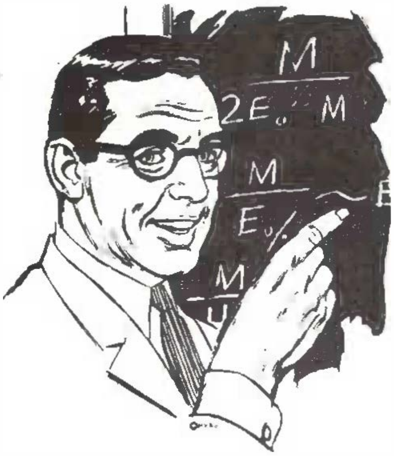

$$\frac{\hat{Y} - \bar{Y}}{s_X} = \cdot z_X ,$$

$$s^2_{(U+V)} = s^2_U + s^2_V + 2r_{UV}s_Us_V$$

and

$$s^2_{(U-V)} = s^2_U + s^2_V - 2r_{UV}s_Us_V.$$

The NEW NEW Mathematics

Dr. Julius H. Hlavaty

The highlight of the 1969 Annual Math Conference in Alberta was the following address by Dr. Hlavaty, president of the National Council of Teachers of Mathematics. It was recorded by Jim Kean, president of MCATA.

Today I should like to talk about three things: first, very briefly, about the old mathematics (the bad old days); second, where we are today (probably labelled the new mathematics); and, last, "Where do we go?" (and that would be the new, new mathematics).

You have heard the old days described often enough. Frankly, sometimes (and I took part in this campaign myself) the criticisms of the old mathematics instruction and curriculum were exaggerated for propaganda purposes and for other reasons. Nevertheless, it is a fact that however poor the instruction was, or however poor the mathematics, or however poor the textbooks, we did manage to train enough mathematicians, teachers of mathematics, physicists, and scientists. To be sure, we need more of them now.

What was the old mathematics? It was drill and more drill, and we shoved decimal points around and inverted fractions, but we didn't really know what it was all about. We borrowed, and we carried, and although we borrowed, we never gave back anything; and then we drilled and we drilled some more. This was the essence of arithmetical instruction. In algebra, we were in the groove by that time for doing the kinds of things that were called for in algebra. There, we always kept changing signs - whenever you saw a sign, you changed it immediately. And we played a sort of put-and-take game. You picked up something here and put it over there (and, of course, you had to change the sign right away). We solved problems, most of them supposedly very practical; they always dealt with these three fellows: A, B, and C - they worked themselves to death! They raced, they filled and emptied cisterns, they built, they mixed things (fortunately they never drank the stuff that resulted from the mixing) and, of course, poor C was always the low man on the totem pole. One of your great Canadian humorists, Stephen Leacock, wrote an essay - some of you perhaps know it - about those great characters, A, B, C. We drilled and drilled incessantly. You couldn't *leave* a thing unsimple anywhere. When I began teaching algebra, I used to teach my youngsters to *un*simplify fractions. As a matter of fact, in many cases is this not exactly what you need to do if you are going to get anywhere at all? Take a simple problem like one-half and one-third. Your first job is to unsimplify those fractions, isn't it? Then you can go ahead.

What did we do in geometry? First, we sprang a great surprise on our students (in the States, usually in the tenth grade.) We told them: "You know, there's such a thing as logic!", as if they had not been reasoning all their lives, right up to that moment - and reasoning logically! Parents know that a three-year-old can out-argue them on purely logical grounds, nine times out of ten! Yet in teaching geometry we pretended that young people didn't know anything about logic - we had to tell them. And what did we use this logic for? To prove an endless series of absolutely obvious things that anybody with half an eye knows are so. You put up a triangle with two equal sides and ask them, "What about those angles?" Anybody can see that they are equal, but we have to *prove* that!

In trigonometry the situation was not any better. We solved oblique triangles, for months on end! It had to be done to so many decimal places, accurately, using tables and so on. Again, in geometry and trigonometry we drilled and drilled, kept going over, incessantly, the same problems, presumably in the expectation that this would give some kind of skill.

The calculus was not exempt from this. Daily you did any number of derivatives and integrals. You had not the slightest idea what a derivative was, or what it was good for, or what an integral could mean.

