Getting the 'rect'-angle on mathematical activity

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There has been a large amount of literature and perhaps an even larger amount of activity packages developed for and by teachers and students of mathematics over the last 5 years. The basis for much of this development came from the assumption of the value of individualized instruction and of such adages as "doing produces understanding", "go from concrete to abstract", and "discovery". The teacher, in face of the proliferation of such educational slogans and complex masses of material, must search out fundamental personal reasons for using activities. Upon finding convincing reasons for having students engage in such activities, the teacher might well ask, "Can this be done without large expenditures of time, money, or both?" It is one purpose of this paper to briefly discuss some issues surrounding the "Why activities?" question. The major portion of the paper will try to cope with the "how" problem by using the simple mathematical creature, the rectangle, and its sub-species, the square.

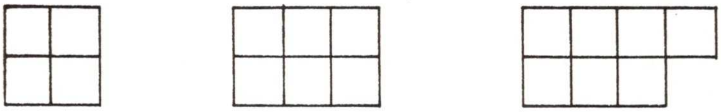
PLAYING AROUND WITH MATHEMATICS

There are many potential benefits of activities in mathematics. From a developmental point of view, it is considered that students, probably through junior high school age (Lovell, 1971) are capable of logical thinking about real or potentially real situations, but not very able to deal with completely hypothetical situations. Thus physical models or pictorial images provide a necessary grist for the logical- mathematical thinking of children perhaps up until the age of 14 or 15.

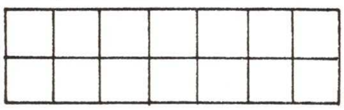
Perhaps as important as the notion of using concrete or pictorial models as starting points for mathematical ideas, is the notion that students can profitably play with such models. For the preschool child, 2 major modes of changing his picture of the world exist, playing and imitating. The latter affords the child the opportunity to reshape his thought to accommodate some phenomenon in the world. Thus a child watching hockey on T.V. finds a model he can imitate in handling a hockey stick, an action which may have been completely foreign to him previously. Playing allows the child to impose his already made ideas on new phenomena. For example, our young "hockey player" finds he can use balls, bottle caps or plastic discs as "pucks" and many things for sticks, goals and even rinks. Thus playing allows for creating powerful general ideas. In school, and perhaps this practice increases with the grade level, we tend to take a one-sided view of acquisition of ideas. Imitation is taken to be the tool, while play becomes a frivolous activity or one which perhaps can take place as a student practices with an idea or skill learned through imitation. In mathematics, this tends to mean that learning becomes being told the "right" way to do something and then verifying through practice that this way is useful. It denies the more playlike mathematical processes such as looking for quantitative or spatial aspects in a situation, looking for relationships, and guessing at patterns. In "playing" with physical materials and pictorial images, the student can bring his own ideas to bear and extend them to include new notions.

If the above arguments are convincing, the teacher is tempted to order a lot of materials or open the mathematics laboratory manuals and get on with the business of playing with mathematics in any form. But the business of learning mathematics is not that simple. Whether the activity is solving a problem, proving a theorem or applying mathematics to everyday life, mathematics means successfully working with symbols. Thus in our "play" (presenting activities-oriented experiences to students), we must ascertain that it will contribute to later symbolic activity. In particular, we must be certain that the student will not have to unlearn what he learned from his activity work in effectively dealing with mathematics symbolically. To insure this, the teacher must see that the mathematical form in the activity should be at least analogous to the later symbolic form. Perhaps an example will be useful.

An activity which one can do with congruent square tiles is to try to make rectangles 2 units on one side out of sets of tiles. For example, sets of 4 and 6 admit such rectangles while 7 does not.



Quickly the student sees numbers as falling into 2 sets, the evens whose tile sets make into "2 x n" rectangles and the odds whose sets make "rectangles with tails".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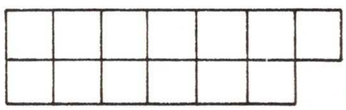
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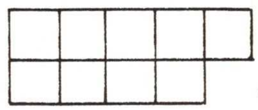
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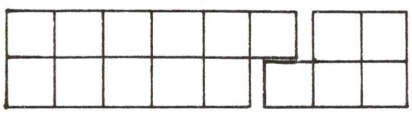
Once this classification is made, the student might "add odds" in the following concrete way and discover that an "odd plus an odd is an even".


