Manipulative aids for developing concepts of three-dimensional space

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In space or 3 dimensions, geometric shapes or figures are determined by surfaces which may be curved (for example, a sphere) or plane (flat, for example, a cube) in which case they are called *faces*. If the figure is determined by faces, these meet in *edges*, which are line segments. These edges may be used to determine the figure, which is then bounded by the plane, or flat faces through the edges. In the case of both 2- and 3-dimensional figures, the intersections of the edges fix the vertices of the figure. If the 3-dimensional figures are closed, then they enclose a region or volume of space [Elliott, MacLean, and Jorden, 1968, p.63].

Aids such as straws emphasize the edges and vertices of a 3-D figure. Paper models emphasize the faces of a polyhedron. Unit cubes are useful for emphasizing the filling of space. Many experiences are needed by children to help them develop the concept of volume.

STRAWS AND PIPE CLEANERS

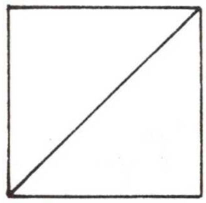
Experience 1

Each child is given a box of straws, pieces of covered wire or pipe cleaners and scissors.

1. Use 3 straws of equal length. What can you make using pipe cleaners to hold the straws together? Is it rigid or flexible? It is rigid. All triangles are rigid figures.

2. Use 3 more straws of equal length. Try to make a tent-like structure on your first figure. Is it rigid or flexible? This figure is called a tetrahedron and is rigid because it is made of triangles.

3. Use 12 straws of equal length. Make the skeleton of a cube. Is it rigid? To make a quadrilateral rigid insert a diagonal. How many diagonals must you insert in your skeleton of a cube to make it rigid? Try it and compare your results with your neighbor's. If you added a diagonal to each of the 6 sides, you have stiffened your cube. It is now divided into tetrahedra. How many?



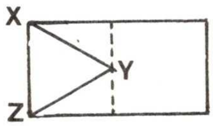
4. Use 8 straws of equal length. Use four of them to make a base. Is this base rigid? Cut another straw and insert it to make the square base rigid. Now use the other 4 straws to make a pyramid. Why is the figure rigid?

PAPER AND TAGBOARD

Experience 2

Give each child an envelope, scotch tape, and colored tagboard.

1. Johnson and Kipps (1970) suggest the following procedure for making a tetrahedron from an envelope. Seal the envelope. Draw an equilateral triangle, XYZ, on the end of the envelope using the end edge as one side. Draw a line through Y and perpendicular to the edge of the envelope. Cut along the dotted line. Fold along XY and ZY and bend back and forth several times. Pull out the open end of the envelope. XYZ will be one side of the tetrahedron. Use scotch tape to fasten the open edges together. Is the tetrahedron rigid? Why?



2. Cut out 16 equilateral triangles of the same size from colored tagboard. Make a tetrahedron. How many triangular shapes did you use? Draw a net or pattern for a tetrahedron. How many different patterns can you find?

3. Now use 8 of the triangular shapes. Fit them together and fasten them with scotch tape to make an octohedron. Make a net or pattern for an octohedron.

4. Cut out a square-shaped base which has a side equal to the side of the equilateral triangles. Make a pyramid using this square base and some triangular shapes. Draw a net or pattern for a square based pyramid.

5. Cut out 5 squares of equal size. Put them together to form an open cube. Try to draw 8 different nets or patterns for an open cube.

CARDBOARD CONTAINERS

Experience 3

Interesting cardboard containers can be found in supermarkets. J-cloths and straws sometimes are sold in hexagonal prisms. Certain types of rolled oats are sold in cardboard cylinders.

A most successful experiment for learning the relationship between the dimensions and volume of rectangular prisms was suggested by Marshal Bye at a Calgary conference. A child who has completed this experiment is not likely to forget it.

Children may work in groups. Each group is given a Rice Krispie carton. Each child is to make a box which has dimensions half as large as the dimensions of the carton. Each child guesses how many of the small boxes will be needed to fill the Rice Krispie carton. Each child records his guess before making his box. After the boxes are made, the children place as many of the small boxes as possible in the carton. How does the volume of each small box compare with the volume of the original Rice Krispie carton? This activity should be given after pupils have studied division of fractions.