What were the results of this? First, for the pupils: The pupils were stultified in any creative or imaginative mathematical interest, especially those who did not resist us sufficiently. Those who resisted (you know, the troublemakers in your class who always ask you questions) were not completely spoiled by this kind of education - they survived it. And my impression is that you were among the people who survived this kind of education. You were among the troublemakers in your class, who eventually learned some mathematics in spite of what was prescribed. The results for the teachers were, in my opinion, even more dangerous and stultifying, because we stopped living intellectually. And after a few years of teaching we were not getting the thrill and the joy that mathematics can and should always give, because, after a while, we were tired of turning these crankhandles and getting out the same old proofs. We ourselves did no more thinking in mathematics. Then, the dull things we were repeating became so obvious to us that we could not understand why those kids did not see it the first time we taught it.

WHAT ARE SOME OF THE CHANGES ALREADY MADE OR AT LEAST ON THE WAY?

In arithmetic, I think we have begun to give some meaning to a number. We know by now that it is not the thing you scribble on the board. It's something else, but it is a something that has real existence in our minds, and it is a very useful kind of symbol. Secondly, we have some idea now of what we mean by an operation. An operation is an association. It is a mapping of pairs of numbers into single numbers. This is a very difficult idea to develop, but we are beginning to develop it now in our elementary instruction. We are also beginning to give some reason so that youngsters know *why* they do some of the things that they do in arithmetic when they find a sum or a product.

You never knew why you shoved a decimal point this way or that way. When my students did addition, they would start in the right-hand column, first units, then tens, then hundreds - I would stop them and say "Wait a minute, why are you doing this problem backwards?" And they would look at me, again in surprise, "What are you talking about - backwards?" "Don't you read this way, from left to right?" I would ask. "Why suddenly do you read numbers from right to left?" They didn't know. Do you? I will let you in on a secret. It is one of those historical accidents: When the Italian merchants stole the Hindu-Arabic system of numeration from the Arabs, they lifted out of the books, in Arabic, the algorithms for all the operations; and, of course, the Arabs write *everything* backwards, including the numbers.

We are at least beginning to raise some of these questions and to discuss them, and students are beginning to see some sense in this world of arbitrary rules and regulations. If taught early, they can understand that the number system somehow or other evolves under our direction in response to expanding needs. First, the whole numbers help us answer questions such as "How many?" Then, in investigating problems about "How large?" or "How much?" we need fractions and decimals. Then with the problem of going this way or that way, signed numbers come in. Children are, I think, learning that these are not arbitrary but natural developments, necessary evolutions in response to expanding needs. Probably that realization is much more important than some of the algorithms which we used to work at so long and so diligently.

WHAT PROGRESS HAVE WE MADE?

We have introduced the concept of what a variable is. Now we use the notion of a set in a conscious way. The notion of sets is one of the things that tie together all of mathematics.

In algebra now, even in the grade schools, youngsters get some idea of open sentences (you may prefer to call them something else) and immediately, from the very beginning, not only equations but inequality. Some of you may recall your own first experience in the calculus, and what a chilling experience it was to use the first inequality there. Never in your life had you seen an inequality before! Through eight years of elementary school and four years of high school you had never *met* things that were not equal. No wonder, then, that when you actually used these inequalities in calculus, it was a traumatic experience, and many people did not survive it! That's why the calculus used to wipe out even more potential mathematicians than geometry - and that's saying a lot.

From seeing this notion of the number system as an evolving thing, we do get into large and substantial mathematical structures. I think the first great surprise is the real number system. You sense that here you have a complete world - it is closed; there seems to be no way out of it. Up to that time, there was always something new you had to do. With just the non-negative numbers, you can ask impossible problems and invent the negative numbers, and so on.