11 + 7

Now of course the mathematical activity would not have to involve physical materials. Students who could divide by 2 could make the original classification and by the following type of exercise:

$$11 + 7 = 18, \quad 19 + 13 = 32, \quad 3 + 5 = 8$$

induce that "an odd plus an odd is even".

Since this symbolic activity is so easy, why all the fuss about concrete activity? The answer is "form". Adding 11 and 7 physically in the manner above is powerfully suggestive of the symbolic proof of the theorem:

$$\overset{\text{odd}}{(2n + 1)} + \overset{\text{odd}}{(2m + 1)} = \underset{\text{even}}{2n + 2m} + "2"$$

$$= 2(n + m + 1).$$

Adding 11 and 7 or 101 and 93 symbolically makes no contribution to the form of the mathematics. Thus though both activities allow for discovery, the physical activity allows for seeing more mathematics than the symbolic activity.

In answering the questions, "Why use physical and pictorial models?", "Why use labs?", or "Why use activities?" a teacher might consider the following guidelines which summarize the section above.

1. Activities with models provide an appropriate setting for development of mathematical ideas for students in elementary and junior high school. They can provide an appropriate bridge to the world of ideas and symbols.

2. Activities provide one opportunity for effective "play". During such "play", students can exercise such processes as seeing the mathematics in a situation, observing possible relationships and guessing and testing personal mathematical ideas.

3. To realize the above in a way which most contributes to further mathematics learning, the teacher must choose activities which best "form" mathematical ideas.

TO POLYNOMIALS AND BACK AGAIN

Once one has a basis for using activities, the question of how to do so effectively arises. Can one use activities in a variety of contexts and can one do so without lots of fancy materials? What follows is an answer to these questions.

The activities designed below relate to mathematics which has traditionally been in the curriculum for Grades IV to X. All of the activities are based on the following physical materials:

1. A large number of squares (1/2 to 3/4 inches on a side) of either oak tag or plastic.

2. A large set of cubes (1 cm. to 3/4 inch on a side).

3. Coordinated sets of squares and rectangles (such as 7 x 7 squares, 7 x 1 rectangles, unit squares) of wood or tag-board.

4. Grid paper.

5. Sets of colored oak-tag rectangles (1" x 2").

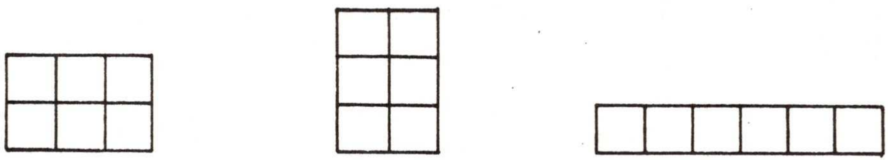
Several of the activities are just described; others are given in the form of activity cards. The methods of use can vary. The whole class can work individually on the same activity interspersed with teacher-direction or class discussion. Or the class may work in small groups, each group working on an independent activity. Above all, the activities are merely suggestive of things you can do to enrich the mathematical experience of your students.

Building up factors

Materials: A set of 25 to 40 squares of plastic or tag for each student or group.

CARD A1

How many rectangles can you make from 6 squares? Here are some.