Experience 4

Walter (1970) cut milk cartons to make open cubes. Make the height equal to the length and width of the base and cut off the top of the carton.

Numbered diagrams of the 8 possible patterns for an open cube are shown to the children. Each child decides which pattern he wishes to make and records its number. He must think how he should cut the carton to produce the pattern or net which he chose. The diagrams may be removed before the children cut the cartons. Have extra cartons available for children who are unsuccessful on the first attempt.

Experience 5

Mark off colored tagboard in square inches. Have each child make several open boxes so that you have a supply of boxes having the following dimensions in inches:

Box	Length	Width	Height
A	2	2	9
B	4	1	9
C	4	3	3
D	6	6	1
E	6	1	6
F	9	2	2
G	18	2	1

Fasten the edges with scotch tape. Have the squared side of the tagboard on the inside of each box. These boxes will be used in an activity described under unit cubes.

UNIT CUBES

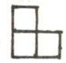
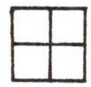
Cubes can be placed together to fill space completely. For this reason they are a valuable aid in the development of the concept of volume as the amount of space a solid occupies and as the amount of space in a hollow container.

Conservation of volume does not occur usually until a child reaches the age of 11 or 12 years. Informal experiences which a child has with unit cubes before that time will help him to reach conservation.

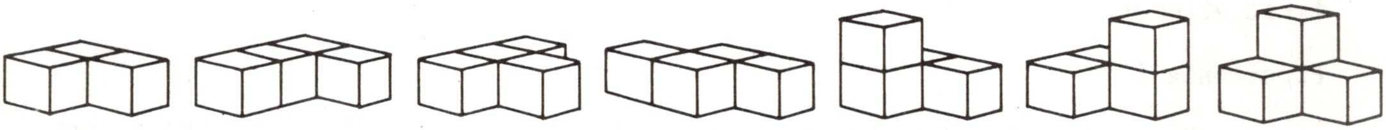
Experience 6

For this experience, each child needs about 33 unit cubes, 24 pieces of circular wood dowling, and 24 square-based wood prisms.

1. Try to build a wall of two thicknesses with each of the materials provided. (Nuffield, 1967). Record your results.

2. Patterns with cubes - How many patterns can you make with 3 cubes? Turn them around and see if they are really all the same. Call this Pattern 1: ⟶  How many different patterns can you make with 4 cubes? Try to find 6, 2 of which are mirror images of each other. If you have time, draw a picture of each of your patterns. These patterns may be found by beginning with Pattern 1 and adding one more cube in 6 different ways. The pattern  where the fourth block forms a square is not used.

In the next activity, these 7 patterns which you have found will be used again. Check yours with the illustration below:



1 2 3 4 5 6 7

Experience 7

The 7 pieces described in the second part of experience 6 form the SOMA puzzle. There are over 200 different ways to arrange the 7 pieces in a 3 x 3 x 3 cube. In order to determine whether a solution is the same as or different from others, always turn your 3 x 3 x 3 cube so that the L shape or Pattern 2 is on the top layer. Now place the L in one of the 4 positions shown below:

2	-	-
2	-	-
2	2	-

A

2	2	-
2	-	-
2	-	-

B

-	2	-
-	2	-
2	2	-

C

-	2	-
-	2	-
-	2	2

D

There are many different Type A solutions. To prove that you have found different solutions, make 3 grids for each solution and indicate in each square the number of the block found there.

Top Layer

Middle Layer

Bottom Layer

See how many different solutions you can find for Types A, B, C, D. Record each one and compare them with your neighbor's solutions.

Experience 8

Filling boxes with cubes.

Each child should have 40 blocks. The 7 boxes having the following dimensions were made in an earlier activity (Experience 5):

Box		Dimensions
A	-	2 x 2 x 9
B	-	4 x 1 x 9
C	-	4 x 3 x 3
D	-	6 x 6 x 1
E	-	6 x 1 x 6
F	-	9 x 2 x 2
G	-	18 x 2 x 1

Give each child the following table:

Box	Estimate of Number of Cubes Needed to Fill Box	Actual Number of Cubes Needed to Fill Box
A		
B		
C		
D		
E		
F		
G		

Have each child fill in all the estimates first, then use blocks to check. No mention need be made of dimensions. This experience highlights the fact that boxes of varied shapes may hold the same number of unit cubes.