In geometry the reform movement and the revolution probably had least success, for a number of reasons. Perhaps it was because we had concentrated on the least exciting part of geometry - that is, the logical, axiomatic development of Euclid's geometry. Euclid's *Elements* is not a geometry textbook. It is a very modern text in integrated mathematics. It contains all the arithmetic, number theory, algebra, and geometry known in Euclid's day. Unfortunately we picked out the dullest theorems of all the 13 books, squashed them into six books, and said, "This is Euclid". We left out the beautiful proof of the infinity of primes; we left out algebraic discussions; we left out formulas of perfect numbers, we left out all the really elegant things in Euclid, and then we were brash enough to say, "This is geometry". What was the result? The reform movement tried to perfect the monster that we had created. It was said, "Euclid made all kinds of mistakes." And, of course, he did, for in his age he did not know what we know about real numbers. He had only the clumsiest numeration system. So he really could not cope with some of the problems. There are a great number of logical fallacies and faults in geometry. You know, for example, that by Euclid's own theorems you can prove that every triangle is isosceles, and you cannot disprove that proof by Euclid alone. In the reform movement, those involved wondered: "How do we eliminate these logical flaws in Euclid?" Therefore, all the changes made in geometry were only to perfect the logical structure. That made geometry even worse - because now, if you're going to talk about continuity, and you want to do it properly, you start at the beginning of September and you don't reach isosceles triangles until the following Easter! Some of the new texts do fall into precisely this trap - they want to be organized, overexact; I think students in Grade X or even in Grade XII don't want to know that much about the foundations of geometry. They do want to know about geometry. Nevertheless, we *did* improve the geometry; we have introduced, to a considerable extent, coordinate methods into geometry instruction, and we have introduced space geometry in our discussions with plane geometry, both of which I think are sound ways of expanding what we used to call plane geometry.

I am now going into some of the good things that we have done with trigonometry. We have moved away from the old computational trigonometry and went into the analytical trigonometry which is more important today and much more exciting. We have started talking not only about functions of angles but also about functions of numbers. We have brought in, for example, the wrapping functions as a beautiful way of introducing young people to what the trigonometric functions were. Even in calculus, there have been a great many improvements, and instruction has moved from just grinding out derivatives and integrals into trying to show that claculus is one of the greatest achievements of mankind - that calculus gave man a tool to analyze the most constant thing in the world, in life, namely, *change*.

The present textbooks, both commercial and experimental, are very much better than textbooks ever were, either in secondary or elementary schools. I think teaching is better than it has ever been, partly because of the all-out teachers' involvement in the last 10 years. We have found it necessary to start learning once again; the thing that can pep up teaching more than anything else is active learning activity on the part of the teacher every single day. Maybe this is over-stating it, but any day that I did not learn something new in the classroom, I considered a failure. We perform our teaching task well only when we ourselves keep learning and remain excited about what we are doing. We have now more teachers who know more about mathematics and we have more mathematicians who know more about education.

Because the reform movement was a joint venture of teachers, mathematicians, and education specialists, all of us have learned something that we did not know before. We have increased our knowledge of our subject matter, and the mathematicians have begun to appreciate and understand some of the educational problems. This is significant, because when we get to the next stage about "Where next?" we have a resource pool of people who will be able to work towards the answer more expertly.

To be altogether fair, there have been exaggerations in this movement, just as there were in the old. The old mathematics was never really quite so bad as I pictured it a few minutes ago. The new mathematics is not quite so beautiful as I have described it. When we discovered sets, we just went wild. It was like a new toy; every new book had two chapters on sets at the beginning; we spent three months on sets and then never mentioned sets again in the rest of the mathematics curriculum. We discovered scales of notation - wonderful! Such a delight, such a new thing to do! Addition and multiplications in scale seven notations! "Oh, great!" Here again we went overboard, and many books just went wild!

I still remember a teacher who, starting to teach SMSG mathematics, fell in love with non-decimal numeration systems, especially three. This could justifiably take perhaps a couple of lessons. This man spent a month and a half on it, and when I last saw him, he was having the class translate the log tables into base three numeration!

Yet despite these exaggerations, a fair balancing of what has happened would show that we have advanced in our mathematics teaching.

WHERE DO WE GO NOW?

Now we do have a number of tentative directions. Europeans, looking at what we were doing, said, "You have a good idea, but why not really reorganize mathematics?" And their reform movement spurted ahead (they are about five years in advance of us right now). Let me explain: Secondary education, in particular, in almost all European countries has been and is a highly selective process. Only five to 10 percent of the school population is in secondary schools. Therefore, they have fewer teachers - and, therefore, those fewer teachers can be expected to fulfill very high requirements, and so can the highly selected pupils. If you looked at texts written in the last four years in Europe, you would not recognize them as secondary school texts. The Europeans have reorganized their mathematics instruction to fit into what mathematics is today. Some attempts are being made here in this direction. The Cambridge Conference Report attempted just this.