3 by 2   2 by 3   6 by 1

The label below the rectangle tells how we describe each rectangle.
Complete the following table:

Number of Squares	Rectangles	How many Rectangles
1		
2		
3		
4		
5		
6	6 by 1, 3 by 2, 2 by 3, 1 by 6	4
7		
8		
9		
10		
11		
12		

(table can be extended to suit your class)

CARD A2

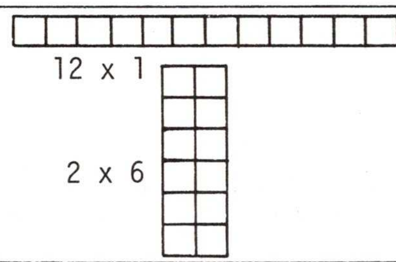
Explorations

1. For what number can you make the fewest rectangles?
2. List those numbers for which you can make only 2 rectangles.
3. Are there any numbers for which you can make squares?
4. Tell the number of squares used in these rectangles:

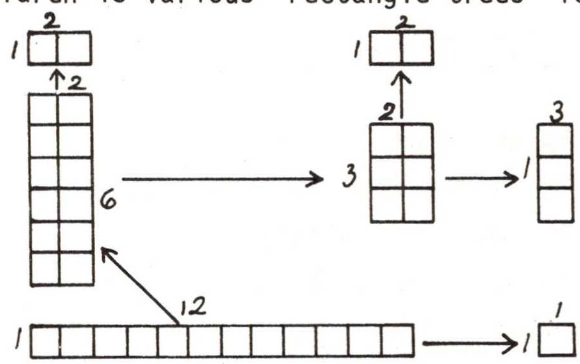
   (a) 4 by 2 ____________ (d) 7 by 3 ____________
   (b) 1 by 5 ____________ (e) 4 by 5 ____________
   (c) 12 by 8 ____________ (f) 30 by 20 ____________

   How do you find the number?
5. There are many other things which you can explore. Collect information or make charts on the following:
   (a) Are the number of squares and the number of rectangles you can make related?
   (b) For 4 squares, the numbers used in describing the rectangles are 1, 2, 4. The sum of these is 7. For 5 squares the numbers are 1,5. This sum is 6. Make this kind of sum for all the numbers in your chart and tell about any patterns you see.

Note: There are many uses for Card A2. Clearly the exercises mentioned in point 5 are more sophisticated and could be used as projects. In 1-4, little verbalization is called for. Yet these activities and their results have the <u>form</u> usable in the symbolization of factors and usable in the definition of primes or perfect squares. An alternative representation for the chart in A1 is a class display board to which students contribute correct rectangles:

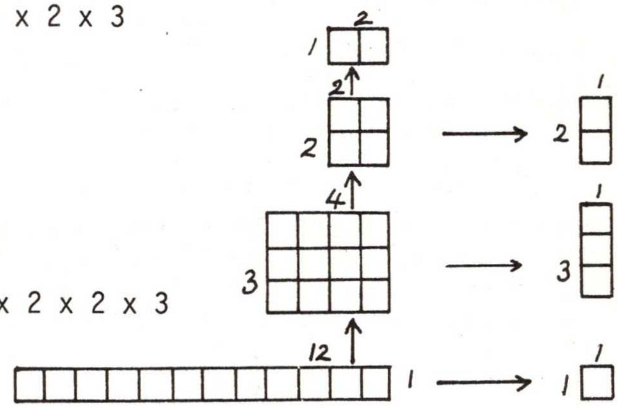
<table>
<tr><th>Number</th><th>Rectangles</th></tr>
<tr><td>12</td><td>
</td></tr>
<tr><td>13</td><td></td></tr>
</table>

This chart could then be used for later discussion. Another display device built by children is various "rectangle trees" for numbers:



12 = 1 x 12 = 1 x 4 x 3 = 1 x 2 x 2 x 3

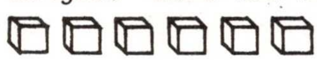
12 = 1 x 12 = 1 x 2 x 6 = 1 x 2 x 2 x 3



An Alternative to Card A1 would use cubes.

Piling Cubes

If you have 6 cubes you can pile them in several ways so that the piles are the same height. Here are some.

1 pile of 6   6 piles of 1  2 piles of 3 

Fill in the chart.

Number	Ways of Piling
1	
2	

This problem,[1] although equivalent, seems simpler than the rectangular problem and can be worked on successfully by children even as young as 6 or 7.