TIN CANS

A variety of tin containers can be assembled by the children. These may include pop cans, small, medium and large fruit juice tins, salmon tins, squat size bean tins, large and small coffee tins, lard tins. Label each tin A, B, C, and so on. The following activity will assist children to understand another concept of conservation of volume, that is, the conservation of displaced volume (Copeland, 1970).

Experience 9

The following activity is adopted from Sawyer and Srawley (1957). Materials needed: 9 labeled cylinders, a large basin of water, balance and weights.

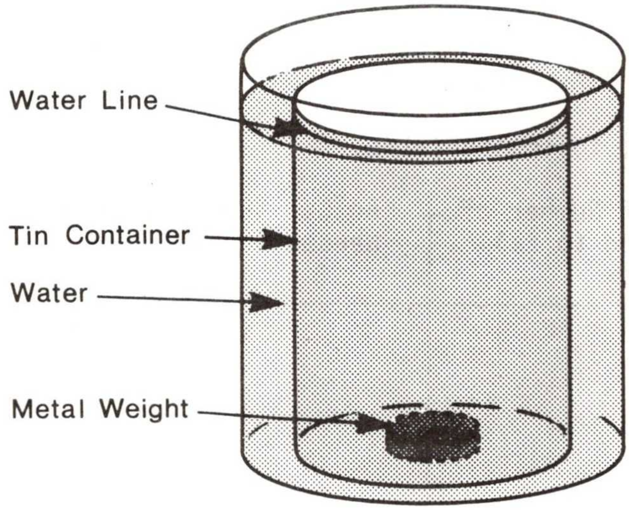
1. Fill each cylinder with water to the water line (see diagram below). Weigh each full cylinder and record the weight in grams in the table.

2. Empty each can and put enough weights in it until it nearly sinks in the basin of water. Mark the level of the water on each can. Record for each cylinder the weight in grams which was needed to just keep the tin afloat.

3. Measure and record the diameter of each cylinder.

4. Measure and record the height of each cylinder up to the water line.

5. Find the connection between the volume of each tin and the weight that almost sinks it. Do you need to consider the weight of the cylinder? Yes?



Cylinder	Wt. in Grams of Empty Cylinder	Wt. in Grams of Cylinder Filled with Water to Water Line	Total Wt. in Grams Needed to Nearly Sink Cylinder	Diameter of Cylinder	Height of Cylinder	Vol. in c.c. of Cylinder
A						
B						
C						
D						
E						
F						
G						
H						
I						

STYROFOAM

Hobby shops sell styrofoam in a variety of 3-D forms such as spheres of various sizes, cones, truncated cones (top cut off) and rectangular prisms.

Experience 10

Materials needed: 5 styrofoam cubes and a sharp knife. One size of cube which works well is a 3" x 3" x 3" but other sizes are possible.

1. Leave cube 1 intact.
2. On Cube 2, measure from each vertex, 3/8" along each side. Cut off the 8 corners as marked. This will be the first truncated cube; TC1.
3. On Cube 3, measure 3/4" from each vertex along each side. Cut off the 8 corners as marked. This will be truncated cube, TC2.
4. On Cube 4, measure 1 1/8" from each vertex along each side. Cut off the 8 corners as marked. This will be truncated cube TC3.
5. On Cube 4, measure 1 1/2" from each vertex along each side. Cut off the 8 corners as marked. This will be a cuboctahedron.

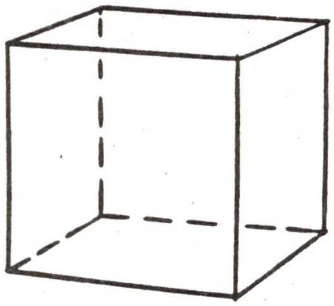
By observing the cube, the three truncated cubes, and the cuboctahedron, complete the following table:

Polyhedron	Shape of Faces			Number of Vertices	Number of Edges
	Square	Hexagonal	Triangle		
Cube					
Truncated Cube 1					
Truncated Cube 2					
Truncated Cube 3					
Octahedron					

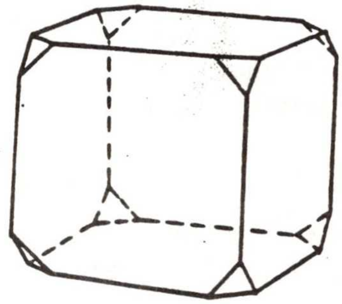
The diagrams on p.56 indicate that it is possible to begin with an octahedron and by paring off the eight vertices, to progress through truncated octahedra back to a cuboctahedron. (Guy, 1968.)