What kinds of mathematics would be indicated for the new era? I would like to direct your attention to at least a few actual experiments. One is the UICSM (University of Illinois Commission on School Mathematics), the University of Illinois program. This was the very first approach to what the new mathematics ought to look like. Second, SMSG (School Mathematics Study Group), is going through what they call "the second round". Some of you may have heard rumors about it; they are writing a series of experimental units for Grade VII - but with this new view of a reorganization of the secondary school curriculum. Third, there is a global operation under Professor Howard Fehr of Teachers' College at Columbia - global in its view about mathematics and what should be taught, but not attempting to impose or introduce a brand-new program on all the schools in the country.

What is essential in this new program? First, new ideas will be introduced early. Perhaps some of you shudder when things like matrix algebra turn up in the third or fourth grade. Naturally, if you think of taking a semester course in matrix algebra, as in graduate school, this is not what is meant at all for the fourth grade! However, some of the ideas in matrix algebra - just a square or rectangular array of numbers to present a single fact - can be grasped by the young. A fraction is a very simple illustration of a matrix: two-thirds - that's two and three - there is a matrix. You have used two numbers to represent a single idea. Now consider the box score of a baseball game which includes runs, hits, and errors. We have a matrix consisting of six numbers. Nothing hard about that! Kids understand. This is what is meant by introducing matrix algebra early. Here are six numbers arranged in a rectangle to communicate a single idea. Such ideas would be introduced early, and after a while, at some stage, they would begin to be organized into a consistent and logical whole. These things would be organized now, not in the categories as we had them: arithmetic, algebra, geometry, trigonometry, and so on, but mathematics, mathematics, mathematics.

As for geometry, it properly always did begin, and should begin consciously, with the kindergarten. Kids know a lot of geometry then. They have been bumping into shapes and they are conversant with sizes - they know a lot of intuitive geometry. Geometry means much more than proving theorems from a set of axioms. It means looking at shapes, moving them around; looking at them through mirrors, trying to fit them (sometimes they do fit, sometimes they don't); turning them over; folding paper. That's geometry. None of this occurred in geometry as we learned it ourselves. Yet this *is* geometry. These are real genuine experiences with real genuine things.

Two major topics must enter early into the curriculum, and they must persist right through the curriculum: probability and statistics, and computers. Probability is perhaps the newest of the major mathematical sciences. It transfuses all thinking today in applications and in pure mathematics. For the first time in this century we have a very substantial tool to deal with phenomena of uncertainty. Where do we teach this? We start it in kindergarten, in the first grade, by collecting data, organizing data, thinking about it, trying to draw conclusions from it and, then, thinking about probabilities. The Commission on Mathematics realized that probability and statistics were important. Many of you will remember, with pleasure, the little grey book that they prepared. The topic is so important that it cannot, and should not, be done in one semester course or a year course. First, an accumulation of experience is needed, and second, a certain amount of maturity is required before one can begin to organize logically and systematically.

The other big and very new thing is the computer. It is an essential part of civilization today. Probably more than 50 percent of the students sitting in front of you every day have a computer in their future, one way or another. Where do you teach about computers? When you need to use them? At the age of 20 or 25? No! Here, again, we must start early and keep going.

I wanted to talk a little about what this would mean in the grades and in the high school. I will just consider a couple of things about trends and directions we should be thinking about. I think we should consciously start to de-emphasize some of the things (such as sets) that we have overemphasized. The basic ideas are important, but let's not overdo them. In the same way, we need de-emphasis on a hangover, namely algorithms and computations. I am *not* advocating not to teach computation, but I think understandings come first.

Let me conclude with a possible outline of what a new textbook for Grade VII would include:

(1) Finite Number Systems
(2) Sets and Operation
(3) Mappings
(4) The Integers
(5) Probability and Statistics
(6) The Integers (again)
(7) Lattice Points in the Plane
(8) Sets and Relations
(9) Transformations of a Plane (this means shoving things around and folding them over)
(10) Segments, Angles, Isometrics
(11) Elementary Number Theory
(12) The Rational Numbers

Some of these topics may sound frightening, but so did the idea of sets and functions when they were first introduced. They will become parts of an ever-improving and ever-expanding curriculum of the future.