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[1]As a teacher, you could give this assignment over a 3-day period and then collect the results on a chart on the board. Try to collect as many different ways as possible for each number.

Prime bag:

The game, prime bag, stretches the concept of using squares, but it is a simple game dealing with primes. The game can be used with a whole class or with small groups.

Each student gets a small bag containing squares, each representing a prime number. There are 5 - 2s; 4 - 3s; 3 - 5s and 2 of each other prime up to and including 23.

Contest: How many numbers from 1 to 100 can you make up using the numbers in the bag and the operation of multiplication only? Make a list of all of your "successes".

Example: $30 = \boxed{5} \times \boxed{3} \times \boxed{2}$ $(5 \times 3 \times 2)$

$35 = \boxed{5} \times \boxed{7}$ (5×7)

Questions:

(a) How many different ways are there to construct each number? (This leads to the Fundamental Theorem of Arithmetic.)
(b) Are there any numbers you can't build? If so, how can you describe these?
(c) What is the largest number you can build using the squares in the bag and multiplication?

Polynomial puzzles

Materials: For each group a set like the following:
3 blue squares - 12 cm. x 12 cm.
10 blue rectangles - 12 cm. x 1 cm.
10 red rectangles - 12 cm. x 1 cm.
30 blue squares - 1 cm. x 1 cm.
30 red squares - 1 cm. x 1 cm.

All of the puzzles have the same direction. Given a certain subset of the set given above, make a blue rectangle. For example: (in the diagrams, unshaded will represent blue; shaded, red.)

$(12 \times 12) + (2 \times 12) + 1 = (12 + 1) \times (12 + 1)$

or $12^2 + 2 \times 12 + 1 = (12 + 1)^2$

or (if the big square is considered x^2, the rectangle x and each unit 1)

$x^2 + 2x + 1 = (x + 1)^2$

These puzzles can be done by individuals guided by an instruction sheet, or the 2-person game Rectangl-it, may be played.

Rectangl-it:

Materials: Like those described above.

Rules:

1. There are 2 positions - Setter and Maker.
2. On each play, the Setter sets the given subset of the playing set and acts as timer.
3. On each play, the Maker attempts to make a rectangle within 3 minutes.
4. Scoring: If the Maker completes a rectangle in less than:
   1 minute: 3 points
   2 minutes: 2 points
   3 minutes: 1 point

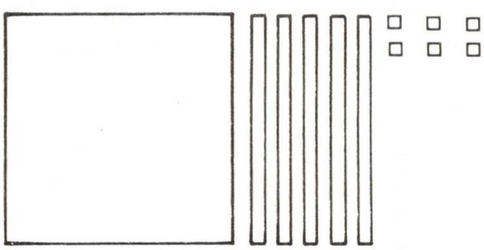
   If not, the Maker either gets no points or calls "no rectangle". If he can show that no rectangle can be made, he gets 3 points. If he calls "no rectangle" and one can be made he loses 3 points.
5. In each round, each player is the Setter and the Maker once.
6. A game is 4 rounds long.

More puzzles:

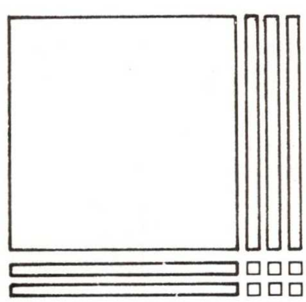
Given below are some puzzles, solutions and records. It is important that students, whether individually or in a game setting, keep accurate records of their attempts. From studying the diagrams and symbolic records, the student will be able to see the forms useful in factoring polynomials.

