CONJUGATE MULTIPLICATIVE INVERSES

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Quite often in a classroom situation it is possible to take an ordinary problem and generalize it to the point of developing good creative effort on the part of students.

In working with complex numbers in an intermediate algebra class, the following textbook problem was encountered:

Find the multiplicative inverse of $\frac{1}{2} + \frac{\sqrt{3}}{2} i$.

The students applied the usual methods and found the inverse to be $\frac{1}{2} - \frac{\sqrt{3}}{2} i$. Someone in the class noticed that, unlike other complex numbers, the multiplicative inverse of $\frac{1}{2} + \frac{\sqrt{3}}{2} i$ was its conjugate.

The question then arose, "What kinds of complex numbers have conjugates for multiplicative inverses?" From working previous problems, the students were aware that this was not true of all complex numbers. What, then, were the characteristics of this particular number that led to this curious fact? Could a general complex number with these characteristics be found? How should the investigation proceed?

An examination of $\frac{1}{2} + \frac{\sqrt{3}}{2} i$ did not provide any clues. The following line of investigation was then suggested: Assume there is such a complex number and let it be $x + yi$. Then $(x + yi)(x - yi) = 1$. This leads to $x^2 + y^2 = 1$. The only requirement then is that $x^2 + y^2 = 1$. A check shows that $\frac{1}{2} + \frac{\sqrt{3}}{2} i$ satisfies this requirement since $(\frac{1}{2}) + (\frac{\sqrt{3}}{2})^2 = 1$.

The next question concerned the problem of generating more of these special complex numbers: How could we find more? After an investigation another complex number was produced: $\frac{3}{5} + \frac{4}{5} i$. Students then noticed that the numbers 3, 4, 5, are a Pythagorean triple. The proof that any Pythagorean triple would work followed easily. Since the Pythagorean relationship $a^2 + b^2 = c^2$ leads immediately to $\frac{a^2}{c^2} + \frac{b^2}{c^2} = 1$, we need only let $x = \frac{a}{c}$ and $y = \frac{b}{c}$ to satisfy our only requirement: $x^2 + y^2 = 1$.

At this point we decided to let $x = \frac{a}{c}$ and $y = \frac{b}{d}$ and to state in general terms what we had concluded. So far we had decided that any complex number of the form $\frac{a}{c} + \frac{b}{d} i$ would have its conjugate for a multiplicative inverse, provided that $c = d$ and $a^2 + b^2 = c^2$.

We now were able to find complex numbers of this special nature where both $\frac{a}{b}$ and $\frac{b}{d}$ were rational. But suppose $\frac{a}{c}$ or $\frac{b}{d}$ were not rational; then what would the relationship have to be? We decided to first investigate complex numbers of the form $\frac{a}{c} + \frac{\sqrt{b}}{c} i$. It is then easy to conclude that $a^2 + b = c^2$ is our requirement. Letting a and c be any integers such that $/c/>/a/$, b is very easy to find: $b = c^2 - a^2$. For example, if we let $a = 5$ and $c = 7$, then $b - 7^2 - 5^2 = 24$. Thus $(\frac{5}{7} + \frac{\sqrt{24}}{7} i)(\frac{5}{7} - \frac{\sqrt{24}}{7} i) = 1$.

We concluded that we are permitted one irrational number, and complex numbers of the form $\frac{a}{c} + \frac{\sqrt{b}}{d} i$ have conjugates for multiplicative inverses when $c = d$ and $a^2 + b = c^2$.

Next we asked, "Are we permitted two irrational numbers?" If we can have two irrationals, then our requirement is $\frac{a}{c^2} + \frac{b}{c^2} = 1$ and $a + b = c^2$. Choosing $c>1$, it is easy to find at least one pair of a,b. For example, let $c = 15$: then $(\frac{1}{15} + \frac{\sqrt{224}}{15}i)$, $(\frac{\sqrt{2}}{15} + \frac{\sqrt{223}}{15}i)$, $(\frac{\sqrt{3}}{15} + \frac{\sqrt{222}}{15}i)$, etc., all have conjugate multiplicative inverses. Our third result was that complex numbers of the form $\sqrt{\frac{a}{c}} + \sqrt{\frac{b}{d}}\, i$ have conjugates for multiplicative inverses when $c = d$ and $a + b = c^2$.

An interesting sidelight might be developed at this point. One of the most famous of unsolved problems in the "Goldbach's Conjecture". In a letter to Euler written in 1742, Goldbach observed that every even integer, excepting 2, seemed representable as the sum of two primes. As yet this conjecture is unproved, but the Russian mathematician Schnirelmann showed that every positive integer can be represented as the sum of not more than 30,000 primes. Later Kloostermann reduced this number to 6. High school students are often surprised to learn that there is any unsolved problem remaining in mathematics, particularly one that is so simply stated.

Our final question was: "Is it necessary for c to equal d?" More generally if the product $(\sqrt{\frac{a}{c}} + \sqrt{\frac{b}{d}}i)(\sqrt{\frac{a}{c}} - \sqrt{\frac{b}{d}}i) = \frac{a}{c} + \frac{b}{d} = f$, here f is an integer, does $/c/ = /d/$? At this point it was necessary to prove the following theorem: If $\frac{a}{b} + \frac{b}{d} = f$, f an integer and $\frac{a}{c}$ and $\frac{b}{d}$ are in reduced form, then $/c/ = /d/$.

This is our final result. We may then characterize a complex number that has its conjugate for a multiplicative inverse as one of the form $\frac{a}{c} + \frac{b}{d}i$, where $a + b = c^2$, $/c/ = /d/$, a and b are real, and c is an integer such that $/c/ = /d/$.

In conclusion, it can be argued that our most general result could have been arrived at in one large step rather than a series of small steps. But this would have destroyed the exact learning situation we wished to create. We should develop the attitude in the student that perhaps each problem holds a hidden pattern that can be identified and pursued relentlessly to its most general form. In this way he has the chance to taste the exhilaration of a creative feat accomplished.

- *MTA News and Views*,
Vol.2, No.2, March 1968.
NSTU (Nova Scotia Teachers' Union)
Mathematics Teachers Association

The *Annual* will be mailed to members shortly.