin(\alpha-\beta)]$ and $P_2(\cos\beta, \sin\beta)$ becomes $P_2[\cos(\beta-\beta), \sin(\beta-\beta)]$ or $P_2(1,0)$ - see Figure 2. By the distance formula we now obtain

$$d^2 = [\cos(\alpha-\beta) - 1]^2 + \sin^2(\alpha-\beta)$$

$$= \cos^2(\alpha-\beta) - 2\cos(\alpha-\beta) + 1 + \sin^2(\alpha-\beta)$$

$$= 2 - 2\cos(\alpha-\beta)$$

Since rotation does not change d, we have

$$2 - 2\cos(\alpha-\beta) = 2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$$

which upon simplification yields

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Furthermore, since the preceding arguments are the same for all values of α and β, the above result is true in general.

TEACHER'S GLOSSARY OF NEW MATHEMATICAL TERMS
(From the *Bulletin of the California Mathematics Council*)

SET: What you do in a chair.
SUBSET: What you do under a chair.
PROPER SUBSET: Sitting straight under a chair.
EMPTY SUBSET: Somebody is absent.
CLOSED SET: Kindergarten teachers.
ELEMENT: Large animal with a trunk.
CLOSURE: Last day of school.
SYMBOL: Part of a brass band.
BINARY: Two-headed canary.
RATIONAL NUMBER: Four-day week.
UNIVERSE: Poems you know.
IRRATIONAL NUMBER: Parent with a complaint.
FRACTION: Broken bones.
PLANE: Not fancy.