PUZZLE 1:

Given



Rectangle



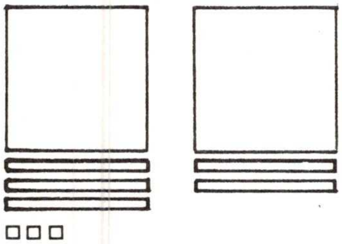
Record

$$12^2 + 5 \times 12 + 6 = (12 + 3)\ (12 + 2)$$

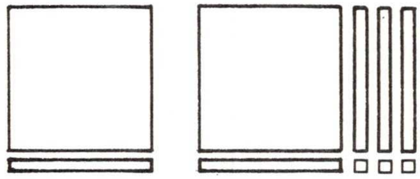
or $$x^2 + 5x + 6 = (x + 3)\ (x + 2)$$

PUZZLE 2:

Given



Rectangle



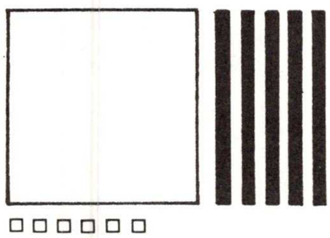
$2x^2 + 5x + 3 = (x + 1)(2x + 3)$

Note: Doing only a few such puzzles gives insufficient experience. Doing a large number allows the students to find relevant ideas such as "the factors of the constant are important". It is important for students to note that just as factoring numbers involved building rectangles, factoring polynomials involves building rectangles. This enables this "puzzle" activity to significantly contribute to polynomial problems.

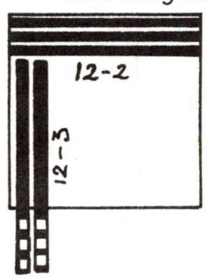
PUZZLE 3:

In this puzzle, the shaded areas stand for red and the unshaded for blue.

Given



Rectangle



Record

$12^2 - 5 \times 12 + 6 = (12 - 3)(12 - 2)$

$x^2 - 5x + 6 = (x - 3)(x - 2)$

Note: This puzzle illustrates the use of negative coefficients and illustrates that these puzzles can and should be challenging puzzles in their own right. It should be noted that red covers blue in these puzzles and that equal numbers of red and blue rectangles can be added without changing the character of the polynomial.

These puzzles represent an activity which is preliminary to symbolic factoring and which allows students to play while preserving the form of later symbolic activity. If the teacher wants a more guided activity, she can construct cards such as the following which makes use of 3 sets of materials like those described at the beginning of this section on polynomial puzzles except based on 5 cm., 7cm., and 12 cm.

CARD P1

1. Choose one card representing 5 x 5; 2 representing 5 x 1; and one unit.
   Build a rectangle.
   What are its dimensions?_________, ____________.
   This can be represented by the following sentence.

   $5^2 + 2 \times 5 + 1 = (5 + \_)\ (5 + \_)$

2. Build a rectangle from one 7 x 7; 2, 7 x 1; and one unit.
   Dimensions: __________, _________.
   Complete the following sentence.

   $7^2 + 2 \times 7 + 1 = (7 + \_)\ (7 + \_)$

3. Build a rectangle from one 12 x 12; 2, 12 x 1, and one unit.

   Dimensions: ________, ________.

   Complete the following sentence

   $12^2 + 2 \times 12 + 1 = (\qquad)\ (\qquad)$

4. Suppose you had to build a rectangle from one, 30 x 30; 2, 30 x 1 and one unit.

   Dimensions: ________, ________.

   $30^2 + 2 \times 30 + 1 = (\qquad)\ (\qquad)$

5. Complete the following chart:

One	Two	One	Dimensions	
5x5	5x1	unit	_ _, _ _,	$5^2 + 2 \times 5 + 1 = (\qquad)(\qquad)$
40x40	40x1	unit	_ _, _ _,	$40^2 + 2 \times 40 + 1 = (\qquad)(\qquad)$
1000x1000	1000x1	unit	_ _, _ _,	$1000^2 + 2 \times 1000 + 1 = (\qquad)(\qquad)$
nxn	nx1	unit	_ _, _ _,	$n^2 + 2n + 1 = (\qquad)(\qquad)$
yxy	yx1	unit	_ _, _ _,	$y^2 + 2y + 1 = (\qquad)(\qquad)$