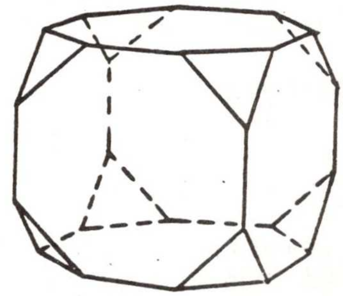
SUMMARY

The 10 activities described have emphasized the idea of the 3-dimensional object or shape occupying a space; the amount of space inside a container; conservation of volume and comparison of volume and capacity. These experiences are performed at the intuitive level as background for later work.

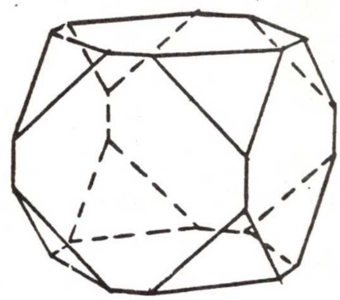


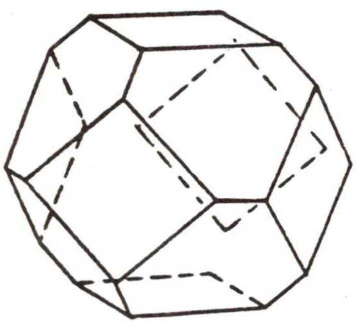
Cube

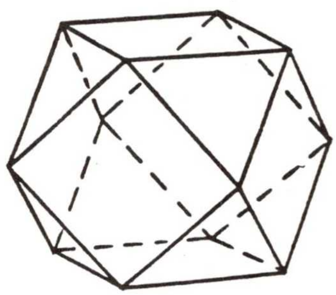




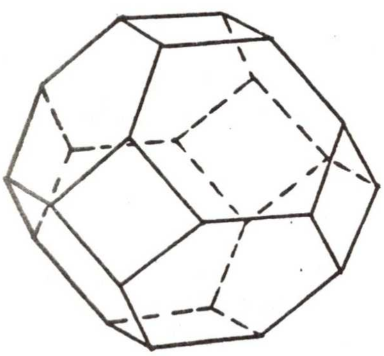
Truncated Cube



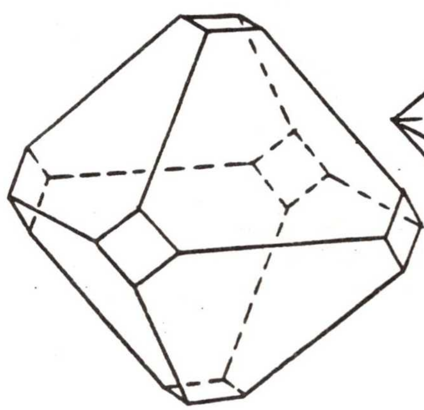


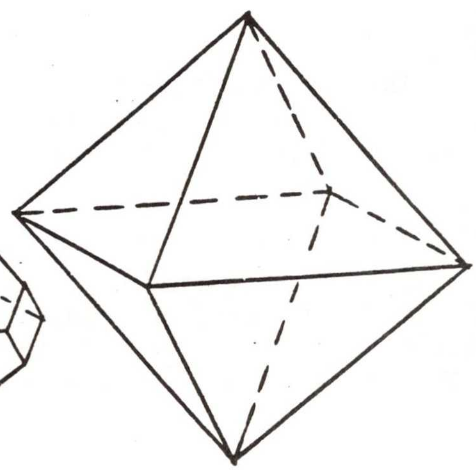


Cuboctahedron



Truncated Octahedron





Octahedron

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