6. Complete the following:

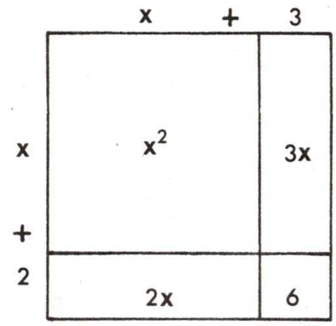
   $x^2 + 2x + 1 = (\qquad)^2$

Note: This card varies the same puzzle over several dimensions or numerical variants. It deliberately attaches the physical activity to the symbolic activity in a tightly prescribed form. It would best be used at a time when you really wished to concentrate on polynomial factoring and special forms. The previous puzzles and games might be profitable earlier. That is, this latter activity would best be used in Grades IX or X while the former could also be used in V, VI, VII or VIII. In order that this kind of activity be effective, cards for other factoring problems such as difference of squares would have to be used. Physical activity as a prelude to symbolic activity is not highly successful on a one-shot basis.

Squares, rectangles and computing

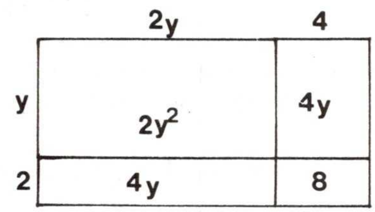
Teachers are probably familiar with rectangular pictures as models for binomial multiplication. These models are based on an area interpretation of multiplication. Some examples are given below.

1



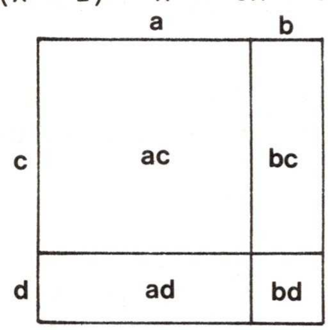
$(x + 3)\ (x + 2) = x^2 + 5x + 6$

2



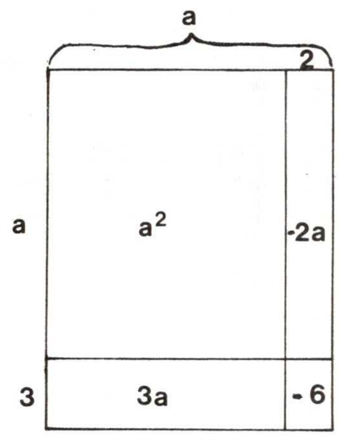
$(2y + 4)\ (y + 2) = 2y^2 + 8y + 8$

3



$(a + b)\ (c + d) = ac + bc + ad + bd$

4



$(a - 2)\ (a + 3)$
$= a^2 - 2a + 3a - 6$
$= a^2 + a - 6$

Making up activities such as these in which students create pictures of binomial multiplication is a means of better understanding the use of the distributive property. As is seen in example 4, this activity is interesting in itself.

This model is even more interesting with polynomials of higher order.

	x^2	$+ 2x$	$+ 3$
$2x^2$	$2x^4$	$4x^3$	$6x^2$
$+ x$	x^3	$2x^2$	$3x$
$+ 5$	$5x^2$	$10x$	15

$(x^2 + 2x + 3)\ (2x^2 + x + 5) = 2x^4 + 5x^3 + 13x^2 + 13x + 15$

One useful pictorial activity is to have students diagram 10 such multiplications. They may do this rather blindly but very likely they will reduce this activity to a kind of algorithm and instead of using rectangles will simply use a grid as on p.82.

	x^2	$+ 2x$	$+ 1$
$3x^2$	$3x^4$	$6x^3$	$3x^2$
$- 5x$	$-5x^3$	$-10x^2$	$-5x$
$+ 7$	$7x^2$	$14x$	7

$$(x^2 + 2x + 1)(3x^2 - 5x + 7)$$
$$= 3x^4 + x^3 + ox^2 + 9x + 7$$

After a few such examples, they will discover that like terms lie on the diagonals and hence invent a neat multiplication algorithm.

The following card might be made up:

> CARD PD1
>
> Using the grid picture, make up a method for dividing polynomials.

Note: This is an interesting open-ended activity especially if the polynomials do not divide evenly.

Another interesting pictorial activity is illustrated by the following card. This activity might be most useful in high school mathematics.

CARD BT1

1. Complete the following diagrams.

	x	1
x	x	x
1	x	1

$(x + 1)^2 = x^2 + 2x + 1$

	x^2	$2x$	1
x			
1			

$(x + 1)(x^2 + 2x + 1) = (x + 1)^3$

2. Continue this process through $(x + 1)^{10}$.

3. Study the rectangles in each of the stages above. How many squares does each contain? Diagonals?

Stage	Squares	Diagonals
$(x + 1)^2$	4	3
$(x + 1)^3$	6	4
$(x + 1)^4$		
"		
"		
$(x + 1)^{10}$		

4. How many squares and diagonals would $(x + 1)^{20}$, $(x + 1)^{100}$, $(x + 1)^n$ have?

   What does this tell you about the number of terms in the product $(x + 1)^n$?

Back to the world of numbers

The last section showed how area presents a nice algorithm for multiplying polynomials. Since numerical representations are polynomials of a sort, it should not be surprising that pictorial algorithms hold here as well. Given below are several illustrations of pictorial multiplication activity.

CARD M1 - Decimals

	.1	.01	.007
.2	.02	.002	.0014
.03	.003	.0003	.00021

.02
.005
.0017
.00021

.02691

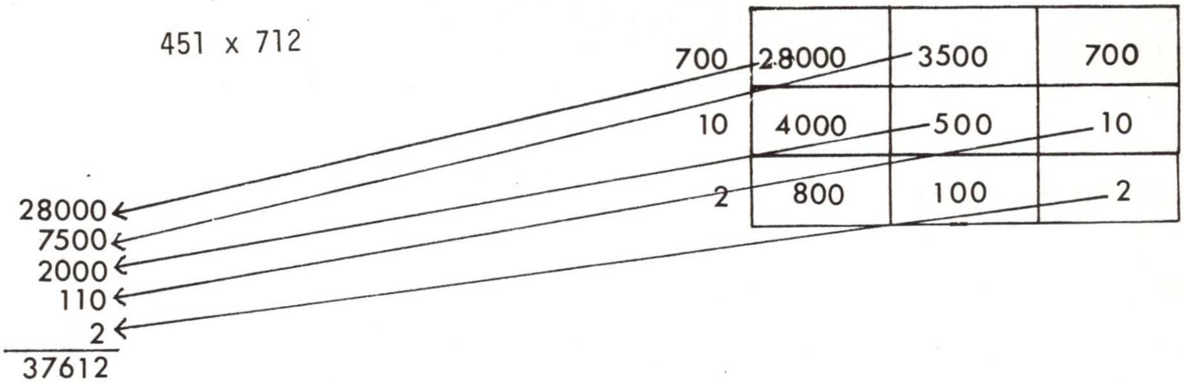
CARD M2 - Rationals greater than 1

2 1/5 x 1 3/7

	2	1/5
1	2	1/5
3/7	6/7	3/35

2 + 1/5 + 6/7 + 3/35 = 2 + 7/35 + 30/35 + 3/35
= 2 + 40/35
= 3 + 5/35
= 3 1/7

CARD M3 - Whole Numbers



Note: These algorithms represent one "understanding" approach to computation. From the last example, one can see that these are just models for a partial products approach. If one wished to add more realism to the problem, the student could construct "scale drawings" of the numbers for the sides of the computational rectangle.

SUMMING UP

The excursion using rectangles from numbers to polynomials and back was done for 2 purposes. Most importantly, it illustrates how physical and pictorial activity contribute to mathematics learning at several grade levels through the use of playing with "form". The second purpose was to illustrate that physical-pictorial activity was easy and inexpensive to use.

From the above, a teacher could expect to use activity extensively and at little cost. If such activity proved effective, it would clearly be cost-effective. There is no case from hard data that can be made that such activity represents a universal success. Yet it is hoped that the range of simple activities suggested above will give you the "rect"-angle on the use of mathematical activity in your classroom. From these suggestions, you may see many more ways to simply design active mathematical experiences which will be fun and productive for your students.

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